International Electronic Journal of Geometry
Volume 4 No. 2 pp. 13–20 (2011) ©IEJG

A NUMERICAL COMPUTATION OF $(k,3)$-ARCS IN THE LEFT SEMIFIELD PLANE OF ORDER 9

ZIYA AKÇA

(Communicated by Yusuf YAYLI)

Abstract. In this paper, an algorithm for the classification of $(k,3)-$arcs in the projective plane of order 9, coordinatized by elements of a left semifield, denoted by $SFPG(2,9)$ is given. Then, some examples of $(k,3)-$arcs are given, GAP, a computer-based exhaustive search.

1. Introduction and Preliminaries

A projective plane π consists of a set $\mathcal{P}$ of *points*, and a set $\mathcal{L}$ of subsets of $\mathcal{P}$, called *lines*, such that every pair of points is contained in exactly one line, every two distinct lines intersect in exactly one point, and there exist four points in such a position that they pairwise define six distinct lines. A subplane of a projective plane π is a set $\mathcal{B}$ of points and lines which is itself a projective plane, relative to the incidence relation given in π.

It is well known that any two projective planes with the same order n, $n \leq 8$, are isomorphic and every projective plane has also an algebraic structure obtained by coordinatization. Conversely, certain algebraic structures can be used to construct projective planes.

There exist at least four non-isomorphic projective planes of order 9. The known four distinct projective planes of order 9 are extensively studied by Room-Kirkpatrick[16]. These are Desarguesian plane, the left nearfield plane, the right nearfield plane and Hughes plane [12]. We will briefly give some information about the algebraic structures of these planes. Let $S = \{0,1,2,a,b,c,d,e,f\}$ and $\oplus$ be the additional operation on field $F = GF(9)$ where $b = a+1$, $c = a+2$, $d = a+a$, $e = d+1$, $f = d+2$ and $1+2 = a+d = 0$. The operation $\otimes$ on S is defined as in Table 1.

2000 *Mathematics Subject Classification.* 51E21, 94B05.

Key words and phrases. $(k,3)-$arcs, Projective plane, Computer search.

⊗	0	1	2	a	b	c	d	e	f
0	0	0	0	0	0	0	0	0	0
1	0	1	2	a	b	c	d	e	f
2	0	2	1	d	f	e	a	c	b
a	0	a	d	2	e	b	1	f	c
b	0	b	f	c	2	d	e	a	1
c	0	c	e	f	a	2	b	1	d
d	0	d	a	1	c	f	2	b	e
e	0	e	c	b	d	1	f	2	a
f	0	f	b	e	1	a	c	d	2

Table 1

Then $(S, \oplus, \otimes)$ is the right nearfield. This nearfield is doneted by $S(9)$. The projective plane $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ whose algebraic structure is the right nearfield plane of order 9 is constructed in following manner [16]: set of the points;

$$\mathcal{P} = \{(x, y, 1) : x, y \in S\} \cup \{(1, x, 0) : x \in S\} \cup \{(0, 1, 0)\},$$

set of the lines;

$$\mathcal{L} = \{[m, 1, k] : m, k \in S\} \cup \{[1, 0, k] : k \in S\} \cup \{[0, 0, 1]\}$$

and incidence relation,

$$\mathcal{I} : (x, y, z) \ o \ [m, n, k] \Leftrightarrow xm + yn + zk = 0.$$

Projective plane of order 9 constructed by $S(9)$ is denoted by $\Pi_S(9)$.

If operation $*$ is defined as $x * y = y \otimes x$ for all $x, y \in S(9)$, $(S, \oplus, *)$ is the left nearfield. The plane constructed over this nearfield is the dual plane of $\Pi_S(9)$ and it is not isomorphic to $\Pi_S(9)$, [17]. This plane is denoted by $\Pi_S^d(9)$. Hughes planes represented by $\Pi_H(9)$ is the other plane of order 9 which has nonlinear ternary ring and hence it is different from projective planes $\Pi_S(9)$ and $\Pi_S^d(9)$.

Let $q = p^h$, where p is odd prime and h is a positive integer. The existence of the right nearfield of order q^2 and the construction methods of the Hughes planes over this nearfield can be seen from [12]. The construction of the smallest Hughes plane for $q = 3$ and self duality of this plane are given in [17].

$\mathbb{P}_2F$, $\Pi_S(9)$, $\Pi_S^d(9)$ and $\Pi_H(9)$ constructing in these manner consist of all known projective planes of order 9. In [16], more detailed information about these distinct four planes is given.

Getting in the search which is done by Lam [13] by computer on projective planes of order 9 is worked on 283.657 non-isomorphic Latin squares, it is note that it can lead the lost a branch of the search because of unknown hardware error or occuring an error in computer; and that there is a possibility that this is a only branch where new plane occurs. Thus, it prevents to definite decision for computer programs. Because of the agreement of the computer results with those obtained by theorical means, it is claimed that the computer program is correct and that there is no another projective plane order 9. It can be seen to [13] for more detail information.

It is essential to characterize certain subsets of the plane. Some of the essential subsets of the plane are Fano planes and arcs. A Fano plane is a projective plane of order 2. A Fano plane also occurs as a subplane of many larger projective planes. Therefore, the discovery of the Fano plane has played an important role in the improvement of the theory of finite geometries. Fibered projective plane and Fano subplanes of some projective planes have been examined by many authors. For instance, Akça-Kaya [1], Akça-Günaltılı & Güney [2], Akpınar [3], Bayar-Ekmekçi & Akça [6], Çifçi-Kaya [7], Room-Kirpatrick [16],etc.

A $(k,2)-$ arc K is a set of k points no 3 of which are collinear in the plane. A $(k,2)-$arc is called simply an arc of size k or a k arc. For a detailed description of the most important properties of these geometric structures, we refer the reader to [10]. A $(k,3)-$ arc K is a set of k points no 4 of which are collinear of this plane. In [11] the relationship between the theory of complete $(k,r)-$arcs, coding theory and mathematical statistics is presented. S. Marcugini - A. Milani and F. Pambianco [14] classificiated all $(k,3)-$arcs in $PG(2,7)$ using MAGMA. R.N. Daskalov and M.E.J. Contreras [8] gave new $(k;r)-$arcs in $PG(2,13)$.

The largest size of a $(k,r)-$arc of $PG(2,q)$ is indicated by $m_r(2,q)$. In [5] and [11], bounds for $m_r(2,q)$ are given. In particular, $m_3(2,q) \leq 2q+1$ for $q \geq 4$, [18].

A general method of generating semifield was given by Hall (1959), [9]. A left semifield of order 9 is defined as follows:

Definition 1.1. A left semifield is a system $(S,+,\cdot)$, where $+$ and $\cdot$ are binary operations on the set S and

i) S is finite
ii) $(S,+)$ is a group, with identity 0
iii) $(S\backslash\{0\},\cdot)$ is a semi-group, with identity 1
iv) $x \cdot 0 = 0$ for all $x \in S$
v) $\cdot$ is left distributive over $+$, that is $x\cdot(y+z) = (x\cdot y)+(x\cdot z)$ for all $x,y,z \in S$
vi) Given $a,b,c \in S$ with $a \neq b$, there exists a unique $x \in S$ such that

$$-a\cdot x + b\cdot x = c.$$

Example 1.1. Let $(F_3,+,.)$ be the field of integers modulo 3. Let S be

$$S = \{a+\lambda b \ : \ a,b \in F_3,\ \lambda \notin F_3\}$$

and we consider the addition and multiplication on S given by

$$(a+\lambda b) \oplus (c+\lambda d) = (a+c) + \lambda(b+d) \tag{1}$$

and

$$(a+\lambda b) \odot (c+\lambda d) = \begin{cases} ac+\lambda(ad), & \text{if} \quad b=0 \\ ac-b^{-1}df(a)+\lambda(bc-ad+d)), & \text{if} \quad b\neq 0 \end{cases} \tag{2}$$

where, $f(t) = t^2 - t - 1$ is a irreducible polynomial on F_3.

For the sake of shortness, if we use ab instead of $a + \lambda b$ in equation (1) and (2) then addition and multiplication tables are as follows:

$\oplus$	00	01	02	10	11	12	20	21	22
00	00	01	02	10	11	12	20	21	22
01	01	02	00	11	12	10	21	22	20
02	02	00	01	12	10	11	22	20	21
10	10	11	12	20	21	22	00	01	02
11	11	12	10	21	22	20	01	02	00
12	12	10	11	22	20	21	02	00	01
20	20	21	22	00	01	02	10	11	12
21	21	22	20	01	02	00	11	12	10
22	22	20	21	02	00	01	12	10	11

$Table$ 2

$\odot$	00	01	02	10	11	12	20	21	22
00	00	00	00	00	00	00	00	00	00
01	00	11	22	01	12	20	02	10	21
02	00	21	12	02	20	11	01	22	10
10	00	01	02	10	11	12	20	21	22
11	00	10	20	11	21	01	22	02	12
12	00	20	10	12	02	22	21	11	01
20	00	02	01	20	22	21	10	12	11
21	00	22	11	21	10	02	12	01	20
22	00	12	21	22	01	10	11	20	02

$Table3$

the system $(S, \oplus, \odot)$ is a left semifield of order 9.

Finally, we consider the projective plane of order 9 coordinatised by elements of the above left semifield and investigate Fano subplanes and $(k, 3)-$arcs of this plane.

A regular quadrangle in a projective plane is a set of four points of which no three are collinear. If $ABCD$ is a regular quadrangle, the six lines AB, AC, AD, BC, BD, CD are called the sides of the quadrangle, and the three points $V = AB \cap CD$, $W = AC \cap BD$, $U = AD \cap BC$ are called the diagonal points of the quadrangle. If the diagonal points of a regular quadrangle are collinear then the incidence structure $(\mathcal{P}, \mathcal{L})$ with

$$\mathcal{P} = \{A, B, C, D, U, V, W\}$$

and

$$\mathcal{L} = \{ABV, ACW, ADU, BCU, BDW, CDV, UVW\}$$

is a Fano plane. Such a Fano plane is called the completion of the regular quadrangle. If the diagonal points V, W, U are not collinear it is said that the quadrangle does not determine a Fano subplane.

The Plane P_2S : The 91 points of P_2S are the elements of the set

$$\{(x, y) \ : \ x, y \in S\} \cup \{(m) : m \in S\} \cup \{(\infty)\}.$$

The points of the form (x, y) are called *proper points*, and the unique point (∞) and the points of the form (m) are called *ideal points*. The 91 lines of P_2S are defined as a set of points satisfying one of the three conditions:

$$[m,k] = \{(x,y) \in S^2 : y = m \odot x \oplus k\} \cup \{(m)\}$$
$$[\lambda] = \{(x,y) \in S^2 : \; x = \lambda\} \cup \{(\infty)\}$$
$$[\infty] = \{(m) \in S\} \cup \{(\infty)\}$$

The 81 lines having form $y = m \odot x \oplus k$ and 9 lines having equation of the form $x = \lambda$ are called the *proper lines* and the unique line $[\infty]$ is called *the ideal line.*

The system of points, lines and incidence relation given above defines a projective plane of order 9 denoted by $SFPG(2,9)$, which is the left semifield plane.

2. Fano Subplanes of P_2S :

We consider the four distinct points $O = (0+\lambda 0, 0+\lambda 0) := (00,00)$, $I = (1+\lambda 0, 1+\lambda 0) := (10,10)$, $X = (0+\lambda 0) := (00)$ and $P_i = (a+\lambda b, c+\lambda d) := (ab, cd)$, $i \in \{1,2,...,6\}$.

A regular quadrangle $OIXP_i$ can be completed to a Fano plane if and only if the diagonal points $OI \cap XP_i = V_i$, $OP_i \cap IX = U_i$, $OX \cap IP_i = W_i$, $i \in \{1,2,...,6\}$, are collinear.

Clearly, each of 48 Fano subplanes of P_2S containing O, I, X has a line passing through (∞). It is also known that every Fano subplane of P_2S has exactly one ideal point. $X = (00)$ is the ideal point of the above 48 Fano subplanes which is paired with (∞). In any Fano subplane let V be an ideal point with V', and let A and B be two proper points such that $V, V' \notin AB$. Then A, B, V can be mapped to O, I, X by a collination mapping the Fano subplane to a Fano subplane containing O, I, X.

Proposition 2.1. *[see 2] The number of Fano subplanes which are completions of* $AVBP$ *is* 414720.

Now, in this paper, an algorithm for the classification of the $(k,3)-$ arcs in the left semifield plane of order 9 is defined and some examples of the $(k,3)-$arcs are given using a computer-based exhaustive search.

3. The algorithm for the classification of the $(k,3)-$ arcs in $SFPG(2,9)$

In this section, the algorithm used in the classification is described. It is known that a finite projective plane is coordinatized by quadrangle $OIXP$. Therefore, $(k,3)-$arcs of $SFPG(2,9)$ are determined with this quadrangle

$$G0 = \{O = [1],\ I = [4],\ X = [28],\ P = [31]\}$$

and $l_1 = OX$, $l_2 = PX$, $l_3 = OP$, $l_4 = IP$, $l_5 = OI$, $l_6 = IX$, using GAP ([4] and [15]).

Step 1: Let $G0 = \{O = [1],\ I = [4],\ X = [28],\ P = [31]\}$ be a quadrangle. Since $SFPG(2,9) \setminus G0$ contains exactly 87 points, the total number of $(5,3)-$arcs which contain $G0$ is 87.

Step 2: Let

$$G_j = G0 \cup \{X_i \; : \; X_i \in l_i,\ 1 \le i \le j\}$$

for each j, $1 \le j \le 14$. One can easily find $(5+t,3)-$arcs which contain Gj, for each j,k, $1 \le j \le 14$, $0 \le t \le 14$ using GAP.

4. Some examples of the $(k,3)-$arcs in $SFPG(2,9)$

Now, we will give some examples of $(5+t,3)-$arcs, for $t \in \{1,2,3\}$, using GAP, respectively.

Example 4.1. $(6,3)-$arcs which contain $G0$

```
gap> Read("zakca.gi");
gap> G2:=[[1],[4],[7],[28],[31]];

gap> Set(G2);
[ [1],[4],[7],[10],[28],[31] ];[ [1],[4],[7],[11],[28],[31] ];
[ [1],[4],[7],[12],[28],[31] ];[ [1],[4],[7],[13],[28],[31] ];
[ [1],[4],[7],[14],[28],[31] ];[ [1],[4],[7],[15],[28],[31] ];
[ [1],[4],[7],[16],[28],[31] ];[ [1],[4],[7],[17],[28],[31] ];
[ [1],[4],[7],[18],[28],[31] ];[ [1],[4],[7],[19],[28],[31] ];
[ [1],[4],[7],[20],[28],[31] ];[ [1],[4],[7],[21],[28],[31] ];
[ [1],[4],[7],[22],[28],[31] ];[ [1],[4],[7],[23],[28],[31] ];
[ [1],[4],[7],[24],[28],[31] ];[ [1],[4],[7],[25],[28],[31] ];
[ [1],[4],[7],[26],[28],[31] ];[ [1],[4],[7],[27],[28],[31] ];
[ [1],[4],[7],[28],[29],[31] ];[ [1],[4],[7],[28],[30],[31] ];
[ [1],[4],[7],[28],[31],[32] ];[ [1],[4],[7],[28],[31],[33] ];
[ [1],[4],[7],[28],[31],[34] ];[ [1],[4],[7],[28],[31],[35] ];
[ [1],[4],[7],[28],[31],[36] ];[ [1],[4],[7],[28],[31],[37] ];
[ [1],[4],[7],[28],[31],[38] ];[ [1],[4],[7],[28],[31],[39] ];
[ [1],[4],[7],[28],[31],[40] ];[ [1],[4],[7],[28],[31],[41] ];
[ [1],[4],[7],[28],[31],[42] ];[ [1],[4],[7],[28],[31],[43] ];
[ [1],[4],[7],[28],[31],[44] ];[ [1],[4],[7],[28],[31],[45] ];
[ [1],[4],[7],[28],[31],[46] ];[ [1],[4],[7],[28],[31],[47] ];
[ [1],[4],[7],[28],[31],[48] ];[ [1],[4],[7],[28],[31],[49] ];
[ [1],[4],[7],[28],[31],[50] ];[ [1],[4],[7],[28],[31],[51] ];
[ [1],[4],[7],[28],[31],[52] ];[ [1],[4],[7],[28],[31],[53] ];
[ [1],[4],[7],[28],[31],[54] ];[ [1],[4],[7],[28],[31],[55] ];
[ [1],[4],[7],[28],[31],[56] ];[ [1],[4],[7],[28],[31],[57] ];
[ [1],[4],[7],[28],[31],[58] ];[ [1],[4],[7],[28],[31],[59] ];
[ [1],[4],[7],[28],[31],[60] ];[ [1],[4],[7],[28],[31],[61] ];
[ [1],[4],[7],[28],[31],[62] ];[ [1],[4],[7],[28],[31],[63] ];
[ [1],[4],[7],[28],[31],[64] ];[ [1],[4],[7],[28],[31],[65] ];
[ [1],[4],[7],[28],[31],[66] ];[ [1],[4],[7],[28],[31],[67] ];
[ [1],[4],[7],[28],[31],[68] ];[ [1],[4],[7],[28],[31],[69] ];
[ [1],[4],[7],[28],[31],[70] ];[ [1],[4],[7],[28],[31],[71] ];
[ [1],[4],[7],[28],[31],[72] ];[ [1],[4],[7],[28],[31],[73] ];
[ [1],[4],[7],[28],[31],[74] ];[ [1],[4],[7],[28],[31],[75] ];
[ [1],[4],[7],[28],[31],[76] ];[ [1],[4],[7],[28],[31],[77] ];
[ [1],[4],[7],[28],[31],[78] ];[ [1],[4],[7],[28],[31],[79] ];
[ [1],[4],[7],[28],[31],[80] ];[ [1],[4],[7],[28],[31],[81] ];
[ [1],[4],[7],[28],[31],[82] ];[ [1],[4],[7],[28],[31],[83] ];
[ [1],[4],[7],[28],[31],[84] ];[ [1],[4],[7],[28],[31],[85] ];
[ [1],[4],[7],[28],[31],[86] ];[ [1],[4],[7],[28],[31],[87] ];
[ [1],[4],[7],[28],[31],[88] ];[ [1],[4],[7],[28],[31],[89] ];
```

```
[ [1],[4],[7],[28],[31],[90] ].
```

Example 4.2. $(7,3)-$arcs which contain $G0$

```
gap> Read("zakca.gi");
gap> G3:=[[1],[4],[7],[28],[31],[58]];

gap> Set(G3);
[ [1],[4],[7],[10],[28],[31],[58] ];[ [1],[4],[7],[11],[28],[31],[58] ];
[ [1],[4],[7],[11],[28],[31],[58] ];[ [1],[4],[7],[12],[28],[31],[58] ];
[ [1],[4],[7],[14],[28],[31],[58] ];[ [1],[4],[7],[15],[28],[31],[58] ];
[ [1],[4],[7],[16],[28],[31],[58] ];[ [1],[4],[7],[18],[28],[31],[58] ];
[ [1],[4],[7],[19],[28],[31],[58] ];[ [1],[4],[7],[20],[28],[31],[58] ];
[ [1],[4],[7],[21],[28],[31],[58] ];[ [1],[4],[7],[23],[28],[31],[58] ];
[ [1],[4],[7],[24],[28],[31],[58] ];[ [1],[4],[7],[25],[28],[31],[58] ];
[ [1],[4],[7],[26],[28],[31],[58] ];[ [1],[4],[7],[28],[29],[31],[58] ];
[ [1],[4],[7],[28],[30],[31],[58] ];[ [1],[4],[7],[28],[31],[32],[58] ];
[ [1],[4],[7],[28],[31],[33],[58] ];[ [1],[4],[7],[28],[31],[34],[58] ];
[ [1],[4],[7],[28],[31],[35],[58] ];[ [1],[4],[7],[28],[31],[36],[58] ];
[ [1],[4],[7],[28],[31],[37],[58] ];[ [1],[4],[7],[28],[31],[39],[58] ];
[ [1],[4],[7],[28],[31],[41],[58] ];[ [1],[4],[7],[28],[31],[42],[58] ];
[ [1],[4],[7],[28],[31],[43],[58] ];[ [1],[4],[7],[28],[31],[44],[58] ];
[ [1],[4],[7],[28],[31],[45],[58] ];[ [1],[4],[7],[28],[31],[46],[58] ];
[ [1],[4],[7],[28],[31],[47],[58] ];[ [1],[4],[7],[28],[31],[50],[58] ];
[ [1],[4],[7],[28],[31],[51],[58] ];[ [1],[4],[7],[28],[31],[52],[58] ];
[ [1],[4],[7],[28],[31],[53],[58] ];[ [1],[4],[7],[28],[31],[54],[58] ];
[ [1],[4],[7],[28],[31],[55],[58] ];[ [1],[4],[7],[28],[31],[56],[58] ];
[ [1],[4],[7],[28],[31],[57],[58] ];[ [1],[4],[7],[28],[31],[58],[59] ];
[ [1],[4],[7],[28],[31],[58],[60] ];[ [1],[4],[7],[28],[31],[58],[61] ];
[ [1],[4],[7],[28],[31],[58],[62] ];[ [1],[4],[7],[28],[31],[58],[63] ];
[ [1],[4],[7],[28],[31],[58],[64] ];[ [1],[4],[7],[28],[31],[58],[65] ];
[ [1],[4],[7],[28],[31],[58],[66] ];[ [1],[4],[7],[28],[31],[58],[69] ];
[ [1],[4],[7],[28],[31],[58],[70] ];[ [1],[4],[7],[28],[31],[58],[71] ];
[ [1],[4],[7],[28],[31],[58],[72] ];[ [1],[4],[7],[28],[31],[58],[73] ];
[ [1],[4],[7],[28],[31],[58],[74] ];[ [1],[4],[7],[28],[31],[58],[75] ];
[ [1],[4],[7],[28],[31],[58],[77] ];[ [1],[4],[7],[28],[31],[58],[79] ];
[ [1],[4],[7],[28],[31],[58],[80] ];[ [1],[4],[7],[28],[31],[58],[81] ];
[ [1],[4],[7],[28],[31],[58],[83] ];[ [1],[4],[7],[28],[31],[58],[84] ];
[ [1],[4],[7],[28],[31],[58],[86] ];[ [1],[4],[7],[28],[31],[58],[87] ];
[ [1],[4],[7],[28],[31],[58],[88] ];[ [1],[4],[7],[28],[31],[58],[89] ];
[ [1],[4],[7],[28],[31],[58],[90] ].
```

Example 4.3. $(8,3)-$arcs which contain $G0$

```
gap> Read("zakca.gi");
gap> G4:=[[1],[4],[7],[28],[31],[58],[61]];

gap> Set(G4);
[ [1],[4],[7],[10],[28],[31],[58],[61] ];[ [1],[4],[7],[12],[28],[31],[58],[61] ];
[ [1],[4],[7],[14],[28],[31],[58],[61] ];[ [1],[4],[7],[16],[28],[31],[58],[61] ];
[ [1],[4],[7],[18],[28],[31],[58],[61] ];[ [1],[4],[7],[19],[28],[31],[58],[61] ];
[ [1],[4],[7],[20],[28],[31],[58],[61] ];[ [1],[4],[7],[24],[28],[31],[58],[61] ];
[ [1],[4],[7],[25],[28],[31],[58],[61] ];[ [1],[4],[7],[26],[28],[31],[58],[61] ];
[ [1],[4],[7],[28],[29],[31],[58],[61] ];[ [1],[4],[7],[28],[30],[31],[58],[61] ];
[ [1],[4],[7],[28],[31],[32],[58],[61] ];[ [1],[4],[7],[28],[31],[33],[58],[61] ];
[ [1],[4],[7],[28],[31],[34],[58],[61] ];[ [1],[4],[7],[28],[31],[35],[58],[61] ];
[ [1],[4],[7],[28],[31],[36],[58],[61] ];[ [1],[4],[7],[28],[31],[37],[58],[61] ];
```

```
[ [1],[4],[7],[28],[31],[42],[58],[61] ];[ [1],[4],[7],[28],[31],[43],[58],[61] ];
[ [1],[4],[7],[28],[31],[44],[58],[61] ];[ [1],[4],[7],[28],[31],[45],[58],[61] ];
[ [1],[4],[7],[28],[31],[46],[58],[61] ];[ [1],[4],[7],[28],[31],[50],[58],[61] ];
[ [1],[4],[7],[28],[31],[52],[58],[61] ];[ [1],[4],[7],[28],[31],[53],[58],[61] ];
[ [1],[4],[7],[28],[31],[54],[58],[61] ];[ [1],[4],[7],[28],[31],[55],[58],[61] ];
[ [1],[4],[7],[28],[31],[56],[58],[61] ];[ [1],[4],[7],[28],[31],[57],[58],[61] ];
[ [1],[4],[7],[28],[31],[58],[59],[61] ];[ [1],[4],[7],[28],[31],[58],[60],[61] ];
[ [1],[4],[7],[28],[31],[58],[61],[62] ];[ [1],[4],[7],[28],[31],[58],[61],[63] ];
[ [1],[4],[7],[28],[31],[58],[61],[64] ];[ [1],[4],[7],[28],[31],[58],[61],[65] ];
[ [1],[4],[7],[28],[31],[58],[61],[66] ];[ [1],[4],[7],[28],[31],[58],[61],[69] ];
[ [1],[4],[7],[28],[31],[58],[61],[70] ];[ [1],[4],[7],[28],[31],[58],[61],[73] ];
[ [1],[4],[7],[28],[31],[58],[61],[74] ];[ [1],[4],[7],[28],[31],[58],[61],[75] ];
[ [1],[4],[7],[28],[31],[58],[61],[77] ];[ [1],[4],[7],[28],[31],[58],[61],[79] ];
[ [1],[4],[7],[28],[31],[58],[61],[83] ];[ [1],[4],[7],[28],[31],[58],[61],[84] ];
[ [1],[4],[7],[28],[31],[58],[61],[86] ];[ [1],[4],[7],[28],[31],[58],[61],[87] ];
[ [1],[4],[7],[28],[31],[58],[61],[89] ];[ [1],[4],[7],[28],[31],[58],[61],[90] ].
```

If $t \in \{4, 5, ..., 14\}$ then other examples can be find similarly as the above examples, using GAP.

References

[1] Z. Akça, -R. Kaya., *On the Subplanes of the Cartesian Group Plane of order 25*, Türk Matematik Derneği X. Ulusal Matematik Sempozyumu, 1-5 Eylül (1997) 1-7.

[2] Z. Akça, İ. Günaltılı & Ö. Güney., *On the Fano Subplanes of the Left Semifield Plane of Order 9*, Hacettepe Journal of Mathematics and Statistics Vol. 35 (1), (2006), 55-61.

[3] A. Akpınar., *On some projective planes of finite order*, G.U. Journal of Science 18 (2) (2005) 315-325.

[4] M. Alp, C. D. Wensley., *XMOD: Crossed Modules and Cat1-groups in GAP*, GAP programı ortak paketi, Bölüm 73 (1997) 1357-1422.

[5] S. Ball, *Multiple blocking sets and arcs in finite planes*, J. London Math. Soc. 54 581-593, (1996).

[6] A. Bayar, S. Ekmekçi & Z. Akça., *A note on fibered projective plane geometry*, Information Science, 178 (2008) 1257-1262.

[7] S. Çiftçi -R. Kaya., *On the Fano Subplanes in the Translation Plane of order 9*, Doğa-Tr. J. of Mathematics 14 (1990), 1-7.

[8] R.N. Daskalov and M.E.J. Contreras., *New $(k; r)-$arcs in the projective plane of order thirteen*, J. Geo. 80 (2004) 10-22.

[9] Jr. M. Hall., *Theory of Groups,* The Macmillan Company, New York (1959).

[10] J.W.P. Hirschfeld., *Projective Geometries over Finite Fields*, second edition, Oxford University Press. Oxford, (1998).

[11] J.W.P. Hirschfeld and L. Storme., *The packing problem in statistics, coding theory and finite projective spaces*: update 2001, in: Finite Geometries, Proceedings of the Fourth Isle of Thorns Conference A. Blokhuis, J.W.P. Hirschfeld, D. Jungnickel and J.A. Thas, Eds., Developments in Mathematics, Kluwer Academic Publishers, Boston, (2000) 201-246.

[12] D.R. Hughes- F.C. Piper, *Projective Planes*, Springer – Verlag, New York Inc, (1973) 196-201 .

[13] C.W.H. Lam-G. Kolesova & L.A. Thiel., *Computer Search for Finite Projective Planes of Order 9*, Discrete Mathematics, 92 (1991) 187-195.

[14] S. Marcugini, A. Milani and F. Pambianco., *Classification of the $(n, 3)-$arcs in $PG(2, 7)$*, J. Geom. 80 (2004) 179-184.

[15] A. Odabas., *Crossed Modules of Algebras with GAP*, Ph.D. Thesis, Osmangazi University (2009).

[16] T.G. Room-P.B. Kirkpatrick., *Miniquaternion Geometry*, Cambridge University Press, 176s (1971).

[17] F.W. Stevenson., *Projective Planes*, W. H. Freeman and Company, San Francisco, 416s (1972).

[18] J.A. Thas, *Some results concerning* $((q+1)(n-1), n)-arcs$, J. Combin Theory Ser. A 19 228-232, (1975).

Department of Mathematics and Computer Sciences, Eskişehir Osmangazi University-TURKEY
E-mail address: zakca@ogu.edu.tr