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SOME TYPES OF WEAKLY SYMMETRIC RIEMANNIAN MANIFOLDS

BANDANA DAS AND A. BHATTACHARYYA

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Abstract. In this paper we study quasi-conformally flat and pseudo projectively flat weakly symmetric Riemannian manifolds. Here we prove a quasi-conformally flat $(WS)_n(n > 3)$ of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature and this manifold of non-vanishing scalar curvature is a quasi-Einstein manifold and manifold of quasi-constant curvature with respect to the 1-form T defined by $T(X) = B(X) - D(X) \neq 0$, $\forall\, X$. Also we obtain that a pseudo-projectively flat $(WS)_n(n > 3)$ of non-zero constant scalar curvature is a manifold of pseudo quasi-constant curvature and with non-vanishing scalar curvature is a quasi-Einstein manifold and manifold of pseudo quasi-constant curvature with respect to above 1-form T.

1. Introduction

L.Tamassy and T.Q.Binh [6] introduced weakly symmetric Riemannian manifold. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric if the curvature tensor R of type (0,4) satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V)\\ &+C(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V)\\ &+E(V)R(Y,Z,U,X) \end{aligned} \tag{1.1}$$

$\forall$ vector fields $X, Y, Z, U, V \in \chi(M^n)$, where A, B, C, D and E are 1-forms (non-zero simultaneously) and ∇ is the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. U.C. De and S. Bandyopadhyay in [3] proved that if the associated 1-forms satisfy $B = C$ and $D = E$, the defining condition of a $(WS)_n$ reduces to the following form

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V)\\ &+B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V)\\ &+D(V)R(Y,Z,U,X). \end{aligned} \tag{1.2}$$

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According to Yano and Sawaki [8] a quasi-conformal curvature tensor $C^\star$ is defined by

$$C^\star(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X$$
$$-S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$
$$-\frac{\gamma}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y] \tag{1.3}$$

where a, b are constants and R, Q, γ are the Riemannian curvature tensor of type (1,3), the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and the scalar curvature respectively.

Chen and Yano in [2] introduced a Riemannian manifold $(M^n, g)(n > 3)$ of quasi-constant curvature which is conformally flat, and its curvature tensor R of type $(0,4)$ has the form

$$R(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$
$$+b[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W)$$
$$+g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)] \tag{1.4}$$

where a, b are non-zero scalars.

According to Shaikh and Jana [4], a Riemannian manifold $(M^n, g)(n > 3)$ is said to be of hyper quasi-constant curvature if it is conformally flat, and its curvature tensor R of type $(0,4)$ satisfies the condition

$$R(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$
$$+g(X,W)P(Y,Z) - g(X,Z)P(Y,W)$$
$$+g(Y,Z)P(X,W) - g(Y,W)P(X,Z) \tag{1.5}$$

where a is non-zero scalar and P is a tensor of type $(0,2)$.

From [5] a pseudo projective curvature tensor $\bar{P}$ is defined by

$$\bar{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y]$$
$$-\frac{r}{n}[\frac{a}{n-1} + b][g(Y,Z)X - g(X,Z)Y] \tag{1.6}$$

where a, b are constants such that $a, b \neq 0$; R, S, r are the Riemannian curvature tensor, the Ricci tensor and scalar curvature, respectively.

In this present paper we like to introduce pseudo quasi-constant curvature.

Definition 1.1. A Riemannian manifold $(M^n, g)(n > 3)$ is said to be of pseudo quasi-constant curvature if it is pseudo projectively flat and its curvature tensor R of type $(0,4)$ satisfies

$$\acute{R}(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$
$$+P(Y,Z)g(X,W) - P(X,Z)g(Y,W) \tag{1.7}$$

where a is constant and P is a tensor of type $(0,2)$.

Section 2 is concerned with preliminary results of $(WS)_n$. In section 3 and 4 we study quasi-conformally flat $(WS)_n$ and pseudo-projectively flat $(WS)_n$.

2. Preliminaries

Let $\{e_i\}$ $i = 1, 2, ..., n$ be an orthonormal basis of the tangent spaces in a neighbourhood of a point of the manifold. Then setting $Y = V = e_i$ in (1.2), and taking summation over i, $1 \le i \le n$, we get

$$\begin{aligned}(\nabla_X S)(Z,U) &= A(X)S(Z,U) + B(Z)S(X,U)\\ &+D(U)S(X,Z) + B(R(X,Z)U)\\ &+D(R(X,U)Z)\end{aligned} \tag{2.1}$$

where S is the Ricci tensor of type $(0,2)$.

From (2.1) it follows that a $(WS)_n(n > 2)$ is weakly Ricci symmetric (briefly $(WRS)_n(n > 2)$) [7] if

$$B(R(X,Z)U) + D(R(X,U)Z) = 0\ ,\ \forall\ X,\ U,\ Z \in \chi(M^n). \tag{2.2}$$

From (2.1) it follows that

$$dr(X) = rA(X) + 2B(QX) + 2D(QX) \tag{2.3}$$

where r is the scalar curvature of the manifold.

From [4] we have, if a $(WS)_n(n > 2)$ is of non-zero constant scalar curvature, the 1-form A can be expressed as

$$A(X) = -\frac{2}{r}[B(QX) + D(QX)]\ ,\ \forall\ X. \tag{2.4}$$

If a $(WS)_n(n > 2)$ is of zero scalar curvature then from (2.4) we get the relation

$$B(QX) + D(QX) = 0\ ,\ \forall\ X. \tag{2.5}$$

Then from (2.1) we obtain

$$T(QX) = \frac{r}{2}T(X) \tag{2.6}$$

where the vector field ρ is defined by

$$T(X) = g(X,\rho) = B(X) - D(X)\ ,\ \forall\ X. \tag{2.7}$$

Also from [4] we get in a $(WS)_n(n > 2)$ the relation

$$T(Z)S(X,U) - T(U)S(X,Z) - T(R(Z,U)X) = 0 \tag{2.8}$$

holds for all vector fields X, Z, U and T is a 1-form.

3. Quasi-conformally flat $(WS)_n$

Let $(M^n, g)(n > 3)$ be a quasi-conformally flat $(WS)_n$. Then from(1.3), we obtain

$$\acute{R}(X,Y,Z,U) = -\frac{b}{a}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) + g(Y,Z)S(X,U)$$

$$-g(X,Z)S(Y,U)] + \frac{\gamma}{n}[\frac{1}{n-1} + \frac{2b}{a}][g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \tag{3.1}$$

where $g(R(X,Y)Z,U) = \acute{R}(X,Y,Z,U)$.

Putting $X = U = e_i$ in (3.1) where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking the summation over i, where $1 \leq i \leq n$, we get

$$S(Y,Z) = \alpha g(Y,Z) \tag{3.2}$$

where $\alpha = \frac{\gamma}{1+\frac{b}{a}(n-2)}[-\frac{b}{a} + \frac{1}{n}(1 + \frac{2b(n-1)}{a})]$

Here we can say that quasi conformally flat $(WS)_n$ is an Einstein manifold.

Now from (3.2), we get

$$(\nabla_X S)(Y,Z) = \alpha' d\gamma(X)g(Y,Z) \tag{3.3}$$

where $\alpha' = \frac{1}{1+\frac{b(n-2)}{a}}[-\frac{b}{a} + \frac{1}{n}(1 + \frac{2b(n-1)}{a})]$

Similarly we can get

$$(\nabla_Z S)(Y,X) = \alpha' d\gamma(Z)g(Y,X) \tag{3.4}$$

Now subtracting (3.4) from (3.3), we get

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \alpha'[d\gamma(X)g(Y,Z) - d\gamma(Z)g(Y,X)] \tag{3.5}$$

Interchanging X and U in (2.1), and then subtracting the resultant from (2.1), we obtain by virtue of (3.5) that

$$[A(X) - D(X)]S(U,Z) - [A(U) - D(U)]S(X,Z) + B(R(X,U)Z) + 2D(R(X,U)Z)$$

$$= \alpha'[d\gamma(X)g(Z,U) - d\gamma(U)g(Z,X)] \tag{3.6}$$

Let ρ_1, ρ_2, ρ_3 be the associated vector fields corresponding to the 1-forms A, B, D respectively; i.e.,

$g(X,\rho_1) = A(X);\ g(X,\rho_2) = B(X);\ g(X,\rho_3) = D(X)$.

Substituting U by ρ_2 in (3.6), and then using (2.3), we get

$$[A(X) - D(X)]B(QZ) - [A(\rho_2) - D(\rho_2)]S(X,Z)$$

$$+R(X,\rho_2,Z,\rho_2) + 2R(X,\rho_2,Z,\rho_3) = \alpha'[B(Z)[\gamma A(X) + 2B(QX) + 2D(QX)]$$

$$-g(X,Z)[\gamma A(\rho_2) + 2B(Q\rho_2) + 2D(Q\rho_2)]] \tag{3.7}$$

If the manifold has non-zero constant scalar curvature, then by virtue of (2.4) equation (3.7) yields that

$$[A(\rho_2) - D(\rho_2)]S(X,Z) - [A(X) - D(X)]B(QZ)$$

$$+R(\rho_2, X, Z, \rho_2) + 2R(\rho_2, X, Z, \rho_3) = 0 \tag{3.8}$$

Again since the manifold is quasi-conformally flat, we have from (3.1)

$$R(\rho_2, X, Z, \rho_2) + 2R(\rho_2, X, Z, \rho_3) = -\frac{b}{a}[S(X,Z)[B(\rho_2) + 2D(\rho_2)]$$
$$-B(QZ)[B(X) + 2D(X)] + g(X,Z)[B(Q\rho_2)$$
$$+2D(Q\rho_2)] - B(Z)[B(QX) + 2D(QX)]]$$
$$+\frac{\gamma}{n}[\frac{1}{n-1} + \frac{2b}{a}][g(X,Z)[B(\rho_2) + 2D(\rho_2)] - B(Z)[B(X) + 2D(X)]] \tag{3.9}$$

Using (3.9) in (3.8) we obtain

$$S(X,Z) = \alpha_1 g(X,Z) + \alpha_2 B(X)B(Z) + \alpha_3 B(Z)D(X)$$
$$+\alpha_4 B(X)\bar{B}(Z) + \alpha_5 D(X)\bar{B}(Z) + \alpha_6 B(Z)\bar{B}(X)$$
$$+\alpha_7 B(Z)\bar{D}(X) + \alpha_8 A(X)\bar{B}(Z) \tag{3.10}$$

where $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_8$ are scalars in terms of $\gamma,\ B(\rho_2),\ D(\rho_2)$ and $\bar{B}(X) = B(QX)$, $\bar{D}(X) = D(QX)\ \ \forall\ X$.

This leads to the following:

Theorem 3.1. *In a quasi-conformally flat $(WS)_n (n > 3)$ of non-zero constant scalar curvature the Ricci tensor S has the form (3.10).*

Again using (2.4) in (3.10), we have

$$S(X,Z) = \alpha_1 g(X,Z) + \alpha_2 B(X)B(Z) + \alpha_3 B(Z)D(X)$$
$$+\alpha_4 B(X)\bar{B}(Z) + \alpha_5 D(X)\bar{B}(Z) + \alpha_6 B(Z)\bar{B}(X)$$
$$+\alpha_7 B(Z)\bar{D}(X) + \alpha_8(-\frac{2}{\gamma})[\bar{B}(X) + \bar{D}(X)]\bar{B}(Z) \tag{3.11}$$

Putting (3.11) in (3.1) we obtain

$$\acute{R}(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$
$$+g(X,W)P(Y,Z) - g(X,Z)P(Y,W)$$
$$+g(Y,Z)P(X,W) - g(Y,W)P(X,Z) \tag{3.12}$$

where

$$P(Y,Z) = (\beta\ BD)(Y,Z) = \beta_1 B(Y)B(Z) + \beta_2 B(Z)D(Y)$$
$$+\beta_3 B(Y)\bar{B}(Z) + \beta_4 \bar{B}(Z)D(Y)$$
$$+\beta_5 \bar{B}(Y)B(Z) + \beta_6 B(Z)\bar{D}(Y)$$
$$+\beta_7 \bar{B}(Z)\bar{D}(Y) + \beta_8 \bar{B}(Y)\bar{B}(Z) \tag{3.13}$$

and a; $\beta_1,\ \beta_2,\ \cdots,\ \beta_8$ are non-zero scalars.

From (3.12), it follows that the manifold is of hyper quasi-constant curvature.

Thus we can state the following:

Theorem 3.2. *A quasi-conformally flat $(WS)_n (n > 3)$ of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.*

Now putting $U = \rho$ in (2.8) and then using (2.6), we get

$$\frac{\gamma}{2}T(X)T(Z) - T(\rho)S(X,Z) + R(\rho, Z, X, \rho) = 0 \tag{3.14}$$

Let us now suppose that a $(WS)_n(n > 3)$ is quasi-conformally flat, and of non-zero scalar curvature. Then (3.1) yields

$$R(\rho, Z, X, \rho) = -\frac{b}{a}[T(\rho)S(X,Z) - \gamma T(X)T(Z) + \frac{\gamma}{2}T(\rho)g(X,Z)]$$

$$+\frac{\gamma}{n}[\frac{1}{n-1} + \frac{2b}{a}][T(\rho)g(X,Z) - T(X)T(Z)] \tag{3.15}$$

Using (3.15) in (3.14), it follows that

$$(1 + \frac{b}{a})T(\rho)S(X,Z) = [\frac{\gamma}{2} + \frac{b\gamma}{a} - \frac{\gamma}{n}(\frac{1}{n-1} + \frac{2b}{a})]T(X)T(Z)$$

$$+[-\frac{b\gamma}{2a} + \frac{\gamma}{n}[\frac{1}{n-1} + \frac{2b}{a}]]T(\rho)g(X,Z) \tag{3.16}$$

We shall now show that $T(\rho) \neq 0$. For if $T(\rho) = 0$, then (3.16) implies

$\gamma[\frac{1}{2} + \frac{b}{a} - \frac{1}{n}(\frac{1}{n-1} + \frac{2b}{a})]T(X)T(Z) = 0$

Since $T(X) \neq 0$ for all X, and $n > 3$, the above relation yields $\gamma = 0$, a contradiction to the assumption that the manifold is of non-zero scalar curvature. Thus we have $T(\rho) \neq 0$.

Consequently, (3.16) yields

$$S(X,Z) = \tilde{\alpha}g(X,Z) + \tilde{\beta}T(X)T(Z) \tag{3.17}$$

where $\tilde{\alpha},\ \tilde{\beta}$ are non-zero scalars and

$\tilde{\alpha} = \frac{1}{(1+\frac{b}{a})}[-\frac{b\gamma}{2a} + \frac{\gamma}{n}(\frac{1}{n-1} + \frac{2b}{a})]$

and

$\tilde{\beta} = \frac{1}{(1+\frac{b}{a})T(\rho)}[\frac{\gamma}{2} + \frac{b\gamma}{a} - \frac{\gamma}{n}(\frac{1}{n-1} + \frac{2b}{a})]$

Again by [1], a Riemannian manifold is said to be quasi-Einstein, if its Ricci tensor is of the form

$$S = pg + q\omega \otimes \omega \tag{3.18}$$

where $p,\ q$ are scalars of which $q \neq 0$ and ω is a 1-form.

In virtue of (3.17) and (3.18) we can state the following theorem:

Theorem 3.3. *A quasi-conformally flat $(WS)_n(n > 3)$ of non-vanishing scalar curvature is a quasi-Einstein manifold with respect to the 1-form T defined by $T(X) = B(X) - D(X) \neq 0\ \forall\ X$.*

Again, using (3.17) in (3.1), it follows that

$$\begin{aligned} \acute{R}(X,Y,Z,W) &= l[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+\delta[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &+g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)] \end{aligned} \tag{3.19}$$

where l and δ are non-zero scalars and

$l = [-\frac{2b}{a}\tilde{\alpha} + \frac{\gamma}{n}(\frac{1}{n-1} + \frac{2b}{a})]$ and $\delta = -\frac{b}{a}\tilde{\beta}$

comparing (3.19) and (1.4), we can state the following theorem:

Theorem 3.4. *A quasi-conformally flat* $(WS)_n(n > 3)$ *of non-vanishing scalar curvature is a manifold of quasi-constant curvature with respect to the 1-form* T *defined by* $T(X) = B(X) - D(X) \neq 0 \ \forall \ X$*.*

Using the expression of T in (3.19), it can be easily seen that

$$\acute{R}(X,Y,Z,W) = l[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + g(X,W)\{\delta BD\}(Y,Z)$$

$$- g(X,Z)\{\delta BD\}(Y,W) + g(Y,Z)\{\delta BD\}(X,W) - g(Y,W)\{\delta BD\}(X,Z)$$

where $\{\delta BD\} = \delta(BB - BD - DB + DD)$

Comparing the above relation with (3.12), we can state:

Corollary 3.1. *A quasi-conformally flat* $(WS)_n(n > 3)$ *of non-zero scalar curvature is a manifold of hyper quasi-constant curvature.*

4. Pseudo-projectively flat $(WS)_n$

Let $(M^n, g)(n > 3)$ be a pseudo-projectively flat $(WS)_n$. Then from (1.6)we obtain

$$\begin{aligned} \acute{R}(X,Y,Z,U) &= -\frac{b}{a}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U)] \\ &+\frac{r}{an}[\frac{a}{(n-1)} + b][g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \end{aligned} \tag{4.1}$$

Putting $X = U = e_i$ in (4.1) where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking the summation over i, where $1 \leq i \leq n$, we get

$$S(Y,Z) = \alpha g(Y,Z) \tag{4.2}$$

where $\alpha = \frac{r}{n}$

The above equation (4.2) indicates, a pseudo-projectively flat manifold is an Einstein manifold.

Now from (4.2) we get

$$(\nabla_X S)(Y,Z) = \alpha_1 dr(X)g(Y,Z) \tag{4.3}$$

where $\alpha_1 = \frac{1}{n}$

Similarly we can get

$$(\nabla_Z S)(Y, X) = \alpha_1 dr(Z) g(Y, X) \tag{4.4}$$

Now subtracting (4.4) from (4.3), we get

$$(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = \alpha_1[dr(X)g(Y, Z) - dr(Z)g(Y, X)] \tag{4.5}$$

Interchanging X and U in (2.1), and then subtracting the resultant from (2.1), we obtain by virtue of (4.5) that

$$\begin{aligned} &[A(X) - D(X)]S(U, Z) - [A(U) - D(U)]S(X, Z) \\ &+B(R(X, U)Z) + 2D(R(X, U)Z) = \alpha_1[g(Z, U)dr(X) - g(Z, X)dr(U)] \end{aligned} \tag{4.6}$$

Let ρ_1, ρ_2, ρ_3 be the associated vector fields corresponding to the 1-forms A, B, D respectively;

i.e., $g(X, \rho_1) = A(X)$; $g(X, \rho_2) = B(X)$; $g(X, \rho_3) = D(X)$.

Substituting U by ρ_2 in (4.6), and then using (2.3), we get

$$\begin{aligned} &[A(X) - D(X)]B(QZ) - [A(\rho_2) - D(\rho_2)]S(X, Z) \\ &+R(X, \rho_2, Z, \rho_2) + 2R(X, \rho_2, Z, \rho_3) = \alpha_1[B(Z)[rA(X) + 2B(QX) + 2D(QX)] \\ &-g(X, Z)[rA(\rho_2) + 2B(Q\rho_2) + 2D(Q\rho_2)]] \end{aligned} \tag{4.7}$$

If the manifold has non-zero constant scalar curvature, then (4.7) yields by virtue of (2.4) that

$$\begin{aligned} &[A(\rho_2) - D(\rho_2)]S(X, Z) - [A(X) - D(X)]B(QZ) \\ &+R(\rho_2, X, Z, \rho_2) + 2R(\rho_2, X, Z, \rho_3) = 0 \end{aligned} \tag{4.8}$$

Again since the manifold is pseudo-projectively flat, we have from (4.1)

$$\begin{aligned} &R(\rho_2, X, Z, \rho_2) + 2R(\rho_2, X, Z, \rho_3) = -\frac{b}{a}[S(X, Z)[B(\rho_2) + 2D(\rho_2)] \\ &-B(QZ)[B(X) + 2D(X)]] + \frac{r}{an}[\frac{a}{(n-1)} + b][g(X, Z)[B(\rho_2) + 2D(\rho_2)] \\ &-B(Z)[B(X) + 2D(X)]] \end{aligned} \tag{4.9}$$

Using (4.9) in (4.8), it follows that

$$\begin{aligned} &S(X, Z) = \alpha_2 g(X, Z) + \alpha_3 B(X)B(Z) + \alpha_4 B(Z)D(X) \\ &+\alpha_5 B(X)\bar{B}(Z) + \alpha_6 D(X)\bar{B}(Z) + \alpha_7 A(X)\bar{B}(Z) \end{aligned} \tag{4.10}$$

where $\alpha_2, \alpha_3, \cdots, \alpha_7$ are scalars in terms of r, $B(\rho_2)$, $D(\rho_2)$ and $\bar{B}(X) = B(QX)$, $\bar{D}(X) = D(QX)\ \forall X$.

This leads to the following theorem:

Theorem 4.1. *In a pseudo-projectively flat $(WS)_n (n > 3)$ of non-zero constant scalar curvature the Ricci tensor S has the form (4.10).*

Again using (2.4) in (4.10) we have

$$S(X,Z) = \alpha_2 g(X,Z) + \alpha_3 B(X)B(Z) + \alpha_4 B(Z)D(X)$$
$$+\alpha_5 B(X)\bar{B}(Z) + \alpha_6 D(X)\bar{B}(Z)$$
$$+\alpha_7[-\frac{2}{r}][\bar{B}(X) + \bar{D}(X)]\bar{B}(Z) \tag{4.11}$$

Putting (4.11) in (4.1) we obtain

$$\acute{R}(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$
$$+P(Y,Z)g(X,W) - P(X,Z)g(Y,W) \tag{4.12}$$

where

$$P(Y,Z) = (\beta\ BD)(Y,Z) = \beta_1 B(Y)B(Z) + \beta_2 B(Z)D(Y)$$
$$+\beta_3 B(Y)\bar{B}(Z) + \beta_4 \bar{B}(Z)D(Y)$$
$$+\beta_5 \bar{B}(Y)\bar{B}(Z) + \beta_6 \bar{B}(Z)\bar{D}(Y) \tag{4.13}$$

and a; $\beta_1,\ \beta_2,\ \cdots,\ \beta_6$ are non-zero scalars.

From (4.12), it follows that the manifold is of pseudo quasi-constant curvature. Thus we can state the following theorem:

Theorem 4.2. *A pseudo- projectively flat $(WS)_n (n > 3)$ of non-zero constant scalar curvature is a manifold of pseudo quasi-constant curvature.*

Now putting $U = \rho$ in (2.8) and then using (2.6), we get

$$(\frac{r}{2})T(X)T(Z) - T(\rho)S(X,Z) + R(\rho,Z,X,\rho) = 0 \tag{4.14}$$

Let us now suppose that a $(WS)_n (n > 3)$ is pseudo projectively flat and non-zero scalar curvature. Then (4.1) yields

$$\acute{R}(\rho,Z,X,\rho) = -\frac{b}{a}[T(\rho)S(X,Z) - \frac{r}{2}T(X)T(Z)]$$
$$+\frac{r}{an}[\frac{a}{(n-1)} + b][g(X,Z)T(\rho) - T(X)T(Z)] \tag{4.15}$$

Using (4.15) in (4.14) it follows that

$$(1 + \frac{b}{a})T(\rho)S(X,Z) = \frac{r}{an}[\frac{a}{(n-1)} + b]T(\rho)g(X,Z)$$
$$+[\frac{r}{2} + \frac{br}{2a} - \frac{r}{an}(\frac{a}{(n-1)} + b)]T(X)T(Z) \tag{4.16}$$

We shall now show that $T(\rho) \neq 0$. For if $T(\rho) = 0$, then (4.16) implies that

$r[\frac{1}{2} + \frac{b}{2a} - \frac{1}{an}(\frac{a}{(n-1)} + b)]T(X)T(Z) = 0$

Since $T(X) \neq 0$ for all X, and $n > 3$, the above relation yields $r = 0$, a contradiction to the assumption that the manifold is of non-zero scalar curvature. Thus we have $T(\rho) \neq 0$.

Consequently (4.16) yields

$$S(X,Z) = \tilde{\alpha}_1 g(X,Z) + \tilde{\beta}_1 T(X)T(Z) \tag{4.17}$$

where $\tilde{\alpha}_1,\ \tilde{\beta}_1$ are non-zero scalars and

$\tilde{\alpha}_1 = \frac{1}{(1+\frac{b}{a})}[\frac{r}{an}(\frac{a}{(n-1)} + b)]$

and

$\tilde{\beta}_1 = \frac{1}{(1+\frac{b}{a})T(\rho)}[\frac{r}{2} + \frac{br}{2a} - \frac{r}{an}(\frac{a}{(n-1)} + b)]$

Hence by virtue of (3.18) we can state:

Theorem 4.3. *A pseudo-projectively flat $(WS)_n (n > 3)$ of non-vanishing scalar curvature is a quasi-Einstein manifold with respect to the 1-form T defined by $T(X) = B(X) - D(X) \neq 0, \forall X$.*

Again, using (4.17) in (4.1), it follows that

$$\acute{R}(X,Y,Z,W) = \gamma_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$

$$+\delta_1 T(Y)T(Z)g(X,W) - \delta T(X)T(Z)g(Y,W) \tag{4.18}$$

where $\gamma,\ \delta$ are non-zero scalars, when

$\gamma_1 = -\frac{b}{a}\tilde{\alpha}_1 + \frac{r}{an}(\frac{a}{(n-1)} + b)$ and $\delta_1 = -\frac{b}{a}\tilde{\beta}_1$

Comparing (4.18) and (1.4), we can state the following theorem:

Theorem 4.4. *A pseudo-projectively flat $(WS)_n (n > 3)$ of non-vanishing scalar curvature is a manifold of pseudo quasi-constant curvature with respect to the 1-form T defined by $T(X) = B(X) - D(X) \neq 0, \forall X$.*

Using the expression of T in (4.18), it can be easily seen that

$\acute{R}(X,Y,Z,W) = \gamma_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$

$+ \{\delta_1 BD\}(Y,Z)g(X,W) - \{\delta_1 BD\}(X,Z)g(Y,W)$

where $\{\delta_1 BD\} = \delta_1(BB - BD - DB + DD)$.

Comparing the above relation with (4.12), we can state:

Corollary 4.1. *A pseudo-projectively flat $(WS)_n (n > 3)$ of non-zero scalar curvature is a manifold of pseudo quasi-constant curvature.*

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Department of Mathematics, Jadavpur University, KOLKATA-700032, INDIA
E-mail address: badan06@yahoo.co.in

Department of Mathematics, Jadavpur University, KOLKATA-700032, INDIA
E-mail address: arin1968@indiatimes.com