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THE CROSS-RATIO MANIFOLD: A MODEL OF CENTRO-AFFINE GEOMETRY

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Abstract. Every non-degenerated Lagrangian immersion in a para-Kähler manifold carries a natural Codazzi structure. For $n \geq 1$, we construct a $2n$-dimensional para-Kähler manifold $\mathbf{M}$ such that every centro-affine hypersurface immersion $f : M \to \mathbb{R}^{n+1}$, equipped with its affine metric, is isometric to a Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$. On the other hand, every Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$ corresponds to a unique homothetic family of centro-affine hypersurface immersions $f : M \to \mathbb{R}^{n+1}$. The Codazzi structure defined by the affine connection and the affine metric of f coincides with the Codazzi structure generated by the Lagrangian immersion $\tilde{f}$. The construction is compatible with the duality defined by the conormal map. The immersion f is a proper affine sphere if and only if the Lagrangian immersion $\tilde{f}$ is minimal. The velocity field of the affine normal flow generated by f and that of the mean curvature flow generated by $\tilde{f}$ are related. The pseudo-Riemannian metric and the symplectic form on $\mathbf{M}$ are generated in the infinitesimal limit by a real-valued symmetric function $(\cdot\,;\cdot)$ on $\mathbf{M} \times \mathbf{M}$. The manifold $\mathbf{M}$ is constructed as a subset of the product $\mathbb{R}P^n \times \mathbb{R}P_n$ of the real projective space and its dual, and the function $(\cdot\,;\cdot)$ is defined by the projective cross-ratio.

1. Introduction

In affine differential geometry, many efforts have been devoted to the study of affine maximal surfaces, see e.g., [3]. An n-dimensional hypersurface immersion $f : M \subset \mathbb{R}^{n+1}$ is affine maximal if it is a critical point of the volume functional, where the volume is computed with respect to the Blaschke metric on M. This metric is the affine metric induced on M by the affine normal field as transversal vector field. Affine maximal surfaces are characterized by the vanishing of the affine mean curvature. Other choices of the transversal vector field on M lead to other affine metrics and hence other volume functionals and notions of minimal surfaces.

None of these surfaces are, however, minimal in the classical sense, i.e., submanifolds of a Riemannian or pseudo-Riemannian manifold with vanishing mean curvature. In [19] Vrancken constructed a $(2n+1)$-dimensional pseudo-Riemannian

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manifold $\tilde{M}$ (as a certain hypersurface in $\mathbb{R}^{2n+2}$, equipped with a flat pseudo-Riemannian metric of neutral signature) carrying a distribution ζ of rank $2n$ and an operator P acting on ζ, with the following properties[1]. An n-dimensional immersion $\tilde{f} : M \to \tilde{M}$ that is integral with respect to ζ is isometric to some n-dimensional centro-affine immersion $f : M \to \mathbb{R}^{n+1}$, and the operator P takes the second fundamental form generated on M by $\tilde{f}$ to the difference tensor generated on M by f. In [19] such immersions $\tilde{f}$ are called *horizontal*. In this way, a horizontal immersion corresponds to a proper affine hypersphere if and only if it is minimal. On the other hand, *every* n-dimensional centro-affine immersion can be represented (at least locally) as a horizontal immersion $\tilde{f} : M \to \tilde{M}$. Thus $\tilde{M}$ serves in some sense as a model for the n-dimensional centro-affine immersions.

In this contribution, we carry out a construction that bears some similarities with the construction in [19], but which generates additional differential-geometric objects that can be related to objects appearing in centro-affine differential geometry, among them the affine connection and its dual connection with respect to the affine metric. Among the most appealing properties of our construction is its symmetry with respect to the duality defined by the conormal map and its transparent geometric interpretation. Actually, we will construct a $2n$-dimensional symplectic manifold $\mathbf{M}$, which we call the *cross-ratio manifold*, with symplectic form ω, equipped with a compatible pseudo-Riemannian metric g, such that the n-dimensional centro-affine immersions in $\mathbb{R}^{n+1}$ correspond to the Lagrangian immersions in $\mathbf{M}$, i.e., those on which ω vanishes identically. Every centro-affine immersion in $\mathbb{R}^{n+1}$ determines a unique Lagrangian immersion in $\mathbf{M}$, while every Lagrangian immersion determines the centro-affine immersion up to homothety. Thus the correspondence between centro-affine immersions in $\mathbb{R}^{n+1}$ and Lagrangian immersions in $\mathbf{M}$ is quite explicit.

The correspondence between centro-affine and horizontal immersions in [19], on the contrary, is based on structural existence theorems and hence is one-to-one only for isometry classes. Actually, one can show that the distribution ζ in [19] defines a contact structure on $\tilde{M}$, and then the horizontal immersions are characterized by the property of being Legendrian. This hints at some relation between the model of Vrancken and ours, but we will not pursue this question in the present paper.

Carrying a pseudo-Riemannian metric and a parallel symplectic form, the cross-ratio manifold belongs to the class of Fedosov manifolds. The metric and the symplectic form interact, however, in a very special way, much similar to the way these objects do in Kähler manifolds. Manifolds of this type are known under the name *para-Kähler manifolds*. The class of para-Kähler manifolds has been explicitly introduced by Libermann in [13]. We shall show below that the cross-ratio manifold is isomorphic to a member of the one-parametric family of *reduced paracomplex projective spaces*, which were introduced and studied in [9]. A recent survey on para-Kähler manifolds can be found in [7]. A compact introduction can also be found in [1, Section 5]. For an introduction to affine differential geometry, we refer to the excellent textbook [15].

[1]Actually, in [19] the distribution ζ is neither defined nor explicitly used. We introduced it here in order to restate the results of [19] in a style that is adapted to our own exposition. The distribution ζ is to be seen as the orthogonal complement of the tangent vector field Pf defined in [19].

The affine connection and the affine metric of an equiaffine hypersurface immersion in $\mathbb{R}^{n+1}$ form a *Codazzi structure* [15, Theorem 2.1, p.32], i.e., a pair (∇, h) consisting of a torsion-free affine connection and a symmetric second order tensor such that ∇h is symmetric in all three indices. For a recent survey on Codazzi tensors see [17]. For centro-affine immersions, which are a particular case of equiaffine immersions, the affine connection is projectively flat [15, Proposition 3.1, p.14 and p.38]. The correspondence between the Lagrangian immersions in $\mathbf{M}$ and the centro-affine hypersurface immersions in $\mathbb{R}^{n+1}$ can accordingly be generalized in two directions. There exists a correspondence between general n-dimensional immersions in $\mathbf{M}$, i.e., not necessarily Lagrangian, and the projectively flat n-dimensional manifolds. On the other side, there exists a correspondence between Lagrangian immersions in general $2n$-dimensional para-Kähler manifolds and n-dimensional Codazzi manifolds, i.e., manifolds carrying a Codazzi structure. These relations are detailed in the companion paper [11]. Namely, every Lagrangian immersion in a para-Kähler manifold, if it satisfies a certain regularity condition, carries two natural affine connections. These connections together with the pseudo-Riemannian metric induced on the immersion form a dual pair of Codazzi structures. The cubic form of this structure is obtained from the second fundamental form of the immersion by multiplication with the symplectic form ω. In particular, the cubic form satisfies the apolarity condition if and only if the immersion is minimal.

The benefits of the theory presented in this paper are two-fold. On the one hand, centro-affine geometry can now be viewed as a particular instance of para-Kählerian, or, more general, pseudo-Riemannian geometry. Methods from these areas can be applied to specific problems in centro-affine geometry, such as classifying centro-affine immersions satisfying certain symmetries. On the other hand, the well-developed theory of centro-affine geometry can serve as a source of inspiration and a touchstone for research in the newly emerging area of Lagrangian immersions in para-Kähler manifolds.

1.1. **Basic construction.** In this subsection we shall motivate and sketch the construction of the cross-ratio manifold $\mathbf{M}$. A detailed description will be given in Subsection 3.1. Consider a smooth centro-affine hypersurface immersion $f : M \to \mathbb{R}^{n+1}$. For each point $y \in M$, the position vector $f(y)$ is by definition nonzero and transversal to the image of the tangent space T_yM under the differential of f. Hence the position vector and this image define a point (x, p) in the direct product $\mathbb{R}P^n \times \mathbb{R}P_n$ of the n-dimensional real projective space and its dual, with the property that x and p are not orthogonal to each other. In this way, the immersion f defines an immersion $\tilde{f}$ of M into the set

$$\mathbf{M} = \{(x, p) \in \mathbb{R}P^n \times \mathbb{R}P_n \mid x \not\perp p\}. \tag{1.1}$$

We will now use and elaborate on a generalization of the projective cross-ratio. This generalization was apparently first introduced in [2]. A pair of points $z = (x, p)$, $z' = (x', p')$ in general position in $\mathbf{M}$ determines two points x, x' in projective space and two points p, p' in the dual projective space. Note that points in the dual projective space can be considered as hyperplanes in the primal projective space. If we draw a projective line l through x, x' and consider the intersection points u, u' of l with the hyperplanes defined by p, p', then together with the original points x, x' we obtain four collinear points on l. We then assign the cross-ratio $(u, x'; u', x)$ of the four points obtained in this way to the pair of points $(z, z') \in \mathbf{M} \times \mathbf{M}$. From

[2, Theorem 2.10] it follows that this function extends continuously to a symmetric function $(\cdot\,;\cdot) : \mathbf{M} \times \mathbf{M} \to \mathbb{R}$.

We show in Subsection 3.1 that on small scales $(z; z')$ behaves up to higher-order terms as a bilinear function in the arguments $x' - x$, $p' - p$, thus defining an invariant bilinear form $\mathbf{Q}$ on $\mathbf{M}$. The symmetric and the skew-symmetric part of $\mathbf{Q}$ define a pseudo-Riemannian metric g and a symplectic form ω on $\mathbf{M}$, respectively, which are mutually compatible. The n-dimensional immersions $\tilde{f} : M \to \mathbf{M}$ which can be obtained in the way described above from n-dimensional centro-affine hypersurface immersions $f : M \to \mathbb{R}^{n+1}$ turn out to be precisely the Lagrangian immersions.

1.2. **Outline.** We shall now give an overview over the contents of the paper.

In Section 2 we consider para-Kähler manifolds in general. In Subsection 2.1 we provide their definition and consider some of their general properties. In view of the applications to affine differential geometry, we restate some results from [11] on Lagrangian immersions into para-Kähler manifolds in Subsection 2.2. In particular, we describe the natural Codazzi structure on such immersions. The study of Lagrangian immersions in para-Kähler manifolds has been initiated by B.-Y. Chen in [4].

In Section 3 we introduce and study the cross-ratio manifold $\mathbf{M}$ as a special para-Kähler manifold. As mentioned in the previous subsection, we show in Subsection 3.1 that the pseudo-Riemannian metric g and the symplectic form ω on the cross-ratio manifold are defined by the infinitesimal limit of an invariant symmetric function on $\mathbf{M} \times \mathbf{M}$. We also establish that $\mathbf{M}$ is highly symmetric, and is in fact a homogeneous para-Kähler Einstein manifold, namely isomorphic to a member of the family of reduced para-complex projective spaces. In Subsection 3.2 we extend the results in [9, Section 2] on the geodesics and totally geodesic submanifolds of the reduced para-complex projective spaces, applied to the cross-ratio manifold $\mathbf{M}$. In particular, we show that Lagrangian immersions in $\mathbf{M}$ at every point have tangent and normal totally geodesic submanifolds.

In Section 4 we investigate the relation between Lagrangian immersions into the cross-ratio manifold $\mathbf{M}$ and n-dimensional centro-affine immersions into $\mathbb{R}^{n+1}$. In Subsection 4.1 we establish the equivalence of objects encountered in centro-affine differential geometry, such as the primal and dual affine connections, the affine metric, the cubic form and the Tchebycheff form, with objects defined on Lagrangian immersions in $\mathbf{M}$ and described in Section 2. In Subsection 4.2 we consider the centro-affine pendants to the objects constructed in Subsection 3.2. We show that for each point of a centro-affine immersion, there exist two distinguished quadrics which are tangent to the immersion at this point. These quadrics are generated by the tangent and normal totally geodesic submanifolds at the corresponding point of the Lagrangian immersion in $\mathbf{M}$. In Subsection 4.3 we study the relation between centro-affine surface flows in $\mathbb{R}^{n+1}$ and Lagrangian surface flows in $\mathbf{M}$, and in Subsection 4.4 we specialize the results to the affine normal flow in $\mathbb{R}^{n+1}$ and the mean curvature flow in $\mathbf{M}$.

Finally, in Section 5 we summarize our results and provide an outlook on possible future research directions and applications.

2. Para-Kähler manifolds

We define para-Kähler manifolds and consider some of their elementary properties in Subsection 2.1. The subject of Subsection 2.2 are Lagrangian immersions into para-Kähler manifolds. The notations introduced in this section will be used also in the rest of the paper. Throughout this section, $\mathbf{M}$ will denote a general para-Kähler manifold. In the next section we specialize $\mathbf{M}$ to be the cross-ratio manifold (1.1) again. Most of the results in this section are restatements from [11], for proofs we refer to this companion paper.

2.1. **Definition and elementary properties.** A *symplectic manifold* is a differentiable manifold carrying a symplectic form ω, i.e., a closed, non-degenerate, skew-symmetric 2-form. A *Fedosov manifold* is a symplectic manifold with a torsion-free affine connection ∇ such that the symplectic form ω is parallel with respect to this connection, $\nabla\omega = 0$. A *para-complex manifold* $\mathbf{M}$ is a $2n$-dimensional manifold with a smooth tensor field J of type $(1,1)$ such that J is an involution of the tangent space $T_z\mathbf{M}$ at each $z \in \mathbf{M}$, $J^2 = 1$, and such that the eigenspaces corresponding to the eigenvalues ± 1 of J form two involutive n-dimensional distributions. We will denote these distributions by $D^{\mathbf{P}}$, $D^{\mathbf{X}}$, respectively, and the projections of the tangent bundle $T\mathbf{M}$ on these distributions by $\Pi_{\mathbf{X}}, \Pi_{\mathbf{P}}$, respectively. The field J is called the *para-complex structure* of the manifold. A *para-Kähler manifold* is a para-complex Fedosov manifold such that the eigenspace distributions of its para-complex structure J are isotropic with respect to the symplectic form ω, and whose affine connection ∇ is the Levi-Civita connection of the pseudo-Riemannian metric g defined by

$$g(X,Y) = \omega(JX,Y), \qquad \omega(X,Y) = g(JX,Y). \tag{2.1}$$

for every two vector fields X, Y on $\mathbf{M}$. Here and in the rest of the paper we denote by parentheses the value of covariant tensors on vectors or vector fields. It is not hard to see that g is necessarily of neutral signature and non-degenerate. The integral submanifolds of the distributions $D^{\mathbf{X}}$, $D^{\mathbf{P}}$ locally form two Lagrangian foliations, which is why a para-Kähler structure is sometimes called a *bi-Lagrangian* structure.

The cross-ratio manifold will be shown to be a particular para-Kähler manifold. Our motivation to consider the whole class of para-Kähler manifolds first is that many objects appearing in affine differential geometry have analogs on a generic Lagrangian immersion in an arbitrary para-Kähler manifold. It is a natural question to ask whether the relations between the cross-ratio manifold and centro-affine hypersurface immersions presented in this paper can be generalized to other kinds of transversal vector fields, involving other para-Kähler manifolds, and we hope that the material of this section provides a valuable point of departure for future research in this direction.

The para-complex structure J of a para-Kähler manifold $\mathbf{M}$ equips $\mathbf{M}$ with a local product structure. Namely, for every $\hat{z} \in \mathbf{M}$, there exists a neighbourhood $U \subset \mathbf{M}$ of $\hat{z}$ and a diffeomorphism $\varphi : U \to U_{\mathbf{X}} \times U_{\mathbf{P}}$ onto the product of simply connected open sets $U_{\mathbf{X}}, U_{\mathbf{P}} \subset \mathbb{R}^n$, with the following property. Let $\varphi^{\mathbf{X}} : U \to U_{\mathbf{X}}$, $\varphi^{\mathbf{P}} : U \to U_{\mathbf{P}}$ be the components of φ, then the distributions $D^{\mathbf{X}}, D^{\mathbf{P}}$ are the kernels of the differentials $D\varphi^{\mathbf{X}}, D\varphi^{\mathbf{P}}$, respectively. Let x, p be coordinates on $U_{\mathbf{X}}, U_{\mathbf{P}}$, respectively, and $z = (x,p)$ coordinates on $\mathbf{M}$. We will call such charts on $\mathbf{M}$ *adapted* to the para-complex structure. In any such chart, the matrix of the para-complex structure J is given by

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.2}$$

where I is the $n \times n$ identity matrix. In the sequel we work only with adapted coordinate charts.

We introduce the following index notation. Indices running from 1 to n will be denoted by lower-case Latin letters, indices running from $n+1$ to $2n$ by upper-case Latin letters, and indices running from 1 to $2n$ will be denoted by lower-case Greek letters. The use of the letters will be consistent, e.g., the indices denoted by α will consist of two groups of indices, denoted by a and A, respectively. The components of a vector field X in an adapted chart can then be written as $X^\alpha = \begin{pmatrix} X^a \\ X^A \end{pmatrix}$. Likewise, we have $z^\alpha = \begin{pmatrix} z^a \\ z^A \end{pmatrix}$ for the coordinates of the chart. From (2.2) we then have in index notation $J_a^b = \delta_a^b$, $J_a^B = J_A^b = 0$, $J_A^B = -\delta_A^B$, δ denoting the Kronecker symbol. We shall also deal with immersions $\tilde{f} : M \to \mathbf{M}$ of lower-dimensional manifolds into $\mathbf{M}$. In this case, the coordinates y on the manifold M will be indexed by upper-case Greek letters. The Einstein summation convention will be applied to all four kinds of indices. For instance, the contraction of a 1-form w with a vector field X on $\mathbf{M}$ is given by $w(X) = w_\alpha X^\alpha = w_a X^a + w_A X^A$. Expressions of the type $w_a X^A$ will also occur and have to be understood as the sum $\sum_{k=1}^n w_k X^{k+n}$.

The metric g and the symplectic form ω of $\mathbf{M}$ can be recovered from their sum $\mathbf{Q} = g + \omega$ as the symmetric and skew-symmetric part of $\mathbf{Q}$, respectively. Note that the covariant second order tensor $\mathbf{Q}$ encodes the para-complex structure J as well, since the distributions $D^{\mathbf{P}}, D^{\mathbf{X}}$ are the right and left kernel of $\mathbf{Q}$, respectively [11, Lemma 2.1]. It follows that in an adapted chart we have $\mathbf{Q}_{ab} = \mathbf{Q}_{Ab} = \mathbf{Q}_{AB} = 0$, and $\mathbf{Q}$ is determined by a smooth field Q of nondegenerate bilinear forms on $D^{\mathbf{P}} \times D^{\mathbf{X}}$ by the relation $\mathbf{Q}(X,Y) = Q(\Pi_{\mathbf{X}}(X), \Pi_{\mathbf{P}}(Y))$ for all vector fields X, Y. Equivalently, the remaining coefficients $\mathbf{Q}_{aB}$ of $\mathbf{Q}$ equal the corresponding coefficients of Q. For convenience, we will index the rows of the matrix of Q from 1 to n and the columns from $n+1$ to $2n$, such that $\mathbf{Q}_{aB} = Q_{aB}$. Let Q^{Ab} denote the coefficients of the inverse matrix Q^{-1}, such that $Q_{aB}Q^{Bc} = \delta_a^c$ and $Q^{Ab}Q_{bC} = \delta_C^A$. The matrices of the tensors $\mathbf{Q}$, g, ω and the inverse of the metric tensor are then given by

$$\mathbf{Q}_{\alpha\beta} = \begin{pmatrix} 0 & Q_{aB} \\ 0 & 0 \end{pmatrix}, \; g_{\alpha\beta} = \frac{1}{2} \begin{pmatrix} 0 & Q_{aB} \\ Q_{bA} & 0 \end{pmatrix}, \tag{2.3}$$

$$\omega_{\alpha\beta} = \frac{1}{2} \begin{pmatrix} 0 & Q_{aB} \\ -Q_{bA} & 0 \end{pmatrix}, \; g^{\alpha\beta} = 2 \begin{pmatrix} 0 & Q^{Ba} \\ Q^{Ab} & 0 \end{pmatrix}. \tag{2.4}$$

The para-complex structure J can then be written as $J_\alpha^\beta = \omega_{\alpha\gamma} g^{\gamma\beta}$. The lowering and raising of indices is as usual performed by the metric tensor and its inverse.

The condition $\nabla\omega = 0$ leads to restrictions on the functions $Q_{aB}(z)$. Namely, these functions must locally be of the form

$$Q_{aB}(z) = \frac{\partial^2 q}{\partial z^a \partial z^B}, \tag{2.5}$$

where the scalar field q is called the *para-Kähler potential* [6, Section 2.2]. This potential is unique up to transformations of the form

$$q(z) \mapsto q(z) + f(x) + h(p) \tag{2.6}$$

for arbitrary smooth functions f, h on $U_{\mathbf{X}}$ and $U_{\mathbf{P}}$, respectively, where $x = \varphi^{\mathbf{X}}(z)$, $p = \varphi^{\mathbf{P}}(z)$. Conversely, let $\mathbf{M}$ be a para-complex manifold such that the eigendistributions $D^{\mathbf{P}}, D^{\mathbf{X}}$ of the para-complex structure J are involutive. If a field Q of bilinear forms on $D^{\mathbf{P}} \times D^{\mathbf{X}}$ is locally given by (2.5) for some smooth scalar function q such that the matrix of Q is everywhere invertible, then (2.3),(2.4) defines a para-Kähler structure on $\mathbf{M}$ [6, Section 2.2, Theorem 2].

The Christoffel symbols of the Levi-Civita connection ∇ of g are given by [1, eq. (6)]

$$\Gamma^c_{ab} = Q^{Dc}\frac{\partial Q_{aD}}{\partial z^b}, \quad \Gamma^C_{AB} = Q^{Cd}\frac{\partial Q_{dA}}{\partial z^B}, \tag{2.7}$$

while the other components of the Christoffel symbol vanish [1, Lemma 5.3]. Let us compute the curvature tensor of the metric g. By [18, eq. (1.27)] the curvature tensor of a Fedosov manifold obeys the relation[2] $\omega_{\alpha\mu}R^{\mu}_{\beta\gamma\delta} = \omega_{\beta\mu}R^{\mu}_{\alpha\gamma\delta}$, which in view of (2.1) is equivalent to $J^{\mu}_{\alpha}R_{\mu\beta\gamma\delta} = J^{\mu}_{\beta}R_{\mu\alpha\gamma\delta}$. The skew-symmetry of the curvature tensor with respect to the first pair of indices now yields $R_{ab\gamma\delta} = R_{AB\gamma\delta} = 0$, and the only nonzero components of the curvature tensor are given by (see also [1, Prop. 5.5, eq. (9)])

$$\begin{aligned} R_{aBcD} = -R_{BacD} = -R_{aBDc} = R_{BaDc} &= \frac{1}{2}Q_{aE}\frac{\partial \Gamma^E_{DB}}{\partial z^c} \\ &= \frac{1}{2}\left(\frac{\partial^2 Q_{aD}}{\partial z^B \partial z^c} - \frac{\partial Q_{aE}}{\partial z^c}Q^{Ef}\frac{\partial Q_{fD}}{\partial z^B}\right). \end{aligned} \tag{2.8}$$

2.2. **Lagrangian immersions.** In this subsection we consider Lagrangian immersions $\tilde{f} : M \to \mathbf{M}$ of an n-dimensional manifold M into a para-Kähler manifold $\mathbf{M}$. The pseudo-Riemannian metric g on $\mathbf{M}$ induces a pseudo-Riemannian metric on M, which we denote by $\hat{g}$. This metric will be used to raise and lower indices of tensors on M. Besides the metric, there are several other differential-geometric objects which arise naturally on such Lagrangian immersions. Similar objects are known from affine differential geometry, which is the main motivation for this subsection.

The following result is a simple consequence of (2.1), see also [4].

Lemma 2.1. *Let $z \in \mathbf{M}$ and let $L \subset T_z\mathbf{M}$ be an n-dimensional subspace of the tangent space at z. Then the following conditions are equivalent.*

i) L is a Lagrangian subspace, i.e., the symplectic form ω vanishes on L.

ii) The para-complex structure J maps L to its orthogonal subspace $L^{\perp}$.

iii) The orthogonal subspace $L^{\perp}$ is Lagrangian.

Let now $\tilde{f} : M \to \mathbf{M}$ be a Lagrangian immersion of some n-dimensional manifold M into $\mathbf{M}$, i.e., such that the differential $D\tilde{f}$ of $\tilde{f}$ maps T_yM to a Lagrangian subspace $L_y \subset T_{\tilde{f}(y)}\mathbf{M}$ for every $y \in M$. Denote by $L_y^{\perp}$ the orthogonal subspace to L_y, and let NM be the normal bundle over M, i.e., a vector bundle such that the fiber over $y \in M$ is given by $N_yM = L_y^{\perp}$. We also introduce the tangent bundle $\tilde{T}M$ with fiber $\tilde{T}_yM = L_y$, which is canonically isomorphic by the differential of $\tilde{f}$ to the usual tangent bundle TM with fiber T_yM. For a vector field X on M, we denote by $\tilde{X}$ the corresponding cross-section of $\tilde{T}M$. Thus $\tilde{X}$ can be seen as a

[2] Note that a general Fedosov manifold need not have a metric tensor. Therefore in the literature on Fedosov manifolds the raising and lowering of indices is performed by means of the symplectic form ω, yielding a somewhat different definition of the purely covariant curvature tensor.

$T\mathbf{M}$-valued vector field on M, i.e., a map associating to every point $y \in M$ a vector in $T_{\tilde{f}(y)}\mathbf{M}$.

For a Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$, the pseudo-Riemannian metric $\hat{g}$ on M equals the pullback $\tilde{f}^*\mathbf{Q}$ on M of the tensor $\mathbf{Q} = g + \omega$. By (2.3) it follows that (cf. [11, eq. (13)])

$$\hat{g} = \left(\frac{\partial x}{\partial y}\right)^T Q \frac{\partial p}{\partial y}. \tag{2.9}$$

Definition 2.1. We call a Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$ *regular* if for every $y \in M$ the subspaces $L_y, L_y^\perp \subset T_{\tilde{f}(y)}\mathbf{M}$ are mutually transversal, i.e., the pseudo-Riemannian metric $\hat{g}$ on M is non-degenerate.

For regular Lagrangian immersions the orthogonal projections on the tangent and the normal bundles $\tilde{T}M$ and NM of a $T\mathbf{M}$-valued vector field $\tilde{X}$ on M are hence well-defined.

Lemma 2.2. [11, Lemma 2.13] *A Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$ is regular if and only if the distributions $D^{\mathbf{X}}, D^{\mathbf{P}}$ are transversal to $\tilde{f}$.*

Suppose now that the distribution $D^{\mathbf{X}}$ is transversal to the immersion $\tilde{f}$. Consider an adapted coordinate chart U on $\mathbf{M}$. Let $U_M \subset M$ be an open subset such that $\tilde{f}[U_M] \subset U$. Then the composition $\varphi^{\mathbf{X}} \circ \tilde{f}|_{U_M} : U_M \to U^{\mathbf{X}}$ is a local diffeomorphism. By possibly shrinking U_M, we can assume without restriction of generality that $\varphi^{\mathbf{X}} \circ \tilde{f}|_{U_M}$ is injective, thereby introducing the coordinates x on U_M. We shall call such a chart on M an *adapted chart*. The adapted charts form an atlas on M. In a similar manner, we can introduce the coordinates p on M if the distribution $D^{\mathbf{P}}$ is transversal to the immersion $\tilde{f}$.

Let $\tilde{f} : M \to \mathbf{M}$ be a regular Lagrangian immersion. The Levi-Civita connection $\hat{\nabla}$ induced on M by the pseudo-Riemannian metric $\hat{g}$ can be considered as an affine connection in the sense of affine differential geometry, induced by the normal bundle NM as transversal subspace distribution. Besides $\hat{\nabla}$, we shall consider two other affine connections $\nabla^{\mathbf{X}}$ and $\nabla^{\mathbf{P}}$, namely those induced by the distributions $D^{\mathbf{X}}$ and $D^{\mathbf{P}}$, respectively, as transversal subspace distributions. By Lemma 2.2 these distributions are indeed transversal. The next result gives an explicit expression of $\nabla^{\mathbf{X}}$.

Lemma 2.3. [11, Lemma 2.10] *Let $\tilde{f} : M \to \mathbf{M}$ be a Lagrangian immersion. Suppose that the distribution $D^{\mathbf{X}}$ is transversal to $\tilde{f}$. Then in an adapted chart on M, the Christoffel symbols of the affine connection $\nabla^{\mathbf{X}}$ are given by $\Gamma_{ab}^c = Q^{Dc}\frac{\partial Q_{aD}}{\partial z^b}$.*

A similar result holds for $\nabla^{\mathbf{P}}$.

A *Codazzi structure* on a differentiable manifold M is a pair (∇, h), consisting of a torsion-free affine connection and a (pseudo-)metric h, such that the tensor ∇h is totally symmetric [16, Def. 2.8, p.33]. If $\hat{\nabla}$ is the Levi-Civita connection of h, then there exists a unique torsion-free connection $\bar{\nabla}$ on M such that $\hat{\nabla} = \frac{1}{2}(\nabla + \bar{\nabla})$. The connection $\bar{\nabla}$ is called the *dual connection* of ∇ with respect to h [15, p.21]. If (∇, h) is a Codazzi structure, then $(\bar{\nabla}, h)$ is also a Codazzi structure, which is called *dual* to the original one [15, Cor.4.4, p.21].

Theorem 2.1. [11, Theorem 2.14] *Let $\tilde{f} : M \to \mathbf{M}$ be a Lagrangian immersion. Then the following assertions hold.*

i) Suppose that the distribution $D^{\mathbf{X}}$ is transversal to the immersion $\tilde{f}$. Then $(\nabla^{\mathbf{X}}, \hat{g})$ is a Codazzi structure on M.

ii) Suppose that the distribution $D^{\mathbf{P}}$ is transversal to the immersion $\tilde{f}$. Then $(\nabla^{\mathbf{P}}, \hat{g})$ is a Codazzi structure on M.

iii) Assume that the conditions of both parts i) and ii) of the theorem are satisfied. Then the Codazzi structures $(\nabla^{\mathbf{X}}, \hat{g})$, $(\nabla^{\mathbf{P}}, \hat{g})$ are dual to each other.

In [11, Corollary 3.4] we showed also the converse result, namely that, given an arbitrary differential manifold M and a Codazzi structure on M, there exists a para-Kähler manifold $\mathbf{M}$ and a regular Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$ such that the given Codazzi structure can be realized as the pair $(\nabla^{\mathbf{X}}, \hat{g})$.

Definition 2.2. Let $\tilde{f} : M \to \mathbf{M}$ be a regular Lagrangian immersion. We call the symmetric tensor $C = \nabla^{\mathbf{X}}\hat{g}$ the *cubic form* of the Lagrangian immersion $\tilde{f}$.

Definition 2.3. Let $\tilde{f} : M \to \mathbf{M}$ be a Lagrangian immersion. Let $\tilde{X}$ be a $T\mathbf{M}$-valued vector field on M. To the field $\tilde{X}$ we associate the 1-form $\omega_{\tilde{X}}$ on M by

$$\omega_{\tilde{X}}(Y) = \omega(\tilde{X}, \tilde{Y}), \tag{2.10}$$

where Y is an arbitrary vector field on M.

The form $\omega_{\tilde{X}}$ depends only on the equivalence class of $\tilde{X}$ modulo the addition of a tangential component. Indeed, let $\tilde{X}$ be a cross-section of $\tilde{T}M$. Then $\omega(\tilde{X}, \tilde{Y}) = 0$ for every vector field Y on M, because $\tilde{f}$ is Lagrangian. Hence $\omega_{\tilde{X}}$ vanishes identically.

We can generalize Definition 2.3 to forms of order k on M which have values in $T\mathbf{M}$. Let ζ be such a form. Then ζ can be associated to a form ω_ζ of order $k+1$ on M by

$$\omega_\zeta(X_1, \dots, X_k, Y) = \omega_{\zeta(X_1,\dots,X_k)}(Y) = \omega(\zeta(X_1, \dots, X_k), \tilde{Y}), \tag{2.11}$$

where $X_1, \dots, X_k, Y$ are arbitrary vector fields on M.

The second fundamental form $II_{\tilde{f}}$ of a regular Lagrangian immersion $\tilde{f}$ is a symmetric quadratic form on M with values in the normal bundle NM. Its value on two vector fields X, Y on M is defined as the orthogonal projection on NM of the covariant derivative $\nabla_{\tilde{X}}\tilde{Y}$, where $\tilde{X}, \tilde{Y}$ have to be thought as smoothly extended to some neighbourhood of the immersion $\tilde{f}$ [12, Section VII.3]. As described above, to the second fundamental form we can associate a 3-form $\omega_{II_{\tilde{f}}}$ on M.

Lemma 2.4. [11, Theorem 2.16] *Let $\tilde{f} : M \to \mathbf{M}$ be a regular Lagrangian immersion. Then the cubic tensor and the 3-form $\omega_{II_{\tilde{f}}}$ on M are related by $C = -2\omega_{II_{\tilde{f}}}$.*

Definition 2.4. Let $\tilde{f} : M \to \mathbf{M}$ be a regular Lagrangian immersion. We call the 1-form $T_\Psi = C^{\Lambda}_{\Lambda\Psi}$ obtained by contraction of the cubic form C with the metric $\hat{g}$ the *Tchebycheff form* of the Lagrangian immersion $\tilde{f}$.

From Lemma 2.4 it follows that

$$T = -2\omega_\varsigma, \tag{2.12}$$

where ς is the mean curvature vector field of the immersion $\tilde{f}$ (recall that the mean curvature vector is the contraction of the second fundamental form with the metric $\hat{g}$).

Corollary 2.1. *A regular Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$ into a para-Kähler manifold $\mathbf{M}$ is minimal if and only if its Tchebycheff form vanishes.*

Proof. By definition, minimal surfaces are characterized by the vanishing of the mean curvature vector field. Hence every minimal regular Lagrangian immersion must have a vanishing Tchebycheff form.

Let us prove the converse implication. If the Tchebycheff form vanishes, then $\omega(\varsigma, \tilde{X}) = 0$ for every cross-section $\tilde{X}$ of $\tilde{T}M$. Since the immersion $\tilde{f}$ is Lagrangian, it follows that ς itself must be a cross-section of $\tilde{T}M$. But ς is a normal vector field by definition. The assertion of the corollary now follows from the regularity of $\tilde{f}$. □

Finally, we shall explicitly compute the cubic form C and the Tchebycheff form T of a regular Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$ in an adapted chart U_M on M. The immersion $\tilde{f}$ defines a vector-valued function $\bar{p}$ on U_M by $\tilde{f}(x) = (x, \bar{p}(x))$, $x \in U_M$. By (2.9) the matrix of $\hat{g}$ equals the product $Q\frac{\partial \bar{p}}{\partial x}$. Denote the elements of the matrix $\frac{\partial \bar{p}}{\partial x}$ by Υ^L_c, with column index $L = n+1, \dots, 2n$ and row index $c = 1, \dots, n$. The elements of the derivative $\frac{\partial^2 \bar{p}}{\partial x^2}$ will correspondingly be denoted by Υ'^L_{bc}. By virtue of Lemma 2.3 the covariant derivative of the metric is then given by

$$\begin{aligned}
\nabla^{\mathbf{X}}_c \hat{g}_{ab} &= \left(\frac{\partial Q_{aK}}{\partial z^c} + \frac{\partial Q_{aK}}{\partial z^L} \Upsilon^L_c \right) \Upsilon^K_b + Q_{aK} \Upsilon'^K_{bc} \\
&\quad - Q_{aK} \Upsilon^K_d Q^{Ld} \frac{\partial Q_{bL}}{\partial z^c} - Q_{dK} \Upsilon^K_b Q^{Ld} \frac{\partial Q_{aL}}{\partial z^c} \\
&= \frac{\partial Q_{aK}}{\partial z^L} \Upsilon^K_b \Upsilon^L_c + Q_{aK} \Upsilon'^K_{bc} - Q_{aK} \Upsilon^K_d Q^{Ld} \frac{\partial Q_{bL}}{\partial z^c}.
\end{aligned}$$

Since the immersion is regular, the inverse metric exists and is given by $\hat{g}^{-1} = \frac{\partial x}{\partial \bar{p}} Q^{-1}$. Denote the elements of the derivative $\frac{\partial x}{\partial \bar{p}}$ by $\tilde{\Upsilon}^c_K$. Then the Tchebycheff form, which by definition is given by the contraction of the cubic form $\nabla^{\mathbf{X}} \hat{g}$, amounts to

$$\begin{aligned}
\hat{g}^{ca} \nabla^{\mathbf{X}}_c \hat{g}_{ab} &= \frac{\partial Q_{aK}}{\partial z^L} \Upsilon^K_b Q^{La} + \Upsilon'^K_{bc} \tilde{\Upsilon}^c_K - Q^{Ld} \frac{\partial Q_{bL}}{\partial z^d} \\
&= -\frac{\partial Q^{La}}{\partial z^L} Q_{aK} \Upsilon^K_b + \Upsilon'^K_{bc} \tilde{\Upsilon}^c_K + Q_{bL} \frac{\partial Q^{Ld}}{\partial z^d}.
\end{aligned} \tag{2.13}$$

3. The cross-ratio manifold

We shall now abandon the consideration of para-Kähler manifolds in general. Instead we construct and study a para-Kähler structure on the manifold (1.1). In this and in the next section, $\mathbf{M}$ will denote this particular manifold, the cross-ratio manifold.

3.1. **Construction.**

The purpose of this subsection is to elaborate an explicit expression of the objects (2.3),(2.4) the Levi-Civita connection ∇ and its curvature on $\mathbf{M}$.

As outlined in Subsection 1.1, $\mathbf{M}$ is a dense open subset of the product $\mathbf{X} \times \mathbf{P}$, where $\mathbf{X}$ is the n-dimensional real projective space $\mathbb{R}P^n$, and $\mathbf{P}$ is the dual projective space $\mathbb{R}P_n$. Denote by $\pi_{\mathbf{X}}, \pi_{\mathbf{P}}$ the projections of $\mathbf{M}$ onto the factors. The space $\mathbf{X}$ is defined as the set of 1-dimensional subspaces of the real vector space $\mathbb{R}^{n+1}$, whereas $\mathbf{P}$ is the set of 1-dimensional subspaces of its dual $\mathbb{R}_{n+1}$. The coordinate indices on the real vector spaces $\mathbb{R}^{n+1}$, $\mathbb{R}_{n+1}$ will run from 0 to n.

Definition 3.1. For a projective point $x \in \mathbf{X}$, we will call a nonzero vector $\tilde{x} \in x \subset \mathbb{R}^{n+1}$ on the line x a *representative* of x. Similarly, for $p \in \mathbf{P}$ a nonzero point $\tilde{p} \in p \subset \mathbb{R}_{n+1}$ will be called a representative of p.

There does not exist a scalar product between points $x \in \mathbf{X}$, $p \in \mathbf{P}$, and the scalar product of representatives $\tilde{x}, \tilde{p}$ of x, p is determined only up to multiplication with a nonzero real number. Therefore between points in $\mathbf{X}$ and $\mathbf{P}$ there exists only an orthogonality relation, $x \in \mathbf{X}$ being orthogonal to $p \in \mathbf{P}$ iff their representatives are orthogonal.

Let us define an atlas of affine coordinate charts on $\mathbf{X} \times \mathbf{P}$. Choose any point $\hat{z} = (\hat{x}, \hat{p}) \in \mathbf{M}$ and a basis in $\mathbb{R}^{n+1}$ such that $\hat{x}$ is represented by the basis vector $e_0 \in \mathbb{R}^{n+1}$ and $\hat{p}$ is represented by the basis vector $e^0 \in \mathbb{R}_{n+1}$ in the corresponding dual basis. Let $z = (x, p) \in \mathbf{X} \times \mathbf{P}$ be such that $x \not\perp \hat{p}$ and $\hat{x} \not\perp p$. Then x and p have representatives $\tilde{x} = (1, x^1, \dots, x^n)^T \in \mathbb{R}^{n+1}$ and $\tilde{p} = (1, p_1, \dots, p_n)^T \in \mathbb{R}_{n+1}$, respectively. We then assign to z the coordinate vector $(x^1, \dots, x^n, p_1, \dots, p_n)^T = (z^1, \dots, z^{2n})^T$. We will say that the affine coordinate chart obtained in this way is *centered on* $\hat{z}$. Indeed, the point $\hat{z}$ lies at the origin of this chart.

The product $\mathbf{X} \times \mathbf{P}$ and hence $\mathbf{M}$ carries a natural para-complex structure. The distributions $D^{\mathbf{P}}, D^{\mathbf{X}}$ of this structure are defined as the kernels of the differentials $D\pi_{\mathbf{P}}, D\pi_{\mathbf{X}}$, respectively. The affine charts on $\mathbf{X} \times \mathbf{P}$ defined in the previous paragraph are adapted to this para-complex structure (see Subsection 2.1), because the coordinates $x = (x^1, \dots, x^n)^T$ and $p = (p_1, \dots, p_n)^T$ define affine charts on the factor manifolds $\mathbf{X}$ and $\mathbf{P}$, respectively. The scalar product of the representatives $\tilde{x}, \tilde{p}$ of x and p is then given by $\langle \tilde{x}, \tilde{p} \rangle = 1 + p^T x = 1 + p_a x^a$. Note that those points which satisfy the relation $1 + p^T x = 0$ do not belong to $\mathbf{M}$, and the introduced charts are, strictly speaking, not charts on $\mathbf{M}$. With a little abuse of notation we will nevertheless speak of affine charts on $\mathbf{M}$, and as long as we respect the condition $1 + p^T x \neq 0$ this detail has no relevance. Note also that the set $\{(x, p) \in \mathbb{R}^{2n} \mid 1 + p^T x > 0\}$ is starlike and hence connected.

As announced in Subsection 1.1, the para-Kähler structure on $\mathbf{M}$ arises as the infinitesimal limit of a symmetric function $(\cdot\,;\cdot) : \mathbf{M} \times \mathbf{M} \to \mathbb{R}$ derived from a generalization of the projective cross-ratio. The central idea in [2] is to generalize the cross-ratio, an $(\mathbb{R} \cup \{\infty\})$-valued function on quadruples of collinear points in projective space, to a function taking a pair of points in projective space and a pair of points in the dual projective space as arguments. Let $(x, x', p, p') \in \mathbf{X} \times \mathbf{X} \times \mathbf{P} \times \mathbf{P}$ be a quadruple of points with representatives $\tilde{x}, \tilde{x}', \tilde{p}, \tilde{p}'$, respectively, such that neither x nor x' are orthogonal to both p and p', and neither p nor p' are orthogonal to both x and x'. Then the ratio

$$[x', x, p, p'] = \frac{\langle \tilde{x}', \tilde{p} \rangle \langle \tilde{x}, \tilde{p}' \rangle}{\langle \tilde{x}', \tilde{p}' \rangle \langle \tilde{x}, \tilde{p} \rangle}$$

is a well-defined number in $\mathbb{R} \cup \{\infty\}$, i.e., it is not dependent on the choice of the representatives $\tilde{x}, \tilde{x}', \tilde{p}, \tilde{p}'$ [2, p.4].

Let now $z = (x, p)$, $z' = (x', p')$ be two points in $\mathbf{M}$. We define the two-point function $(\cdot\,;\cdot) : \mathbf{M} \times \mathbf{M} \to \mathbb{R}$ by

$$(z; z') = 1 - [x', x, p, p']. \tag{3.1}$$

Clearly this function is symmetric in the arguments z, z'. The relations $x \not\perp p$ and $x' \not\perp p'$ assure that $(z; z')$ cannot assume the value ∞. The equivalence of this definition with the construction given in Subsection 1.1 is assured by [2, Theorem 2.10] and the symmetries of the cross-ratio.

We shall now give an explicit expression for function (3.1). Choose an affine coordinate chart on $\mathbf{M}$ such that both $z=(x,p)$, $z'=(x',p')$ are contained in it. Then

$$(z;z') = 1 - \frac{(1+p^Tx')(1+p'^Tx)}{(1+p'^Tx')(1+p^Tx)} = (x'-x)^T\frac{(1+p'^Tx)I - p'x^T}{(1+p^Tx)(1+p'^Tx')}(p'-p). \tag{3.2}$$

If we now fix the first argument z and consider $(z;z')$ as a scalar function $f(z') = f(x',p')$ on $\mathbf{M}$, then the gradients $\frac{\partial f}{\partial x'}, \frac{\partial f}{\partial p'}$ vanish at $z'=z$ and hence the second derivative

$$Q(z) = \left.\frac{\partial^2(z;z')}{\partial x'\partial p'}\right|_{z'=z} \tag{3.3}$$

defines an invariant bilinear form $Q : T_x\mathbf{X}\times T_p\mathbf{P}\to\mathbb{R}$ on $\mathbf{M}$. From (3.2) it follows that the matrix of this form is given by

$$Q(z) = \frac{(1+p^Tx)I - px^T}{(1+p^Tx)^2}. \tag{3.4}$$

We then have $\det Q = (1+p^Tx)^{-(n+1)}$, and Q is non-degenerated for all $z\in\mathbf{M}$, with inverse $Q^{-1} = (1+p^Tx)(I+px^T)$. Moreover, if we define the scalar $q(z) = \log|1+p^Tx|$, then $Q = \frac{\partial^2 q}{\partial x\partial p}$.

This allows us to define a para-Kähler structure (2.3),(2.4) on the manifold $\mathbf{M}$, with the scalar $q = \log|1+p^Tx|$ playing the role of the para-Kähler potential. Note that q is not an invariant scalar field. It depends on the affine chart chosen on $\mathbf{M}$ and undergoes a transformation (2.6) if we pass to an affine chart centered on a different point. From (3.4) we obtain for the elements of Q and its inverse

$$Q_{aB}(z) = \frac{(1+p_cx^c)\delta_a^b - p_ax^b}{(1+p_cx^c)^2}, \qquad Q^{Ab} = (1+p_cx^c)(\delta_a^b + p_ax^b). \tag{3.5}$$

Recall that by the notation convention introduced in Subsection 2.1 we have $b = B-n$ and $a = A-n$.

Definition 3.2. We will call the manifold $\mathbf{M}$ defined by (1.1) and equipped with the structures (2.3) defined by (3.3), (3.1) the *cross-ratio manifold*.

In [9] Gadea and Montesinos Amilibia introduced a one-parametric family of para-Kähler manifolds $P_n(B)/\mathbb{Z}_2$, the *reduced para-complex projective spaces*, and an isomorphic family of spaces $P(\mathbb{R}^{n+1}\oplus\mathbb{R}_{n+1})/\mathbb{Z}_2$. A comparison of (3.5) with [9, eq. (1.3)] and of (2.2) with [9, eq. (1.4)] yields the following result.

Theorem 3.1. *The cross-ratio manifold* $\mathbf{M}$ *is canonically isomorphic to the space* $P(\mathbb{R}^{n+1}\oplus\mathbb{R}_{n+1})/\mathbb{Z}_2$ *with parameter value* $c=4$*, which in turn is isomorphic to the reduced para-complex projective space* $P_n(B)/\mathbb{Z}_2$ *with parameter value* $c=4$.

While the underlying differentiable manifold, which is common for all spaces $P(\mathbb{R}^{n+1}\oplus\mathbb{R}_{n+1})/\mathbb{Z}_2$, is constructed in [9] in a way resembling definition (1.1), the para-Kähler structure is defined by a very different procedure.

Let us now compute the Levi-Civita connection and the curvature of the pseudo-Riemannian metric g. Inserting (3.5) into (2.7) and (2.8), one easily verifies

$$\Gamma^c_{ab} = -\frac{z^B\delta^c_a + z^A\delta^c_b}{1+p_dx^d}, \qquad \Gamma^C_{AB} = -\frac{z^b\delta^C_A + z^a\delta^C_B}{1+p_dx^d}, \tag{3.6}$$

$$R_{aBcD} = -R_{BacD} = -R_{aBDc} = R_{BaDc} = -\frac{1}{2}\left(Q_{aB}Q_{cD} + Q_{aD}Q_{cB}\right),$$

the other components of $\Gamma^{\gamma}_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$ being zero. From the last relation we get the following expression for the curvature tensor[3],

$$R_{\alpha\beta\gamma\delta} = g_{\beta\delta}g_{\alpha\gamma} - \omega_{\beta\delta}\omega_{\alpha\gamma} - g_{\alpha\delta}g_{\beta\gamma} + \omega_{\alpha\delta}\omega_{\beta\gamma} - 2\omega_{\alpha\beta}\omega_{\gamma\delta}.$$

By (2.1), the Ricci tensor is then given by

$$R_{\beta\delta} = g^{\alpha\gamma}R_{\alpha\beta\gamma\delta} = 2(n+1)g_{\beta\delta}.$$

Corollary 3.1. *The cross-ratio manifold* $\mathbf{M}$ *is a pseudo-Einstein manifold.*

Definition 3.3. We call a diffeomorphism $f : \mathbf{M} \to \mathbf{M}$ an *automorphism* if it preserves the bilinear form $\mathbf{Q}$ encoding the para-Kähler structure.

Clearly f is an automorphism if and only if it is at the same time an isometry and a symplectomorphism. The following result shows that the cross-ratio manifold is homogeneous.

Theorem 3.2. *The automorphism group of* $\mathbf{M}$ *is given by the projective general linear group* $PGL(n+1,\mathbb{R})$. *This group acts transitively and faithfully on* $\mathbf{M}$ *and preserves the two-point function (3.1).*

Proof. The group $PGL(n+1,\mathbb{R})$ is the automorphism group of the projective space $\mathbf{X} = \mathbb{R}P^n$, upon which it acts by projective transformations. Since points in the dual projective space $\mathbf{P}$ can be interpreted as hyperplanes in $\mathbf{X}$, the action on $\mathbf{X}$ induces also an action on $\mathbf{P}$ and hence on the product $\mathbf{X}\times\mathbf{P}$. This action preserves the projective cross-ratio as well as the orthogonality relation between points in primal and dual projective space. Thus $PGL(n+1,\mathbb{R})$ indeed acts on the set $\mathbf{M}$ and preserves the function (3.1). Moreover, since $PGL(n+1,\mathbb{R})$ acts on $\mathbf{X}$ and $\mathbf{P}$ separately, it preserves the distributions $D^{\mathbf{P}}, D^{\mathbf{X}}$ introduced in Section 2 and hence by (3.3) also the form $\mathbf{Q}$. The group $PGL(n+1,\mathbb{R})$ acts transitively on $\mathbf{M}$, because for every two pairs of complementary linear subspaces (in this case, of dimensions n and 1, respectively) in $\mathbb{R}^{n+1}$ with consistent dimensions there exists a linear transformation that takes one pair to the other. Since $PGL(n+1,\mathbb{R})$ acts faithfully on $\mathbf{X}$, it acts also faithfully on $\mathbf{M}$.

Let us now show that $PGL(n+1,\mathbb{R})$ exhausts the automorphism group of $\mathbf{M}$. Consider an affine chart on $\mathbf{M}$ and an automorphism $f : \mathbf{M} \to \mathbf{M}$. Our aim is to show that f is an element of $PGL(n+1,\mathbb{R})$. By the transitivity of the action of $PGL(n+1,\mathbb{R})$, we can without restriction of generality assume that f leaves the origin of the affine coordinate chart fixed. By (3.5), the form $\mathbf{Q}$ is at the origin given by the matrix $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$. Let $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be the matrix representation of the differential of f at the origin, partitioned into blocks of size $n\times n$. Since f preserves the form $\mathbf{Q}$, we have

$$\begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix}\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^TA_{21} & A_{11}^TA_{22} \\ A_{12}^TA_{21} & A_{12}^TA_{22} \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}.$$

It follows that $A_{12} = A_{21} = 0$ and $A_{22} = A_{11}^{-T}$. Note that there exists an element of $PGL(n+1,\mathbb{R})$ which leaves the origin of the affine chart fixed and whose differential

[3]The expression for the curvature tensor given in [9, p.267] differs from ours by a sign, which can be explained by a nonstandard definition of the curvature operator in [10, p.86].

at the origin equals that of f, namely, the transformation $x \mapsto A_{11}x$, $p \mapsto A_{11}^{-T}p$. Hence we can assume without restriction of generality that the differential of f at the origin is given by the identity matrix. But then f must equal the identity mapping on a whole neighbourhood of the origin. Namely, it must leave fixed all geodesics going through the origin. The proof is completed by noting that $\mathbf{M}$ is connected, which implies that f must equal the identity on the whole manifold $\mathbf{M}$. $\square$

The fact that the cross-ratio manifold is homogeneous is not new. In [9, p.265] the reduced para-complex projective spaces $P_n(B)/\mathbb{Z}_2$ are characterized as homogeneous spaces $SL(n+1,\mathbb{R})/S(GL(1,\mathbb{R})\times GL(n,\mathbb{R}))$. However, for odd n the group $SL(n+1,\mathbb{R})$ acts neither faithfully on $P_n(B)/\mathbb{Z}_2$ (the element $-I$ leaves the whole space fixed) nor does it exhaust the full automorphism group (the transformation $\mathrm{diag}(1,-I)$ induces a nontrivial automorphism).

The spaces $P_n(B)/\mathbb{Z}_2$ and hence also the cross-ratio manifold are symmetric spaces [9, p.265], i.e., for every $z \in \mathbf{M}$ there exists an involutive automorphism with z as an isolated fixed point.

Remark 3.1. For any given affine chart on $\mathbf{M}$, it is also of interest to consider the map $z = (x,p) \mapsto (p,x)$. The pseudo-Riemannian metric g and the symplectic form ω transform under this map as $g \mapsto g$, $\omega \mapsto -\omega$. This map is hence not an automorphism, but it is an isometry and it preserves the two-point function (3.1) as well as the property of an immersion to be Lagrangian.

The topology of the spaces $P_n(B)/\mathbb{Z}_2$ and hence also the cross-ratio manifold has been determined in [9, Prop. 1.1] to be that of the tangent bundle $T\,\mathbb{R}P^n$. The cross-ratio manifold $\mathbf{M}$ is thus homeomorphic to the tangent bundles $T\mathbf{X}$ and $T\mathbf{P}$.

3.2. **Geodesics.** In this subsection we consider properties of the geodesic flow in $\mathbf{M}$. In particular, we will show that totally geodesic submanifolds are either locally products of subspaces in $\mathbf{X}$ and $\mathbf{P}$ or isotropic with respect to the symplectic form ω. The material in this subsection builds on the results in [9, Section 2].

Fix an affine chart on $\mathbf{M}$. Consider the geodesic $\gamma(t)$ through the origin, $\gamma(0) = 0$, with nonzero velocity vector $\dot\gamma(0) = v$. By [9, p.268, eq. (2.5)][4] we have $\gamma(t) = \lambda(t)v$ with

$$\lambda(t) = \begin{cases} t, & g(v,v) = 0, \\ \frac{1}{\sqrt{g(v,v)}}\tan(\sqrt{g(v,v)}t), & g(v,v) > 0, \\ \frac{1}{\sqrt{-g(v,v)}}\tanh(\sqrt{-g(v,v)}t), & g(v,v) < 0. \end{cases}$$

Here $g(v,v)$ is the squared length of the vector v. We obtain the following classification of the geodesics in $\mathbf{M}$.

If $g(v,v) = 0$, i.e., the vector v is isotropic, then we have two subcases. If $v^a \neq 0$ and $v^A \neq 0$, i.e., v does not belong to the distributions $D^{\mathbf{P}}$ or $D^{\mathbf{X}}$, then the projections of the geodesic γ on $\mathbf{X}$ and $\mathbf{P}$ are given by a projective line minus a point. If $v^a = 0$ ($v^A = 0$), then the projection of γ on $\mathbf{X}$ ($\mathbf{P}$) becomes a point. In both cases, the limits $\lim_{t\to\pm\infty}\gamma(t)$ exist and coincide in $\mathbf{X}\times\mathbf{P}$.

If $g(v,v) > 0$, then γ is closed and projects to a whole projective line in both $\mathbf{X}$ and $\mathbf{P}$. The period of one orbit is given by $\frac{\pi}{\sqrt{g(v,v)}}$.

[4]The definition of λ in this formula should read $\mathrm{sgn}(\alpha(\nu))\sqrt{|\alpha(\nu)|}$ instead of $\sqrt{\alpha(\nu)}$.

If $g(v,v) < 0$, then γ projects to segments of projective lines in $\mathbf{X}$ and $\mathbf{P}$ whose complements have a nonempty interior. The limits $\lim_{t\to\pm\infty}\gamma(t)$ exist in $\mathbf{X}\times\mathbf{P}$ and do not coincide.

Definition 3.4. Depending on whether the sign of $g(v,v)$ is $0, 1$ or -1, we call the corresponding geodesics of *parabolic*, *elliptic* and *hyperbolic type*, respectively.

In all cases the geodesics are complete, and hence $\mathbf{M}$ is complete.

We shall now pass to the study of totally geodesic submanifolds. For a point $z \in \mathbf{M}$ and a k-dimensional subspace $L \subset T_z\mathbf{M}$, let M_L^z be the geodesic submanifold formed by the geodesics going through z with velocities in L. The following result is a direct consequence of the above considerations.

Lemma 3.1. *Pass to an affine chart on $\mathbf{M}$ centered on $\hat z \in \mathbf{M}$, and let $L \subset T_{\hat z}\mathbf{M}$ be a k-dimensional subspace, given by the column space of a $2n\times k$ matrix F. Then the intersection of $M_L^{\hat z}$ with the affine chart is given by the set $M_o = \{z = (x,p) = Fy \,|\, y \in \mathbb{R}^k,\ 1 + p^Tx > 0\}$.*

By [12, Theorem 4.3, p.237], the property of M_L^z to be a totally geodesic manifold depends only on the subspace L. In [9, Theorem 2.2], the subspaces L possessing the property of generating a totally geodesic submanifold are characterized. Either such a subspace L is the product $L_{\mathbf{X}}\times L_{\mathbf{P}}$ of two subspaces $L_{\mathbf{X}} \subset T_x\mathbf{X}$, $L_{\mathbf{P}} \subset T_p\mathbf{P}$, where $z = (x,p)$, or it can be decomposed into the direct sum of three subspaces which are related by some technical condition. For convenience we will say that the subspaces of the latter type are "Type 1" subspaces. The subspaces of Type 1 can be only of dimension not exceeding n [9, p.270], whereas the subspaces of product type can have arbitrary dimension.

Below we will give a similar, but slightly different and more transparent characterization of totally geodesic subspaces. Namely, we show that such a subspace has to be either of the product type described in the previous paragraph, or it has to be isotropic with respect to the symplectic form ω. Rather than performing the tedious linear-algebraic exercise of showing that subspaces of Type 1 are isotropic, and isotropic subspaces which are not of product type have to be of Type 1, we find it more instructive to present an independent proof that uses the isotropic property directly.

Lemma 3.2. *Let $\hat z \in \mathbf{M}$, let $L \subset T_{\hat z}\mathbf{M}$ be a k-dimensional linear subspace, and let $M_L^{\hat z}$ be the geodesic manifold through $\hat z$ with velocities in L. If the symplectic form ω vanishes on the subspace L, then $M_L^{\hat z}$ is totally geodesic.*

Proof. Fix an affine chart on $\mathbf{M}$ centered on $\hat z$ and let the $2n\times k$-matrix $F_\Lambda^\alpha = \begin{pmatrix} F_\Lambda^a \\ F_\Lambda^A \end{pmatrix}$, $\Lambda = 1,\dots,k$, be such that its columns form a basis of L. By Lemma 3.1, the intersection of $M_L^{\hat z}$ with the affine chart is given by the set $M_o = \{z = (x,p) = Fy \,|\, y \in \mathbb{R}^k,\ 1 + p^Tx > 0\}$. Define $M = \{y \in \mathbb{R}^k \,|\, 1 + p^Tx > 0,\ (x,p) = Fy\}$ and consider M_o as an immersion of M into $\mathbf{M}$, defined by $y \mapsto Fy$. Every vector field X on M then defines a tangent vector field $\tilde X$ on M_o by $\tilde X^\alpha = F_\Lambda^\alpha X^\Lambda$. We suppose that such vector fields are smoothly extended to a neighbourhood of M_o in $\mathbf{M}$.

Totally geodesic submanifolds are characterized by the vanishing of the second fundamental form. The second fundamental form of M_o vanishes if and only if for all vector fields X, Y on M we have that $\nabla_{\tilde{X}}\tilde{Y}$ is tangent to M_o. We have

$$\nabla^{\alpha}_{\tilde{X}}\tilde{Y} = \frac{\partial \tilde{Y}^{\alpha}}{\partial z^{\beta}}\tilde{X}^{\beta} + \Gamma^{\alpha}_{\beta\gamma}\tilde{X}^{\beta}\tilde{Y}^{\gamma} = F^{\alpha}_{\Lambda}\frac{\partial Y^{\Lambda}}{\partial y^{\Theta}}X^{\Theta} + \Gamma^{\alpha}_{\beta\gamma}F^{\beta}_{\Lambda}F^{\gamma}_{\Xi}X^{\Lambda}Y^{\Xi}.$$

The first summand is clearly in the image of F. By virtue of (3.6) the second summand, which we denote by u^{α}, at $z = (x, p) = Fy$ equals

$$\begin{aligned} u^{a} &= -(1 + p^T x)^{-1}\left(G_{\Lambda\Theta}F^{a}_{\Xi} + G_{\Xi\Theta}F^{a}_{\Lambda}\right)X^{\Lambda}Y^{\Xi}y^{\Theta}, \\ u^{A} &= -(1 + p^T x)^{-1}\left(G_{\Theta\Lambda}F^{A}_{\Xi} + G_{\Theta\Xi}F^{A}_{\Lambda}\right)X^{\Lambda}Y^{\Xi}y^{\Theta}, \end{aligned}$$

where $G_{\Lambda\Theta} = \sum_{i=1}^{n} F^{i}_{\Lambda}F^{i+n}_{\Theta}$. With $G^{\mathbf{sym}}_{\Lambda\Theta} = \frac{1}{2}(G_{\Lambda\Theta} + G_{\Theta\Lambda})$, $G^{\mathbf{asym}}_{\Lambda\Theta} = \frac{1}{2}(G_{\Lambda\Theta} - G_{\Theta\Lambda})$, $\bar{F}^{\alpha}_{\Lambda} = J^{\alpha}_{\beta}F^{\beta}_{\Lambda} = \begin{pmatrix} F^{a}_{\Lambda} \\ -F^{A}_{\Lambda}\end{pmatrix}$ we have $u^{\alpha} = U^{\alpha}_{\Lambda\Theta\Xi}X^{\Lambda}Y^{\Xi}y^{\Theta}$, where

$$U^{\alpha}_{\Lambda\Theta\Xi} = \begin{pmatrix} G_{\Lambda\Theta}F^{a}_{\Xi} + G_{\Xi\Theta}F^{a}_{\Lambda} \\ G_{\Theta\Lambda}F^{A}_{\Xi} + G_{\Theta\Xi}F^{A}_{\Lambda}\end{pmatrix} = (G^{\mathbf{sym}}_{\Lambda\Theta}F^{\alpha}_{\Xi} + G^{\mathbf{sym}}_{\Xi\Theta}F^{\alpha}_{\Lambda}) + \left(G^{\mathbf{asym}}_{\Lambda\Theta}\bar{F}^{\alpha}_{\Xi} + G^{\mathbf{asym}}_{\Xi\Theta}\bar{F}^{\alpha}_{\Lambda}\right).$$

Note that the term in the first parentheses on the right-hand side is in the image of F for all values of the indices Λ, Θ, Ξ. Thus $\nabla_{\tilde{X}}\tilde{Y}$ is tangent to M_o for all vector fields X, Y on M if and only if the object

$$G^{\mathbf{asym}}_{\Lambda\Theta}\bar{F}^{\alpha}_{\Xi} + G^{\mathbf{asym}}_{\Xi\Theta}\bar{F}^{\alpha}_{\Lambda} \tag{3.7}$$

is in the image of F for all values of the indices Λ, Θ, Ξ.

Now note that the form ω is given by the matrix $\frac{1}{2}\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}$ at the origin. Hence ω vanishes on L if and only if $G^{\mathbf{asym}}_{\Lambda\Theta} = 0$. But in this case (3.7) vanishes. Thus the isotropy of L implies that the second fundamental form vanishes identically on M_o, and hence on $M^{\hat{z}}_{L}$, because M_o is dense in $M^{\hat{z}}_{L}$. This completes the proof. □

Corollary 3.2. *Let $\tilde{f} : M \to \mathbf{M}$ be a Lagrangian immersion. Then for every point $y \in M$ there exist Lagrangian totally geodesic submanifolds $S_T, S_N \subset \mathbf{M}$ which are tangent and orthogonal, respectively, to the immersion $\tilde{f}$ at the point $\tilde{f}(y)$.*

Proof. Let $L_y = \tilde{T}_y M$, $L^{\perp}_y = N_y M$ be the tangent and normal subspaces at y. Then the geodesic submanifold $M^{\tilde{f}(y)}_{L_y}$ is totally geodesic by the previous lemma and tangent to the immersion $\tilde{f}$ at $\tilde{f}(y)$ by construction. Likewise, the geodesic submanifold $M^{\tilde{f}(y)}_{L^{\perp}_y}$ is totally geodesic by Lemma 2.1 and the previous lemma and orthogonal to the immersion $\tilde{f}$ at $\tilde{f}(y)$ by construction. It rests to show that $M^{\tilde{f}(y)}_{L_y}$ and $M^{\tilde{f}(y)}_{L^{\perp}_y}$ are Lagrangian. This follows from the fact that ω is parallel, because if the tangent space of a connected totally geodesic submanifold is isotropic with respect to ω at some point, it has to be isotropic everywhere. □

Lemma 3.3. *Let $S \subset \mathbf{M}$ be a connected totally geodesic submanifold. Then either the form ω vanishes on S, or there exist projective subspaces $\mathbf{X}_S \subset \mathbf{X}$, $\mathbf{P}_S \subset \mathbf{P}$ such that $S \subset \mathbf{X}_S \times \mathbf{P}_S$ and at each point $z = (x, p) \in S$ we have $T_z S = T_x\mathbf{X}_S \times T_p\mathbf{P}_S$.*

Proof. Assume that the form ω does not vanish on S in the neighbourhood of some point $\hat{z} = (\hat{x}, \hat{p}) \in S$. Let $L \subset T_{\hat{z}}\mathbf{M}$ be the tangent space to S at $\hat{z}$. Then S locally coincides with the geodesic manifold $M^{\hat{z}}_{L}$. Pass to an affine chart centered on $\hat{z}$ and assume the notations in the proof of Lemma 3.2. Since L is not isotropic, we can find values of the indices Λ, Θ such that $G^{\mathbf{asym}}_{\Lambda\Theta} \neq 0$. Fixing Λ, Θ at these values and letting Ξ run from 1 to k, one easily verifies that expression (3.7) runs through a basis of the image of $\bar{F}$.

Now since S is totally geodesic, the second fundamental form of S vanishes, and (3.7) must be in the image of F for all values of the indices Λ, Θ, Ξ. Therefore the images of F and $\bar{F} = JF$ coincide, and must be an invariant subspace of the inversion J. Any such invariant subspace can be represented as the sum of subspaces $L_{\mathbf{X}} \subset J_+$, $L_{\mathbf{P}} \subset J_-$ of the eigenspaces of J corresponding to the eigenvalues $+1, -1$, respectively. These eigenspaces are precisely the subspaces of $T_{\hat{z}}\mathbf{M}$ defined by the distributions $D^{\mathbf{P}}, D^{\mathbf{X}}$. It follows that there exist projective subspaces $\mathbf{X}_S \subset \mathbf{X}$, $\mathbf{P}_S \subset \mathbf{P}$ and a neighbourhood $U \subset \mathbf{M}$ of $\hat{z}$ such that $S \cap U = (\mathbf{X}_S \times \mathbf{P}_S) \cap U$. But then $S \subset \mathbf{X}_S \times \mathbf{P}_S$, because S is connected. The proof is completed by a dimensional argument. □

Clearly a connected submanifold possessing the local product structure described in Lemma 3.3 is totally geodesic. Thus we get the following pendant of [9, Theorem 2.3, (iv)].

Theorem 3.3. *Let $S \subset \mathbf{M}$ be a connected submanifold, coinciding with a geodesic manifold M^z_L for some $z \in S$ and $L = T_zS$. Then S is totally geodesic if and only if at least one of the following two conditions holds.*

i) The form ω vanishes on S.

ii) There exist projective subspaces $\mathbf{X}_S \subset \mathbf{X}$, $\mathbf{P}_S \subset \mathbf{P}$ such that $S \subset \mathbf{X}_S \times \mathbf{P}_S$ and at each point $z = (x,p) \in S$ we have $T_zS = T_x\mathbf{X}_S \times T_p\mathbf{P}_S$.

3.3. **Example.** In this subsection we consider the 2-dimensional cross-ratio manifold, i.e., for $n = 1$. The projective line $\mathbb{R}P^1$ is homeomorphic to S^1, hence the product manifold $\mathbf{X}\times\mathbf{P}$ is homeomorphic to the 2-dimensional torus. Let us parameterize $\mathbf{X}$ and $\mathbf{P}$ by angles $\phi, \xi \in (-\pi/2, \pi/2]$, such that the representatives $\tilde{x} \in \mathbb{R}^2$, $\tilde{p} \in \mathbb{R}_2$ of $x \in \mathbf{X}$, $p \in \mathbf{P}$ have the form $r\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}$, $r\begin{pmatrix}\cos\xi\\ \sin\xi\end{pmatrix}$, respectively. Then the complement of $\mathbf{M}$ in $\mathbf{X}\times\mathbf{P}$ is given by all pairs (ϕ,ξ) with $|\phi-\xi| = \pi/2$ and is homeomorphic to S^1. The manifold $\mathbf{M}$ itself is homeomorphic to $T\mathbf{X} \cong S^1 \times \mathbb{R}$. Let $z = (\phi,\xi)$, $z' = (\phi',\xi')$. The two-point function (3.1) is given by

$$(z;z') = \frac{\sin(\phi'-\phi)\sin(\xi'-\xi)}{\cos(\xi-\phi)\cos(\xi'-\phi')},$$

and the bilinear form Q is given by $Q = \mathbf{Q}_{\phi\xi} = \cos^{-2}(\xi-\phi)$, all other components of $\mathbf{Q}$ being zero. The Christoffel symbols (3.6) amount to

$$\Gamma^{\phi}_{\phi\phi} = -\Gamma^{\xi}_{\xi\xi} = 2\tan(\phi-\xi),$$

all other Christoffel symbols being zero. The geodesics going through $(0,0)$ are explicitly given by

$$\begin{pmatrix}\arctan t\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \arctan t\end{pmatrix}, \begin{pmatrix}\arctan(c^{-1}\tan(ct))\\ \arctan(c\tan(ct))\end{pmatrix}, \begin{pmatrix}\arctan(c^{-1}\tanh(ct))\\ -\arctan(c\tanh(ct))\end{pmatrix},$$

up to multiplication of the time parameter by a constant factor. Here $c \neq 0$ is a parameter.

By Theorem 3.1 and [8] the two-dimensional cross-ratio manifold is isomorphic to the ruled hyperboloid $H = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2-z^2 = \frac{1}{4}\right\}$, whose para-Kähler structure is defined by the almost para-complex structure determined by the straight lines and the pseudo-Riemannian metric induced by the pseudo-Riemannian metric $dx^2 + dy^2 - dz^2$ on $\mathbb{R}^3$.

4. A model of centro-affine immersions

In this section we investigate the relations between the Lagrangian immersions into the cross-ratio manifold with the centro-affine hypersurface immersions into $\mathbb{R}^{n+1}$. In the first subsection we establish the coincidence of the Codazzi structures defined by the two immersions. In Subsection 4.2 we consider the centro-affine pendants of the totally geodesic Lagrangian submanifolds of Corollary 3.2. In Subsection 4.3 we investigate the relation between centro-affine hypersurface flows in $\mathbb{R}^{n+1}$ and Lagrangian surface flows in $\mathbf{M}$. Finally, in Subsection 4.4 we consider the relation between the affine normal flow in $\mathbb{R}^{n+1}$ and the mean curvature flow in $\mathbf{M}$.

4.1. **Equivalence theorems.** Let $f : M \to \mathbb{R}^{n+1}$ be a centro-affine hypersurface immersion. In the way outlined in Subsection 1.1 we construct from f an immersion $\tilde{f} : M \to \mathbf{M}$ into the cross-ratio manifold. Then we show that the immersion $\tilde{f}$ is Lagrangian, the pseudo-Riemannian metric $\hat{g}$ induced on M by $\tilde{f}$ coincides with the centro-affine metric defined by f, and the connection $\nabla^{\mathbf{X}}$ on M coincides with the centro-affine connection induced by f. In the opposite direction, we show that $\tilde{f}$ determines f up to homothety, and the two-point function (3.1) defines an invariant for centro-affine immersions.

Let M be an n-dimensional manifold and $f : M \to \mathbb{R}^{n+1}$ a centro-affine immersion. We adopt the convention that the transversal vector field equals the negative of the position vector. Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbf{X}$ be the map taking a nonzero vector to its linear span, which is an element of the projective space $\mathbf{X}$. By definition the map $\pi \circ f : M \to \mathbf{X}$ is a local diffeomorphism. By virtue of this diffeomorphism, the atlas of affine charts on $\mathbf{X}$ induces an atlas of charts on M. Assume that some neighbourhood $U_M \subset M$ can be covered by such a chart and choose a corresponding affine chart on $\mathbf{M}$. Let $y \in U_M$ and $f(y) = (r^0, r^1, \dots, r^n)^T$. Then the coordinate vector $(y^1, \dots, y^n)^T$ of y is defined by

$$y^\Lambda = \frac{r^\Lambda}{r^0}. \tag{4.1}$$

Let us further consider r^0 as a function of y on U_M, denote this function by $\tilde{r}^0$ and put $u = -(\tilde{r}^0)^{-1}$. Then we can express the map f in terms of the function $\tilde{r}^0$, namely

$$f(y) = (\tilde{r}^0(y), \tilde{r}^0(y)y^1, \dots, \tilde{r}^0(y)y^n)^T. \tag{4.2}$$

Assume without restriction of generality that $\tilde{r}^0 > 0$ on U_M.

If the function $\tilde{r}^0$ would be identically equal to 1, then $f[U_M]$ would be a subset of the affine hyperplane $r^0 = 1$. The induced affine connection would be flat, and since the coordinate system would parameterize the hyperplane affinely, the Christoffel symbols of the connection would identically vanish. It then follows from [15, (3.14), p.16] that the Christoffel symbols of the actual centro-affine connection ∇' induced on M by the immersion f are given by

$$\Gamma'^{\Phi}_{\Lambda\Xi} = \rho_\Lambda \delta^\Phi_\Xi + \rho_\Xi \delta^\Phi_\Lambda, \tag{4.3}$$

where $\rho = -u\frac{\partial \tilde r^0}{\partial y}$ is the differential of the scalar $\log \tilde r^0$. The centro-affine metric h induced on M by the immersion f is given by [14, eq. (7)][5]

$$h = u^{-1}\frac{\partial^2 u}{\partial y^2} = -(\tilde r^0)^{-1}\frac{\partial^2 \tilde r^0}{\partial y^2} + 2(\tilde r^0)^{-2}\left(\frac{\partial \tilde r^0}{\partial y}\right)^T\frac{\partial \tilde r^0}{\partial y}. \tag{4.4}$$

We now construct an immersion $\tilde f : M \to \mathbf{M}$ from f. For $y \in U_M$ we define $\tilde f(y) = z = (x,p)$ by

$$x = y, \qquad p = -\frac{\left(\frac{\partial \tilde r^0}{\partial y}\right)^T}{\tilde r^0 + \frac{\partial \tilde r^0}{\partial y}y}. \tag{4.5}$$

We then have $\frac{\partial \tilde r^0}{\partial y} + p^T\frac{\partial(\tilde r^0 y)}{\partial y} = 0$, and the linear hyperplane in $\mathbb{R}^{n+1}$ which is the kernel of the nonzero linear functionals in p is just the tangent plane to $f[M]$ at y. It follows that $\tilde f$ is independent of the chosen affine chart and that $x \not\perp p$, i.e., $\tilde f(y)$ is indeed an element of $\mathbf{M}$. Clearly the differential of $\tilde f$ is full rank, and $\tilde f$ is indeed an immersion.

Theorem 4.1. *Assume above definitions. The immersion $\tilde f : M \to \mathbf{M}$ is Lagrangian, the pseudo-Riemannian metric $\hat g$ induced on M by $\tilde f$ is equal to the centro-affine metric h, and the connection $\nabla^{\mathbf{X}}$ on M equals the centro-affine connection ∇'.*

Proof. Assume above notations. Then we have

$$\frac{\partial^2 \tilde r^0}{\partial y^2} = \frac{-\left(\tilde r^0 + \left(\frac{\partial \tilde r^0}{\partial y}\right)^T y\right)\frac{\partial^2 \tilde r^0}{\partial y^2} + \left(\frac{\partial \tilde r^0}{\partial y}\right)^T\left(2\frac{\partial \tilde r^0}{\partial y} + y^T\frac{\partial^2 \tilde r^0}{\partial y^2}\right)}{\left(\tilde r^0 + \left(\frac{\partial \tilde r^0}{\partial y}\right)^T y\right)^2}$$

and

$$1 + p^T x = \frac{\tilde r^0}{\tilde r^0 + \left(\frac{\partial \tilde r^0}{\partial y}\right)^T y}.$$

[5] In the reference the metric differs form ours in sign, because the transversal vector field was assumed equal to the position vector, see [14, eq. (4)].

By (3.5) and (4.5) the pullback $\hat{\mathbf{Q}}$ of the bilinear form $\mathbf{Q}$ on M is then given by the formula

$$\begin{aligned}
\hat{\mathbf{Q}}_{\Lambda\Xi} &= \frac{\partial z^\alpha}{\partial y^\Lambda}\mathbf{Q}_{\alpha\beta}\frac{\partial z^\beta}{\partial y^\Xi} = \frac{\partial z^a}{\partial y^\Lambda}Q_{aB}\frac{\partial z^B}{\partial y^\Xi} \\
&= \delta^a_\Lambda \frac{(1+p_c x^c)\delta^b_a - p_a x^b}{(1+p_c x^c)^2} \frac{\frac{\partial \tilde r^0}{\partial y^b}\left(2\frac{\partial \tilde r^0}{\partial y^\Xi} + y^\Phi \frac{\partial^2 \tilde r^0}{\partial y^\Phi \partial y^\Xi}\right) - \left(\tilde r^0 + \frac{\partial \tilde r^0}{\partial y^\Phi} y^\Phi\right)\frac{\partial^2 \tilde r^0}{\partial y^b \partial y^\Xi}}{\left(\tilde r^0 + \frac{\partial \tilde r^0}{\partial y^\Phi}y^\Phi\right)^2} \\
&= \frac{-\tilde r^0 \frac{\partial^2 \tilde r^0}{\partial y^\Lambda \partial y^\Xi} + (1+p_c x^c)\frac{\partial \tilde r^0}{\partial y^\Lambda}\left(2\frac{\partial \tilde r^0}{\partial y^\Xi} + y^\Phi \frac{\partial^2 \tilde r^0}{\partial y^\Phi \partial y^\Xi}\right)}{(\tilde r^0)^2} \\
&\quad - p_\Lambda \frac{-\tilde r^0 y^\Phi \frac{\partial^2 \tilde r^0}{\partial y^\Phi \partial y^\Xi} + 2y^\Phi \frac{\partial \tilde r^0}{\partial y^\Phi}\frac{\partial \tilde r^0}{\partial y^\Xi}}{(\tilde r^0)^2} \\
&= \frac{-\tilde r^0 \frac{\partial^2 \tilde r^0}{\partial y^\Lambda \partial y^\Xi} + 2\frac{\partial \tilde r^0}{\partial y^\Lambda}\frac{\partial \tilde r^0}{\partial y^\Xi}}{(\tilde r^0)^2},
\end{aligned}$$

which is identical to $h_{\Lambda\Xi}$ by (4.4). Thus the skew-symmetric part of $\hat{\mathbf{Q}}$ vanishes identically on M, while the symmetric part equals the affine metric h. This proves the first two assertions of the theorem.

Now let us compute the Christoffel symbols of the connection $\nabla^{\mathbf{X}}$. By Lemma 2.3 and (3.6) these are given by

$$\Gamma^\Phi_{\Lambda\Xi} = -\frac{p_\Xi \delta^\Phi_\Lambda + p_\Lambda \delta^\Phi_\Xi}{1+p^T x} = \frac{\frac{\partial \tilde r^0}{\partial y^\Xi}\delta^\Phi_\Lambda + \frac{\partial \tilde r^0}{\partial y^\Lambda}\delta^\Phi_\Xi}{\tilde r^0},$$

which coincides with expression (4.3). This completes the proof. □

If the centro-affine immersion f is non-degenerate, i.e., the affine metric is everywhere full rank, then by Definition 2.1 the corresponding Lagrangian immersion $\tilde f$ is regular. Theorem 4.1 then says that the Codazzi structures induced by the immersions f and $\tilde f$ on M coincide. This has the following implications.

Corollary 4.1. *Let $f : M \to \mathbb{R}^{n+1}$ be a non-degenerate centro-affine hypersurface immersion and $\tilde f : M \to \mathbf{M}$ the corresponding regular Lagrangian immersion into the cross-ratio manifold. Then the affine connection which is induced on M by the image $f^* : M \to \mathbb{R}_{n+1}$ of the immersion f under the conormal map equals the connection $\nabla^{\mathbf{P}}$. The cubic form induced on M by f coincides with the cubic form given by Definition 2.2, and the Tchebycheff form T induced by f equals the Tchebycheff form given by Definition 2.4. In particular, if f is an affine hypersphere with centre in the origin, then the Lagrangian immersion $\tilde f$ is minimal.*

Proof. The coincidence of $\nabla^{\mathbf{P}}$ with the dual affine connection follows from the fact that the latter is dual to the affine connection ∇' with respect to the affine metric h, and $\nabla^{\mathbf{P}}$ is dual to $\nabla^{\mathbf{X}}$ with respect to the pseudo-Riemannian metric $\hat g$. But since $\hat g = h$ and $\nabla^{\mathbf{X}} = \nabla'$ by the preceding theorem, $\nabla^{\mathbf{P}}$ must equal the dual affine connection.

The cubic forms are given by the covariant derivatives $\nabla^{\mathbf{X}}\hat g$ and $\nabla' h$, respectively, and must hence coincide by Theorem 4.1. The Tchebycheff forms, which are the contractions of the cubic forms with the respective metrics, then also coincide. The last assertion of the theorem follows from Corollary 2.1. □

Formula (4.5) determines the Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$ as a function of the centro-affine immersion $f : M \to \mathbb{R}^{n+1}$. The following result specifies what happens if we go the opposite way.

Theorem 4.2. *Let M be a simply connected n-dimensional smooth differentiable manifold, and let $\tilde{f} : M \to \mathbf{M}$ be a Lagrangian immersion into the cross-ratio manifold such that the map $\pi_{\mathbf{X}} \circ \tilde{f} : M \to \mathbf{X}$ is a local diffeomorphism. Then there exists a centro-affine immersion $f : M \to \mathbb{R}^{n+1}$ such that $\tilde{f}$ can be recovered from f by formula (4.5). The immersion f is determined up to multiplication of the position vector by a nonzero global constant.*

Proof. Let $U_M \subset M$ be a simply connected open set such that $\tilde{f}$ maps U_M injectively to some affine chart on $\mathbf{M}$. We shall now determine a scalar function $\tilde{r}^0$ on U_M and locally construct the centro-affine immersion f with the desired properties from $\tilde{r}^0$ by (4.2). In order for f to be a valid centro-affine immersion, the function $\tilde{r}^0$ must be nonzero everywhere. In order for f to recover $\tilde{f}$ via formula (4.5), the function $\tilde{r}^0$ must satisfy the integrability condition for this equation. Resolving (4.5) with respect to $\tilde{r}^0$, we get

$$\left(\frac{\partial \log|\tilde{r}^0|}{\partial x}\right)^T = -(I + px^T)^{-1}p = -(1 + p^T x)^{-1}p. \tag{4.6}$$

This equation is integrable if and only if the right-hand side is the gradient of some scalar function, which happens if and only if

$$\frac{\partial}{\partial x}\left(\frac{-p}{1 + p^T x}\right) = \frac{pp^T - \left((1 + p^T x)I - px^T\right)\frac{\partial p}{\partial x}}{(1 + p^T x)^2} \tag{4.7}$$

is symmetric. But by (3.4) the expression $\frac{(1+p^T x)I - px^T}{(1+p^T x)^2}\frac{\partial p}{\partial x}$ is the matrix of the pullback $\hat{\mathbf{Q}}$ of the bilinear form $\mathbf{Q}$ on M, and its being symmetric is equivalent to the immersion $\tilde{f}$ being Lagrangian. Thus a solution for $\log|\tilde{r}^0|$ exists and is determined up to an additive constant. It follows that $\tilde{r}^0$ is nonzero and determined up to a nonzero multiplicative constant.

This proves the assertion of the theorem for the set U_M. But since M is simply connected and can be covered with sets having the properties of U_M, the centro-affine immersion f can be constructed globally. $\square$

We shall now consider the two-point function (3.1). A centro-affine immersion $f : M \to \mathbb{R}^{n+1}$ defines a two-point function $(\cdot\,;\cdot) : M \times M \to \mathbb{R}$ by

$$(y; y') = (\tilde{f}(y); \tilde{f}(y')), \tag{4.8}$$

where $\tilde{f} : M \to \mathbf{M}$ is the Lagrangian immersion induced by f. Clearly this function is a centro-affine invariant.

Theorem 4.3. *The two-point function $(\cdot\,;\cdot)$ defined by (4.8) is symmetric and compatible with the centro-affine metric h, i.e.,*

$$(y; y) = 0, \qquad \left.\frac{\partial(y; y')}{\partial y'}\right|_{y'=y} = 0, \qquad \left.\frac{\partial^2(y; y')}{\partial {y'}^2}\right|_{y'=y} = h \tag{4.9}$$

for all $y \in M$.

Proof. Symmetricity of (4.8) follows from the symmetricity of (3.1). Compatibility with the affine metric follows from definition (3.3), the fact that $\tilde{f}$ is Lagrangian, which implies that the pullback of the form $\mathbf{Q}$ on M equals the pseudo-Riemannian metric $\hat{g}$, and Theorem 4.1 which states that $\hat{g} = h$. □

Function (4.8) allows to measure distances on centro-affine immersions with definite centro-affine metric by much simpler means than the geodesic length, yet staying compatible with the metric. In fact, since the affine metric is generated by (4.8) via (4.9), the expression $\sqrt{|(y;y')|}$ can be considered as the natural distance measure, while the geodesic length just integrates the behaviour of (4.8) on small scales. It should be noted, however, that (4.8) does not generate a true distance function.

4.2. **Tangent and normal geodesic manifolds.** Recall that by Corollary 3.2 at every point $y \in M$ of a regular Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$ there exist tangent and normal totally geodesic Lagrangian submanifolds S_T, S_N. We will consider them as immersions given by the identity inclusion. Let now $f : M \to \mathbb{R}^{n+1}$ be a non-degenerate centro-affine immersion and $\tilde{f}$ the corresponding regular Lagrangian immersion. By Theorem 4.2 the submanifolds S_T, S_N at a point $y \in M$ correspond, a priori at least locally, to families of centro-affine immersions $f_T : S_T \to \mathbb{R}^{n+1}$, $f_N : S_N \to \mathbb{R}^{n+1}$. The members of each family are related by homothety, and in each of these two families there is a distinguished representative defined by $f_T(\tilde{f}(y)) = f_N(\tilde{f}(y)) = f(y)$. The study of these induced centro-affine immersions is the subject of this subsection. We will work with the same charts and parameterizations as in the previous subsection.

Let $f : M \to \mathbb{R}^{n+1}$ be a non-degenerate centro-affine immersion, $\tilde{f}$ the corresponding regular Lagrangian immersion, and $\hat{y} \in M$ an arbitrary point. Pass to an affine chart on $\mathbf{M}$ centered on $\hat{z} = \tilde{f}(\hat{y})$, introduce a corresponding coordinate system in $\mathbb{R}^{n+1}$, and define a chart on M by (4.1) on a neighbourhood $U_M \subset M$ of $\hat{y}$. Then $\hat{r} = f(\hat{y})$ is given by the basis vector e_0. Locally around $\hat{y}$ the immersion $\tilde{f}$ is defined by a function $p = p(x)$. Denote the derivative $\frac{\partial p}{\partial x}$ at the origin by B. Since $\tilde{f}$ is regular and Lagrangian, the matrix B is non-singular and symmetric. By Lemma 3.1, the intersection of the affine chart on $\mathbf{M}$ with the totally geodesic manifold S_T is given by $S_T^o = \{z = (x, Bx) \,|\, x \in \mathbb{R}^n,\ x^T Bx > -1\}$, and, by virtue of Lemma 2.1, the intersection of this chart with S_N is given by $S_N^o = \{z = (x, -Bx) \,|\, x \in \mathbb{R}^n,\ x^T Bx < 1\}$.

The immersions $f_T : S_T \to \mathbb{R}^{n+1}$, $f_N : S_N \to \mathbb{R}^{n+1}$ can be represented by non-zero scalar functions $\tilde{r}^0$ in analogy with (4.2). Note that $f_T(\hat{y}) = f_N(\hat{y}) = f(\hat{y}) = e_0$, and hence both these functions satisfy $\tilde{r}^0(\hat{y}) = 1$. By (4.6) we then get $\log|\tilde{r}^0| = -\frac{1}{2}\log(1 \pm x^T Bx)$, or equivalently, $(\tilde{r}^0)^2 \pm (\tilde{r}^0 y)^T B(\tilde{r}^0 y) = 1$, where the sign depends on whether we consider S_T or S_N. By (4.2) the position vectors r in the images $f_T[S_T], f_N[S_N]$ obey

$$r^T \begin{pmatrix} 1 & 0 \\ 0 & \pm B \end{pmatrix} r = 1.$$

The matrices $\mathrm{diag}(1, \pm B)$ define two non-degenerate quadratic forms B_T, B_N on $\mathbb{R}^{n+1}$. The immersions f_T, f_N are then the connected components containing $f(\hat{y})$ of the 1-level set of these forms, and they are actually embeddings. That f_T, f_N are quadrics follows already by the Pick-Berwald theorem (see [15, Remark 4.3, p.53]) from the fact that S_T, S_N are totally geodesic, and hence their second fundamental form and the cubic form vanish.

Note that by (4.6) and (4.7) the position vectors of the immersion f itself satisfy the relation $\tilde{r}^0 = 1 - \frac{1}{2}y^T By + O(||y||^3)$, hence $(\tilde{r}^0)^{-2} = 1 + y^T By + O(||y||^3)$ and

$$r^T \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} r = 1 + O(||y||^3).$$

In the neighbourhood of $\hat{y} = 0$ the immersions f and f_T thus coincide up to an error term of third order. On the other hand, the arithmetic mean of the quadratic forms B_T, B_N is a rank one form, such that the 1-level set of this form contains the tangent plane to $f[M]$ at $y = \hat{y}$. This yields the following coordinate-free characterization of the centro-affine immersions f_T, f_N induced by the tangential and normal totally geodesic submanifolds.

Theorem 4.4. *Let $f : M \to \mathbb{R}^{n+1}$ be a non-degenerate centro-affine immersion, let $\tilde{f} : M \to \mathbf{M}$ be the corresponding regular Lagrangian immersion into the cross-ratio manifold and let $y \in M$. Let S_T, S_N be the tangential and normal totally geodesic Lagrangian submanifolds to $\tilde{f}[M]$ at y, and let $f_T : S_T \to \mathbb{R}^{n+1}$, $f_N : S_N \to \mathbb{R}^{n+1}$ be the unique centro-affine hypersurface immersions such that $f_T(\tilde{f}(y)) = f(y)$, $f_N(\tilde{f}(y)) = f(y)$, and the Lagrangian immersions $\tilde{f}_T : S_T \to \mathbf{M}$, $\tilde{f}_N : S_N \to \mathbf{M}$ induced by f_T, f_N are the identity inclusions. Then the images of the immersions f_T, f_N are the connected components containing $f(y)$ of the level sets $W_T = \{r \,|\, r^T B_T r = 1\}$, $W_N = \{r \,|\, r^T B_N r = 1\}$ of two quadratic forms on $\mathbb{R}^{n+1}$ and f_T, f_N are embeddings. The form B_T is the unique quadratic form such that the level set W_T approximates the image $f[M]$ at y up to an error term of third order, and B_N is defined by the condition that the level set $\{r \,|\, \frac{1}{2}r^T(B_T + B_N)r = 1\}$ contains the tangent plane to $f[M]$ at y.*

Proof. It suffices to pass to an affine chart on $\mathbf{M}$ centered on $\tilde{f}(y)$ and to a corresponding chart on M and to apply above considerations. □

4.3. **Velocity fields of surface flows.** Let $f : M \to \mathbb{R}^{n+1}$ be a centro-affine hypersurface immersion, embedded in a smooth 1-parametric family f_t of hypersurface immersions such that $f_0 = f$. To this family corresponds a 1-parametric family $\tilde{f}_t : M \to \mathbf{M}$ of regular Lagrangian immersions, such that $\tilde{f}_0 = \tilde{f}$ is the Lagrangian immersion generated by f. In this subsection we show that the $T\mathbf{M}$-valued velocity field $\tilde{\xi} = \frac{\partial \tilde{f}_t}{\partial t}\Big|_{t=0}$ on M depends only on the initial immersion f and the $\mathbb{R}^{n+1}$-valued velocity field $\xi = \frac{\partial f_t}{\partial t}|_{t=0}$, and compute this dependence explicitly. We will then give a simple characterization of the $T\mathbf{M}$-valued vector fields $\tilde{\xi}$ on M which can be obtained in this manner.

Let $\hat{y} \in M$ be an arbitrary point and let $U_M \subset M$ be a neighbourhood of $\hat{y}$ such that $f[U_M]$ can be represented as the graph of a function F in an appropriate basis of $\mathbb{R}^{n+1}$. By choosing a corresponding chart on U_M, we can assume that the immersion f can be represented like $f(y) = (F(y), y^T)^T$ for $y \in U_M$. Pass to the affine chart on $\mathbf{M}$ corresponding to the chosen basis of $\mathbb{R}^{n+1}$. Then it is not hard to see that the Lagrangian immersion $\tilde{f}$ is given by $\tilde{f}(y) = z = (x, p)$ with

$$x = \frac{y}{F(y)}, \qquad p = -\left(\frac{\partial F}{\partial y}\right)^T. \tag{4.10}$$

Lemma 4.1. *Assume above notations. Let $f : M \to \mathbb{R}^{n+1}$ be a centro-affine immersion given by $f(y) = (F(y), y^T)^T$, $y \in U_M \subset M$, and let $\tilde{f}$ be the corresponding Lagrangian immersion given by (4.10). Let further $\{f_t\}_{t\in I\subset\mathbb{R}}$ be a 1-parametric smooth family of centro-affine immersions such that $f_0 = f$, and $\xi = (\xi^0, \dots, \xi^n)^T = \frac{\partial f_t}{\partial t}|_{t=0}$ the corresponding velocity field. If $\tilde{f}_t$ is the corresponding family of Lagrangian immersions in $\mathbf{M}$, then the velocity field $\tilde{\xi} = \frac{\partial \tilde{f}_t}{\partial t}|_{t=0}$ is given by*

$$\tilde{\xi}^a = \frac{\xi^a}{F} - \frac{\xi^0 y^a}{F^2}, \qquad \tilde{\xi}^A = -\frac{\partial \xi^0}{\partial y^a} + \frac{\partial \xi^b}{\partial y^a}\frac{\partial F}{\partial y^b}. \tag{4.11}$$

Proof. Let us introduce functions $\bar{F} : I \times U_M \to \mathbb{R}$, $\bar{y} : I \times U_M \to \mathbb{R}^n$ by $f_t(y) = (\bar{F}(t,y), \bar{y}^T(t,y))^T$. Then $\frac{\partial \bar{F}(t,y)}{\partial t}|_{t=0} = \xi^0(y)$, $\frac{\partial \bar{y}(t,y)}{\partial y}|_{t=0} = I$, $\frac{\partial \bar{y}^a(t,y)}{\partial t}|_{t=0} = \xi^a(y)$. By (4.10) we have $\tilde{f}_t(y) = \bar{z}(t,y) = (\bar{x}(t,y), \bar{p}(t,y))$, where

$$\bar{x}(t,y) = \frac{\bar{y}(t,y)}{\bar{F}(t,y)}, \qquad \bar{p}(t,y) = -\left(\frac{\partial \bar{y}(t,y)}{\partial y}\right)^{-T}\left(\frac{\partial \bar{F}(t,y)}{\partial y}\right)^T.$$

Evaluation of the expressions $\tilde{\xi}^a = \left.\frac{\partial \bar{z}^a(t,y)}{\partial t}\right|_{t=0}$, $\tilde{\xi}^A = \left.\frac{\partial \bar{z}^A(t,y)}{\partial t}\right|_{t=0}$ then leads to the formulas asserted in the lemma. $\square$

Given a centro-affine immersion $f : M \to \mathbb{R}^{n+1}$ and the corresponding Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$, relation (4.11) defines a linear operator $\mathbf{V}_f : \xi \mapsto \tilde{\xi}$, taking smooth $\mathbb{R}^{n+1}$-valued vector fields on M to smooth $T\mathbf{M}$-valued vector fields. If a $\mathbb{R}^{n+1}$-valued vector field ζ on M is tangential to the immersion f, then it can be lifted to a vector field Z on M and its coordinate vector is given by $\zeta = \frac{\partial f}{\partial y}Z = \begin{pmatrix} \frac{\partial F}{\partial y} \\ I \end{pmatrix} Z$. The corresponding $T\mathbf{M}$-valued vector field is then given by

$$\tilde{\zeta} = \mathbf{V}_f(\zeta) = \begin{pmatrix} \frac{I}{F} - \frac{y\frac{\partial F}{\partial y}}{F^2} \\ -\frac{\partial^2 F}{\partial y^2} \end{pmatrix} Z = \frac{\partial z}{\partial y} Z = \tilde{Z}.$$

It follows that $\tilde{\zeta}$ is a cross-section of $\tilde{T}M$, namely it is the image of Z under the differential of $\tilde{f}$. Denote the restriction of the operator $\mathbf{V}_f$ to the subspace of tangential vector fields by $\mathbf{V}_f^{\mathbf{t}}$. Thus $\mathbf{V}_f^{\mathbf{t}}$ is the canonical isomorphism between $\mathbb{R}^{n+1}$-valued vector fields which are tangent to the immersion f, and $T\mathbf{M}$-valued vector fields which are tangent to the immersion $\tilde{f}$.

The operator $\mathbf{V}_f$ between $\mathbb{R}^{n+1}$-valued and $T\mathbf{M}$-valued vector fields on M can thus be lifted to an operator relating equivalence classes modulo additive components that are tangential to the immersions f and $\tilde{f}$, respectively. This is nothing else than the infinitesimal version of the relation between surface flows in $\mathbb{R}^{n+1}$ and $\mathbf{M}$, because the time derivative of the surface flow is determined only by the corresponding equivalence class of the velocity field. Since the immersion f is centro-affine, every $\mathbb{R}^{n+1}$-valued vector field ξ on M can be decomposed into a tangential component ζ and a radial component sf, which is proportional to the position vector with proportionality factor s. The equivalence class of ξ can then be characterized by the scalar field s.

Theorem 4.5. *Assume above notations. Let $f : M \to \mathbb{R}^{n+1}$ be a centro-affine hypersurface immersion and let ξ be a $\mathbb{R}^{n+1}$-valued vector field on M. Decompose ξ into a tangential component ζ and a radial component sf, which is proportional to the position vector f with coefficient s. Let $\tilde f : M \to \mathbf{M}$ be the corresponding Lagrangian immersion and $\tilde\xi = \mathbf{V}_f(\xi)$. Then the 1-form $\omega_{\tilde\xi}$ given by Definition 2.3 equals $\frac{1}{2}ds$. The kernel of the operator $\mathbf{V}_f$ is given by the radial vector fields of the form λf, where the scalar λ is constant on every connected component of M.*

Proof. Let us compute the vector field $\tilde\eta = \mathbf{V}_f(sf)$ corresponding to the radial component of ξ. By virtue of (4.11) and (4.10) we have

$$\begin{aligned}(4.12)\qquad \tilde\eta &= \begin{pmatrix} 0 \\ -(F - \frac{\partial F}{\partial y}y)\left(\frac{\partial s}{\partial y}\right)^T\end{pmatrix} = \begin{pmatrix} 0 \\ -F(1+p^Tx)\left(\frac{\partial x}{\partial y}\right)^T\left(\frac{\partial s}{\partial x}\right)^T\end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -(1+p^Tx)(I+px^T)\left(\frac{\partial s}{\partial x}\right)^T\end{pmatrix}.\end{aligned}$$

Let now X be a vector field on M. Denote $\tilde\zeta = \mathbf{V}_f(\zeta)$. Then $\omega(\tilde\zeta,\tilde X) = 0$, because both $\tilde\zeta, \tilde X$ are tangential and the immersion $\tilde f$ is Lagrangian. Hence by virtue of $\tilde\xi = \tilde\eta + \tilde\zeta$ we have

$$\omega(\tilde\xi,\tilde X) = \omega(\tilde\eta,\tilde X) = \omega_{\alpha\beta}\tilde\eta^\alpha\frac{\partial z^\beta}{\partial y^\Lambda}X^\Lambda = \frac{1}{2}\left(Q_{aB}\tilde\eta^a\frac{\partial z^B}{\partial y^\Lambda} - Q_{bA}\tilde\eta^A\frac{\partial z^b}{\partial y^\Lambda}\right)X^\Lambda,$$

where the last relation is due to (2.4). Inserting the value for $\tilde\eta$ and using (3.4), we get

$$\omega(\tilde\xi,\tilde X) = \frac{1}{2}\frac{\partial s}{\partial x}(I+xp^T)\frac{(1+p^Tx)I - xp^T}{1+p^Tx}\frac{\partial x}{\partial y^\Lambda}X^\Lambda = \frac{1}{2}\frac{\partial s}{\partial y^\Lambda}X^\Lambda.$$

Hence $\omega_{\tilde\xi} = \frac{1}{2}ds$.

Let us compute the kernel of $\mathbf{V}_f$. For any vector field ξ in this kernel, $\tilde\xi = 0$ and the form $\omega_{\tilde\xi} = \frac{1}{2}ds$ vanishes. Hence the radial component sf of ξ must be such that s is constant on every connected component of M. Let now $\xi = sf$ be a radial vector field with such a scalar s. Since the operator $\mathbf{V}_f^{\mathbf{t}}$ is injective, there exists a unique tangent vector field ζ such that $\mathbf{V}_f(sf+\zeta) = 0$. But by (4.12) the relation $\mathbf{V}_f(sf+\zeta) = 0$ is satisfied by the tangent field $\zeta = 0$. This completes the proof. $\square$

Corollary 4.2. *Assume above notations. Let $f : M \to \mathbb{R}^{n+1}$ be a centro-affine hypersurface immersion and let $\tilde f : M \to \mathbf{M}$ be the corresponding Lagrangian immersion into the cross-ratio manifold. A smooth $T\mathbf{M}$-valued vector field $\tilde\xi$ on M is of the form (4.11) for some $\mathbb{R}^{n+1}$-valued vector field ξ on M if and only if the 1-form $\omega_{\tilde\xi}$ is exact.*

Proof. Let first ξ be a $\mathbb{R}^{n+1}$-valued vector field on M and set $\tilde\xi = \mathbf{V}_f(\xi)$. By Theorem 4.5 the corresponding form $\omega_{\tilde\xi}$ is exact.

On the other hand, let $\tilde\xi'$ be a smooth $T\mathbf{M}$-valued vector field on M such that $\omega_{\tilde\xi'}$ is exact. Then $\omega_{\tilde\xi'}$ can be expressed as $\frac{1}{2}ds$ for some scalar field s. Define the radial vector field $\xi = sf$ and let $\tilde\xi = \mathbf{V}_f(\xi)$. Let $\omega_{\tilde\xi}$ be the corresponding 1-form on M. By construction and by virtue of Theorem 4.5 we then have $\omega_{\tilde\xi'} = \omega_{\tilde\xi}$, and $\omega(\tilde\xi' - \tilde\xi, \tilde X) = 0$ for all cross-sections $\tilde X$ of $\tilde TM$. Since $\tilde f$ is Lagrangian, the difference $\tilde\zeta = \tilde\xi' - \tilde\xi$ must then also be a cross-section of $\tilde TM$, and hence has a preimage $\zeta = (\mathbf{V}_f^{\mathbf{t}})^{-1}(\tilde{\zeta})$. Then $\mathbf{V}_f(\xi + \zeta) = \tilde{\xi} + \tilde{\zeta} = \tilde{\xi}'$, which completes the proof. □

Exchanging the roles of $\mathbb{R}^{n+1}$ and its dual $\mathbb{R}_{n+1}$, we can carry out the same construction for centro-affine hypersurface immersions $f^* : M \to \mathbb{R}_{n+1}$. Then there exists a linear operator $\mathbf{V}_f^*$ taking $\mathbb{R}_{n+1}$-valued vector fields ξ^* on M to $T\mathbf{M}$-valued vector fields $\tilde{\xi}^* = \mathbf{V}_f^*(\xi^*)$. The restriction of $\mathbf{V}_f^*$ to tangent vector fields is a canonical isomorphism between the spaces of vector fields which are tangent to the immersions f^* and $\tilde{f}^*$, respectively. The kernel of $\mathbf{V}_f^*$ consists of those radial vector fields $s^* f^*$ such that the scalar coefficient s^* is constant on every connected component of M.

Corollary 4.3. *Assume above notations and fix a volume form on $\mathbb{R}^{n+1}$ which is compatible with the affine structure. Let $f_t : M \to \mathbb{R}^{n+1}$ be a smooth family of non-degenerate centro-affine hypersurface immersions and let $f_t^* : M \to \mathbb{R}_{n+1}$ be the image of f_t under the conormal map. Let further $\xi = \left.\frac{\partial f_t}{\partial t}\right|_{t=0}$ and $\xi^* = \left.\frac{\partial f_t^*}{\partial t}\right|_{t=0}$ be the corresponding velocity fields at $t = 0$. Decompose these fields as $\xi = s f_0 + \zeta$, $\xi^* = s^* f_0^* + \zeta^*$ into radial and tangent components. Then the proportionality factors at the position vectors obey the relation $s + s^* = 0$.*

Proof. Clearly both families f_t, f_t^* give rise to the same family of Lagrangian immersions, so $\tilde{f}_t = \tilde{f}_t^*$. It follows that $\tilde{\xi} = \tilde{\xi}^*$. However, the 1-forms $\omega_{\tilde{\xi}}$, $\omega_{\tilde{\xi}^*}$ have opposite sign, because ω changes sign under the map $(x, p) \mapsto (p, x)$ (see Remark 3.1). By Theorem 4.5 it then follows that $ds + ds^* = 0$ and hence $s + s^*$ is constant on every connected component of M. The value of this constant on a given connected component cannot depend on the vector field ξ, because it must remain invariant under deformations of ξ whose support has a complement with nonempty interior, and hence under any deformations of ξ. Since $s^* \equiv 0$ whenever $s \equiv 0$, it follows that $s + s^* = 0$. □

4.4. **Affine normal flow and mean curvature flow.** In this subsection we apply the results of the previous subsection to determine the relation between the affine normal vector field ξ of a non-degenerate centro-affine immersion $f : M \to \mathbb{R}^{n+1}$ and the mean curvature vector field ς of the corresponding regular Lagrangian immersion $\tilde{f} : M \to \mathbf{M}$.

The mean curvature flow of para-Kähler pseudo-Einstein manifolds was investigated in [5]. It was shown that this flow is well-defined for positive time up to some final, possibly infinite, time $T > 0$, and that it preserves the property of the surface to be Lagrangian. Theorem 4.2 then implies that for regular Lagrangian immersions the mean curvature flow is, at least locally, generated by some surface flow in $\mathbb{R}^{n+1}$. By (2.12) and Corollary 4.2 we have the following result.

Corollary 4.4. *Let $\tilde{f} : M \to \mathbf{M}$ be a regular Lagrangian immersion into the cross-ratio manifold. Then the Tchebycheff form T of $\tilde{f}$ is exact.*

We adopt the same notations, parametrization of M and representation of f as in the previous subsection. Suppose that in some subset $U_M \subset M$ we have $f(y) = (F(y), y^T)^T$ for some smooth function F and for all $y \in U_M$. Then $\tilde{f}$ is given by (4.10). We assume that $\mathbb{R}^{n+1}$ is endowed with a volume element det such that $\det(e_0, \dots, e_n) = 1$. The affine normal field $\xi = (\xi^0, \dots, \xi^n)^T = (\xi^0, \bar{\xi}^T)^T$ of f

obeys the relations [15, (3.4), p.48]

$$\bar{\xi} = -\left(\frac{\partial^2 F}{\partial y^2}\right)^{-1}\left(\frac{\partial \phi}{\partial y}\right)^T, \qquad \xi^0 = \frac{\partial F}{\partial y}\bar{\xi} + \phi,$$

where $\phi = \left|\det \frac{\partial^2 F}{\partial y^2}\right|^{\frac{1}{n+2}}$.

Let us compute the corresponding form $\omega_{\tilde{\xi}}$ using Theorem 4.5. We decompose ξ into a tangential part and a radial part,

$$\xi = \begin{pmatrix} \frac{\partial F}{\partial y} \\ I \end{pmatrix} Z + s\begin{pmatrix} F \\ y \end{pmatrix}.$$

From this we get

$$s = \frac{\xi^0 - \frac{\partial F}{\partial y}\bar{\xi}}{F - \frac{\partial F}{\partial y}y} = \frac{\phi}{F - \frac{\partial F}{\partial y}y}. \tag{4.13}$$

Note that the scalar field s is nowhere zero and remains invariant only under equiaffine coordinate changes, i.e., those which preserve the volume element in $\mathbb{R}^{n+1}$. Under a general coordinate change, s will be multiplied with a nonzero constant. From Theorem 4.5 it now follows that the coordinate vector of the form $\omega_{\tilde{\xi}}$ is given by

$$\begin{aligned} 2\omega_{\tilde{\xi}} &= \left(\frac{\partial s}{\partial y}\right)^T = \frac{(F - \frac{\partial F}{\partial y}y)\left(\frac{\partial \phi}{\partial y}\right)^T + \phi\frac{\partial^2 F}{\partial y^2}y}{(F - \frac{\partial F}{\partial y}y)^2} \\ &= \frac{\frac{1}{n+2}(F - \frac{\partial F}{\partial y}y)\langle(\frac{\partial^2 F}{\partial y^2})^{-1}, \frac{\partial^3 F}{\partial y^3}\rangle + \frac{\partial^2 F}{\partial y^2}y}{(F - \frac{\partial F}{\partial y}y)^2}\phi, \end{aligned} \tag{4.14}$$

where $\langle(\frac{\partial^2 F}{\partial y^2})^{-1}, \frac{\partial^3 F}{\partial y^3}\rangle$ denotes the vector obtained by contraction of the third derivative of F with the inverse of the second.

On the other hand, let us compute the Tchebycheff form given by Definition 2.4. Expression (2.13) becomes

$$\frac{\partial y^c}{\partial x^b}\left[-\frac{\partial Q^{La}}{\partial z^L}Q_{aK}\frac{\partial p^k}{\partial y^c} + \frac{\partial}{\partial y^c}\left(\frac{\partial p^k}{\partial x^d}\right)\frac{\partial x^d}{\partial p^k} + Q_{aL}\frac{\partial Q^{Ld}}{\partial z^d}\frac{\partial x^a}{\partial y^c}\right]. \tag{4.15}$$

Note that (2.13) expresses the Tchebycheff form in x-coordinates. To obtain its expression in y-coordinates, one has to remove the factor $\frac{\partial y^c}{\partial x^b}$. Differentiating (3.5), we get

$$\frac{\partial Q^{Ab}}{\partial z^c} = p_c(\delta_a^b + p_a x^b) + (1 + p_d x^d)p_a\delta_c^b, \qquad \frac{\partial Q^{Ab}}{\partial z^C} = x^c(\delta_a^b + p_a x^b) + (1 + p_d x^d)\delta_a^c x^b,$$

$$\frac{\partial Q^{Ab}}{\partial z^b} = (n+1)(1 + p_b x^b)p_a, \qquad \frac{\partial Q^{Ab}}{\partial z^A} = (n+1)(1 + p_a x^a)x^b.$$

From (4.10) we get

$$\begin{aligned}
\frac{\partial x}{\partial y} &= F^{-2}\left(F \cdot I - y\frac{\partial F}{\partial y}\right), \\
\frac{\partial y}{\partial x} &= F\left(I + \left(F - \frac{\partial F}{\partial y}y\right)^{-1} y\frac{\partial F}{\partial y}\right), \\
\frac{\partial p}{\partial x} &= -F\frac{\partial^2 F}{\partial y^2}\left(I + \left(F - \frac{\partial F}{\partial y}y\right)^{-1} y\frac{\partial F}{\partial y}\right), \\
\frac{\partial x}{\partial p} &= -F^{-2}\left(F \cdot I - y\frac{\partial F}{\partial y}\right)\left(\frac{\partial^2 F}{\partial y^2}\right)^{-1}, \\
Q &= F\left(F - \frac{\partial F}{\partial y}y\right)^{-1}\left(I + \left(F - \frac{\partial F}{\partial y}y\right)^{-1} y\frac{\partial F}{\partial y}\right)^{T}.
\end{aligned}$$

Inserting these expressions into the brackets in (4.15), we obtain after some calculations

$$T = (n+2)\left(F - \frac{\partial F}{\partial y}\right)^{-1} \frac{\partial^2 F}{\partial y^2}y + \left\langle \left(\frac{\partial^2 F}{\partial y^2}\right)^{-1}, \frac{\partial^3 F}{\partial y^3} \right\rangle. \tag{4.16}$$

From (4.13),(4.14) it then follows that

$$2\omega_{\tilde{\xi}} = \frac{s}{n+2}T \tag{4.17}$$

and from (2.12),(4.14) that

$$T = -2\omega_{\varsigma} = (n+2)d\log|s|. \tag{4.18}$$

This again proves Corollary 4.4 by providing an explicit expression of the Tchebycheff form in terms of the affine normal field.

We shall now show that the velocity field of the Lagrangian surface flow in $\mathbf{M}$ defined by the affine normal flow can be chosen to be pointwise proportional to the mean curvature vector field, and construct a surface flow in $\mathbb{R}^{n+1}$ which produces the mean curvature flow in $\mathbf{M}$.

Theorem 4.6. *Let $f : M \to \mathbb{R}^{n+1}$ be a non-degenerate centro-affine immersion, and let $\tilde{f} : M \to \mathbf{M}$ be the corresponding Lagrangian immersion. Let further $\xi : M \to \mathbb{R}^{n+1}$ be the affine normal field of f with respect to some fixed volume element in $\mathbb{R}^{n+1}$ which is compatible with the affine structure, and let $\tilde{\xi} = \mathbf{V}_f(\xi)$ as defined in Subsection 4.3. Decompose ξ into a tangential component ζ and a radial component sf, which is proportional to the position vector with proportionality factor s. Then the normal component of $\tilde{\xi}$ is given by $\tilde{\xi}^N = -\frac{s}{n+2}\varsigma$, where ς is the mean curvature vector field of $\tilde{f}$. Define further the $\mathbb{R}^{n+1}$-valued vector field $\xi' = -(n+2)s^{-1}\log|s|\xi$ and let $\tilde{\xi}' = \mathbf{V}_f(\xi')$. Then the mean curvature flow coincides with the surface flow induced in $\mathbf{M}$ by the velocity field $\tilde{\xi}'$.*

Proof. Assume the notations of the theorem. Since $\xi = \zeta + sf$, it follows that $\xi' = -(n+2)s^{-1}\log|s|\zeta - (n+2)\log|s|f$, and the radial part of ξ' is proportional to the position vector with proportionality factor $-(n+2)\log|s|$. From Theorem 4.5 and relation (4.18) it then follows that $\omega_\varsigma = \omega_{\tilde{\xi}'}$. Hence the velocity fields ς and

$\tilde{\xi}'$ are identical modulo an additive tangential component, and the surface flows induced by these velocity fields coincide.

From (4.17) and (2.12) we have $\omega_{\tilde{\xi}} = -\frac{s}{n+2}\omega_\varsigma$, and hence the vector fields $\tilde{\xi}$ and $-\frac{s}{n+2}\varsigma$ coincide modulo an additive tangential component. In particular, the normal component $\tilde{\xi}^N$ must be equal to $-\frac{s}{n+2}\varsigma$, because ς is a normal vector field. □

Corollary 4.5. *Assume above notations and fix a volume form on $\mathbb{R}^{n+1}$ which is compatible with the affine structure. Let $f : M \to \mathbb{R}^{n+1}$ be a non-degenerate centro-affine hypersurface immersion, and let $f^* : M \to \mathbb{R}_{n+1}$ be the image of f under the conormal map. Let further ξ, ξ^* be the corresponding affine normal fields. Decompose these fields as $\xi = sf + \zeta$, $\xi^* = s^*f^* + \zeta^*$ into radial and tangent components. Then the proportionality factors at the position vectors obey the relation $ss^* = const$.*

Proof. By Theorem 4.6, the velocity fields $\xi' = -(n+2)s^{-1}\log|s|\xi$, $\xi^{*\prime} = -(n+2)(s^*)^{-1}\log|s^*|\xi^*$ are such that $\mathbf{V}_f(\xi'), \mathbf{V}_f(\xi^{*\prime})$ are in the same equivalence class modulo additive tangential components, namely they are both equivalent to the mean curvature vector field ς. By Theorem 4.5 and Remark 3.1, the differentials of the proportionality factors $-(n+2)\log|s|$, $-(n+2)\log|s^*|$ of the radial components of the fields ξ', $\xi^{*\prime}$ add up to zero, which yields the desired assertion. □

A direct calculation shows that in fact $ss^* = \pm 1$, depending on the definition of the affine normal field. Corollary 4.3 then implies that the velocity fields $\xi', \xi^{*\prime}$ in the above proof define surface flows in $\mathbb{R}^{n+1}, \mathbb{R}_{n+1}$, respectively, which are related by the conormal map. The affine normal fields ξ, ξ^*, on the contrary, do not possess such a symmetry. This suggests that the surface flows defined by $\xi', \xi^{*\prime}$ are worthwhile objects to study in centro-affine geometry.

5. Conclusions

In this paper we constructed a special para-Kähler pseudo-Einstein manifold, the *cross-ratio manifold* (Subsections 1.1 and 3.1). The name cross-ratio manifold reflects that the para-Kähler structure is determined by the behaviour of the projective cross-ratio on small scales. Manifolds which are isomorphic to the cross-ratio manifold were studied before. In particular, one of the reduced para-complex projective spaces $P_n(B)/\mathbb{Z}_2$ defined in [9] is in this isomorphism class (Theorem 3.1).

In [9] it is emphasized that a family of reduced para-complex projective spaces can be naturally associated to every real vector space E. The same holds also for the cross-ratio manifold, when the real projective spaces $\mathbf{X} = \mathbb{R}P^n$, $\mathbf{P} = \mathbb{R}P_n$ in the construction presented in Subsection 3.1 are replaced by the isomorphic projective spaces $P(E)$, $P(E^*)$ over the $(n+1)$-dimensional real vector space E and its dual, respectively. The cross-ratio manifold associated to E has an intimate relation with the centro-affine geometry of E. This is the main result of the paper and is detailed in Section 4. Namely, the Lagrangian immersions in the cross-ratio manifold exhaustively model the geometry of centro-affine hypersurface immersions in E. Every Lagrangian immersion corresponds to a one-parametric family of centro-affine hypersurface immersions related by homothety, and every centro-affine hypersurface immersion corresponds to a unique Lagrangian immersion (Subsection 4.1).

In particular, the dual pair of Codazzi structures formed by the primal and dual centro-affine connections and the centro-affine metric of a centro-affine hypersurface immersion appear in a natural way on the corresponding Lagrangian immersion in the cross-ratio manifold (Theorem 4.1). The cubic form of the centro-affine immersion is proportional to the second fundamental form of the Lagrangian immersion (see Lemma 2.4 for the exact relation), which explains why the behaviour of the cubic form in many respects resembles that of the second fundamental form of Euclidean geometry. In the same way, the Tchebycheff form of the centro-affine immersion is proportional to the mean curvature vector of the Lagrangian immersion (see eq. (2.12) for the exact relation), which yields a transparent characterization of the proper affine hyperspheres as minimal Lagrangian immersions in the cross-ratio manifold.

The close relation between the centro-affine hypersurface immersions in a real vector space E and the Lagrangian immersions in the cross-ratio manifold constructed from E offers a two-fold advantage. Firstly, centro-affine geometry can now be studied as a particular instance of Riemannian geometry, and methods from Riemannian geometry can be applied to centro-affine geometry. Moreover, the structure defined on the cross-ratio manifold by the projective cross-ratio (see (3.1) and (3.2)) can be pulled back to centro-affine immersions, providing a new centro-affine invariant (Theorem 4.3). Secondly, centro-affine geometry can serve as an inspiration and a starting point for the emerging theory of Lagrangian submanifolds in para-Kähler spaces. We have made some steps in this direction in the companion paper [11].

In Subsection 3.2 we established a close link between totally geodesic submanifolds of the cross-ratio manifold and the behaviour of the symplectic form on these submanifolds. In particular, a geodesic submanifold is totally geodesic if and only if it either possesses a certain product structure (this was already established in [9]), or is isotropic with respect to the symplectic form. This implies that isotropic submanifolds of the cross-ratio manifold can locally be well approximated by totally geodesic submanifolds. The cubic form on Lagrangian submanifolds serves as a measure of this deviation, which offers the possibility of obtaining results on the global structure and topology of such manifolds if bounds on the cubic form are known.

In Subsection 4.3 we investigated the relation between surface flows in E and in the cross-ratio manifold associated to E. In Subsection 4.4 we concretized these results to the relation between the affine normal flow in E and the mean curvature flow in the cross-ratio manifold. In particular, we showed that the mean curvature flow also corresponds to a surface flow in E (Theorem 4.6), and that this flow, in contrast to the affine normal flow, can be chosen to be symmetric with respect to the duality defined by the conormal map.

Graph immersions can be modeled in a similar way as centro-affine immersions. In this case the role of the cross-ratio manifold is played by the flat para-Kähler space. This will be the subject of a future publication.

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