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HALF-DIMENSIONAL IMMERSIONS IN PARA-KÄHLER MANIFOLDS

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Abstract. We show that every in a certain sense non-degenerated Lagrangian immersion in a para-Kähler manifold naturally carries a dual pair of Codazzi structures. On the other hand, every manifold carrying a dual pair of Codazzi structures can be represented as a non-degenerated Lagrangian submanifold of a para-Kähler manifold. We derive this equivalence from a similar, but more general one, relating non-degenerated half-dimensional immersions in para-Kähler manifolds to dual pairs of what we call pre-Codazzi structures. We specialize this equivalence in two cases. Firstly, we show that every projectively flat manifold carries a natural pre-Codazzi structure, and can be, at least locally, represented as a half-dimensional immersion in a special para-Kähler manifold, which we call the cross-ratio manifold. Secondly, we show that manifolds carrying pre-Codazzi structures with flat connections are represented by half-dimensional immersions in the flat para-Kähler space. Our results have applications in affine differential geometry. Namely, centro-affine geometry can be seen as the geometry of Lagrangian immersions in the cross-ratio manifold, while the geometry of graph immersions is equivalent to the geometry of Lagrangian immersions in the flat para-Kähler space. We also obtain a natural duality relation between projectively flat connections on a manifold, extending the duality induced by the conormal map of centro-affine immersions to connections which are not equiaffine.

1. Introduction and overview

One of the main goals of this contribution is to uncover a close relation that exists between Lagrangian submanifolds of para-Kähler manifolds and Codazzi manifolds, i.e., manifolds carrying a Codazzi structure. Codazzi manifolds are sometimes also called statistical manifolds. We will show that every Lagrangian submanifold of a para-Kähler manifold, subject to a non-degeneracy condition, can be naturally seen as a Codazzi manifold. On the other hand, every Codazzi manifold possesses a corresponding representation as a Lagrangian submanifold of a para-Kähler manifold. Codazzi structures also naturally appear on certain affine hypersurface immersions in affine differential geometry. Indeed, we will show that certain classes of such

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hypersurface immersions are linked via their Codazzi structures to the Lagrangian immersions in specific para-Kähler manifolds. In order to give an overview of the technical details in this paper, we continue with a short introduction into para-Kähler manifolds, affine differential geometry and Codazzi structures.

Para-Kähler manifolds. A *symplectic manifold* is a differentiable manifold carrying a symplectic form ω, i.e., a closed, non-degenerate, skew-symmetric 2-form. A *Fedosov manifold* is a symplectic manifold with a torsion-free affine connection ∇ such that the symplectic form ω is parallel with respect to this connection, $\nabla\omega = 0$. A *para-complex manifold* $\mathbf{M}$ is a $2n$-dimensional manifold with a smooth tensor field J of type $(1,1)$ such that J is an involution of the tangent space $T_z\mathbf{M}$ at each $z \in \mathbf{M}$, $J^2 = 1$, and such that the eigenspaces corresponding to the eigenvalues ± 1 of J form two involutive (completely integrable) n-dimensional distributions. We will denote these distributions by $D^{\mathbf{P}}$, $D^{\mathbf{X}}$, respectively. The field J is called the *para-complex structure* of the manifold. A *para-Kähler manifold* is a para-complex Fedosov manifold such that the eigenspace distributions $D^{\mathbf{X}}$, $D^{\mathbf{P}}$ are isotropic with respect to the symplectic form ω, and whose affine connection ∇ is the Levi-Civita connection of the pseudo-Riemannian metric g defined by

$$g[X,Y] = \omega[JX,Y], \qquad \omega[X,Y] = g[JX,Y] \tag{1.1}$$

for every two vector fields X, Y on $\mathbf{M}$. Here and in the rest of the paper we denote by brackets the value of covariant tensors on vectors or vector fields. It is not hard to see that g is necessarily of neutral signature and non-degenerate. The integral submanifolds of the distributions $D^{\mathbf{X}}$, $D^{\mathbf{P}}$ locally form two Lagrangian foliations, which is why a para-Kähler structure is sometimes called a *bi-Lagrangian structure.*

The simplest para-Kähler manifold is the flat para-Kähler space $\mathbb{E}^{2n}_n$, defined as the space $\mathbb{R}^{2n}$ endowed with the structures

$$g = \frac{1}{2}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \qquad \omega = \frac{1}{2}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \tag{1.2}$$

where I_n is the $n \times n$ identity matrix.

Para-Kähler manifolds differ from Kähler manifolds by the property that the automorphism J of the tangent bundle is an anti-isometry, $g[JX, JY] = -g[X,Y]$, while in Kähler manifolds it is an isometry, and that it is an involution, $J^2 = 1$, while in Kähler manifolds it squares to -1. The class of para-Kähler manifolds has been explicitly introduced by Libermann in [14]. Recent expositions of para-Kähler manifolds can be found in [10] or [11]. A compact introduction can also be found in [1, Section 5].

Affine differential geometry. Consider an immersion $f : M \to \mathbf{M}$ of an n-dimensional differentiable manifold M into an $(n+k)$-dimensional differentiable manifold $\mathbf{M}$ carrying a torsion-free affine connection $\bar\nabla$. Suppose further that for every $y \in M$, we are given a k-dimensional linear subspace $D_y \subset T_{f(y)}\mathbf{M}$, depending smoothly on y, such that $T_{f(y)}\mathbf{M} = D_y \oplus f_*[T_yM]$, where f_* is the differential of the immersion f. In other words, the distribution D is required to be transversal to the immersion. Let now X, Y be vector fields on M and let $\tilde X, \tilde Y$ be (locally) extensions of their images under the differential f_* to a neighbourhood of the immersion f in $\mathbf{M}$. On the immersion we can then decompose the covariant derivative

$$\bar\nabla_{\tilde X}\tilde Y = f_*(\nabla_X Y) + \alpha[X,Y] \tag{1.3}$$

into a component $f_*(\nabla_X Y)$ which is tangent to the immersion and a component $\alpha[X,Y]$ which lies in the distribution D. One can show that $\bar{\nabla}_{\tilde{X}}\tilde{Y}$ depends only on the original vector fields X, Y, the preimage under the differential f_* of the tangential component defines a torsion-free affine connection ∇ on M, the *induced connection*, while α is a D-valued symmetric bilinear form on M, the *affine fundamental form* [17, p.28]. If we imagine M to be equipped a priori with the connection ∇, we say that f is an *affine immersion* with transversal distribution D. In the case when $k = 1$, the distribution D will amount to a transversal vector field ξ on the immersion, and α can be written as

$$\alpha[X,Y] = h[X,Y]\xi,$$

where h is now a symmetric covariant tensor field of second order on M, the *affine metric*. In this case we speak of an *affine hypersurface immersion*.

In affine differential geometry, special attention is dedicated to affine hypersurface immersions into the flat space $\mathbb{R}^{n+1}$. We will consider two special classes of such immersions, the *centro-affine immersions* and the *graph immersions*. In the first case, the transversal vector field is given by the negative position vector of the immersion, $\xi(y) = -f(y)$ [17, Example 2.2, p.37], while in the second case, the transversal vector field equals a constant vector [17, Example 2.4, p.39]. The affine connection induced by a centro-affine hypersurface immersion is projectively flat (a torsion-free affine connection ∇ on a manifold M is said to be *projectively flat* if there exists a flat connection ∇' on M such that the geodesics of ∇ can be obtained from the geodesics of ∇' by a reparametrization [17, Def. 3.3, p.17 and Proposition A1.1, p.236]) with symmetric Ricci tensor equal to $n-1$ times the affine metric [17, Proposition 3.1, p.14 and p.38], while the affine connection induced by graph immersions is flat [17, p.40]. The converse also holds, at least locally [17, Proposition 2.7, p.38 and Proposition 2.8, p.40].

Let the ambient space $\mathbb{R}^{n+1}$ of an affine hypersurface immersion $f : M \to \mathbb{R}^{n+1}$ with transversal vector field ξ be equipped with an invariant volume element $\tilde{\theta}$. Then we can define an induced volume element θ on M by

$$\theta(X_1, \dots, X_n) = \tilde{\theta}(f_*(X_1), \dots, f_*(X_n), \xi).$$

If θ is ∇-parallel, $\nabla\theta = 0$, then ξ is said to be *equiaffine*. The centro-affine immersions and the graph immersions are special cases of affine hypersurface immersions with equiaffine transversal vector field. For affine hypersurface immersions $f : M \to \mathbb{R}^{n+1}$ with equiaffine transversal vector field the induced affine connection ∇ and the affine metric h on M satisfy the *Codazzi equation* [17, Theorem 2.1, p.32]. The Codazzi equation for an affine connection ∇ and a symmetric covariant second order tensor P is given by

$$(\nabla_X P)[Y,Z] = (\nabla_Z P)[Y,X] \tag{1.4}$$

for every triple (X,Y,Z) of vector fields. It follows that the covariant derivative ∇h of the affine metric with respect to the induced affine connection is a totally symmetric 3-form.

Codazzi structures. A *Codazzi structure* on a differentiable manifold M is a pair (∇, g) of a torsion-free affine connection ∇ and a pseudo-Riemmannian metric g such that the covariant derivative ∇g is totally symmetric. If ∇ is flat, then (∇, g) is called a *Hessian structure*, and the manifold M a *Hessian manifold*. Clearly a pair (∇, g) is a Codazzi structure if and only if it satisfies the Codazzi equation (1.4)

[19, p.33], [17, p.21]. The totally symmetric tensor ∇g is called the *cubic form* of the Codazzi structure [17, p.21]. As explained in the previous paragraph, affine hypersurface immersions $f : M \to \mathbb{R}^{n+1}$ with equiaffine transversal vector field carry a natural Codazzi structure (∇, h), composed of the induced affine connection and the affine metric.

If ∇ is an affine connection and P is non-degenerate symmetric covariant second order tensor, then there exists a unique affine connection $\tilde{\nabla}$ such that [17, p.20]

$$(XP)[Y, Z] = P[\nabla_X Y, Z] + P[Y, \tilde{\nabla}_X Z] \tag{1.5}$$

for all vector fields X, Y, Z on M. Here XP denotes the derivative of P in the direction of X. The connection $\tilde{\nabla}$ is called the *conjugate connection* (or *dual connection*) of ∇ relative to P. The conjugate connection has a simple interpretation. Namely, if $X(t), \tilde{Y}(t)$ are vector fields along a curve $\gamma(t)$ on M, such that X is ∇-parallel, and $\tilde{Y}$ is $\tilde{\nabla}$-parallel, then $P[X, \tilde{Y}]$ is constant along the curve [17, Proposition 4.5, p.21].

Let now (∇, g) be a Codazzi structure on M. Then the connection ∇ is torsion-free. The Codazzi equation (1.4) guarantees that the conjugate connection $\tilde{\nabla}$ of ∇ relative to g is also torsion-free [17, Cor.4.3, p.21]. Moreover, $(\tilde{\nabla}, g)$ is also a Codazzi structure, and $\frac{1}{2}(\nabla + \tilde{\nabla})$ is the Levi-Civita connection $\hat{\nabla}$ for g [17, Cor.4.4, p.21], [19, Lemma 2.3]. The Codazzi structure $(\tilde{\nabla}, g)$ is called the *dual Codazzi structure* to (∇, g).

As mentioned in the exposition of affine differential geometry, Codazzi structures arise naturally on affine hypersurface immersions with equiaffine transversal vector field into the flat space $\mathbb{R}^{n+1}$ and have first been considered in this context [17, p.22]. For an introduction to Codazzi structures, see [19, Section 2.5] or [17, Section I.4]. Independently, Codazzi manifolds have been used in statistics, where they are called *statistical manifolds*. An overwiew over this field of application is given in [2].

The subject of this contribution are n-dimensional immersions in $2n$-dimensional para-Kähler manifolds. For each such immersion $f : M \to \mathbf{M}$, the pseudo-Riemannian metric g and the symplectic form ω of the ambient para-Kähler manifold $\mathbf{M}$ define on M a pseudo-Riemannian metric $\hat{g}$ and a closed skew-symmetric 2-form $\hat{\omega}$, respectively. If $\hat{\omega}$ vanishes, then f is a Lagrangian immersion. The study of Lagrangian immersions in para-Kähler manifolds has been initiated by Chen in [5] and pursued in [4]. It is shown in [5, Lemma 3.2, (iii)] that a Lagrangian immersion $f : M \to \mathbf{M}$ in a para-Kähler manifold carries a natural completely symmetric 3-form σ induced by the second fundamental form II_f of M. Note that for an isometric immersion $f : M \to \mathbf{M}$ of pseudo-Riemannian manifolds, the second fundamental form II_f is only well-defined if the metric $\hat{g}$ on M is non-degenerate. It was omitted in [5, Lemma 3.2] to impose this non-degeneracy condition, and this Lemma is hence formally not correct as stated.

In Subsection 2.3 we will show that the non-degeneracy condition on $\hat{g}$ is equivalent to the condition that the distributions $D^{\mathbf{X}}, D^{\mathbf{P}}$ are transversal to the Lagrangian immersion $f : M \to \mathbf{M}$ (Lemma 2.6). For Lagrangian immersions satisfying this condition, the transversal distributions $D^{\mathbf{X}}, D^{\mathbf{P}}$ induce affine connections $\nabla^{\mathbf{X}}, \nabla^{\mathbf{P}}$ on M. One of the main results in this contribution is that the pairs $(\nabla^{\mathbf{X}}, \hat{g})$ and $(\nabla^{\mathbf{P}}, \hat{g})$ are dual Codazzi structures on M (Theorem 2.2). Thus for every non-degenerate Lagrangian immersion $f : M \to \mathbf{M}$ in a para-Kähler manifold, M is naturally a Codazzi manifold. Moreover, we will identify the totally symmetric 3-form defined in [5, Lemma 3.2, (iii)], up to the multiplicative factor 2, as the cubic form of the Codazzi structure $(\nabla^{\mathbf{X}}, \hat{g})$ (Theorem 2.3). Thus there is a close relation between f, considered as an affine immersion with transversal distribution $D^{\mathbf{X}}$ or $D^{\mathbf{P}}$, and f, considered as an isometric immersion of pseudo-Riemannian manifolds.

One can then ask whether every n-dimensional Codazzi manifold M with non-degenerate pseudo-Riemannian metric can be realized as a Lagrangian immersion of a para-Kähler manifold. We will answer this question in the affirmative, constructing a para-Kähler manifold $\mathbf{M}$ and a Lagrangian immersion $f : M \to \mathbf{M}$ such that $(\nabla^{\mathbf{X}}, \hat{g})$, defined as in the previous paragraph, coincides with the given Codazzi structure on M (Corollary 3.1).

Specific choices of the para-Kähler manifold $\mathbf{M}$ will lead to the Codazzi structures $(\nabla^{\mathbf{X}}, \hat{g})$ on the Lagrangian immersions in $\mathbf{M}$ having specific properties. We will consider two such choices, the flat para-Kähler space $\mathbb{E}^{2n}_n$ and a special para-Kähler manifold which we call the *cross-ratio manifold*. The cross-ratio manifold is isomorphic to a member of the one-parametric family of *reduced paracomplex projective spaces*, which were introduced and studied in [12] (Theorem 4.1). We will show that the Codazzi structures on the Lagrangian immersions of the cross-ratio manifold have a projectively flat equiaffine connection with Ricci tensor equal to $(n-1)$ times the metric (Corollary 4.1), and every such Codazzi structure can be locally obtained in this way (Corollary 4.2). The Codazzi structures on the Lagrangian immersions of the flat para-Kähler space are actually Hessian structures (Corollary 5.1), and every Hessian structure can locally be obtained in this way (Corollary 5.2). Note that the Codazzi structures having these properties are exactly the Codazzi structures generated by centro-affine hypersurface immersions and by graph immersions into $\mathbb{R}^{n+1}$, respectively. This suggests an intimate relation between the geometry of Lagrangian immersions in the flat para-Kähler space $\mathbb{E}^{2n}_n$ and that of graph immersions into $\mathbb{R}^{n+1}$, and between the geometry of Lagrangian immersions in the cross-ratio manifold and centro-affine geometry. Indeed, the links between these concepts go far beyond the scope of this contribution, and will be the subject of a companion paper.

Links between affine differential geometry and Lagrangian submanifolds of para-Kähler manifolds have been found, albeit in a very different framework, in [9], [7]. Connections between affine differential geometry and Kähler manifolds have also appeared in the literature [15]. Let σ be a symmetric 3-form and $\hat{g}$ a metric on some manifold M. Conditions on $\sigma, \hat{g}$ have been given in [4, Section 3] such that $\sigma, \hat{g}$ can be realized as the corresponding structures on a Lagrangian immersion $f : M \to \mathbb{E}^{2n}_n$. A proof can be found in [6]. Similar conditions are given if $\mathbb{E}^{2n}_n$ is replaced by another para-Kähler space form (a para-Kähler space form is a homogeneous para-Kähler manifold with constant para-sectional curvature, see, e.g., [1]). Note that the cross-ratio manifold, being isomorphic to a reduced paracomplex projective space, is also a para-Kähler space form.

Our results described above will actually be obtained as special cases of more general analogs. The generalization consists in dropping the condition that the considered n-dimensional immersions $f : M \to \mathbf{M}$ in $2n$-dimensional para-Kähler manifolds be Lagrangian. We will show that for such general half-dimensional immersions the pair $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$, where $\hat{\mathbf{Q}} = \hat{g} + \hat{\omega}$, still satisfies the Codazzi equation (1.4) (Theorem 2.1). However, the pair $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ is no more a Codazzi structure, because $\hat{\mathbf{Q}}$ is no more symmetric. We will call such a pair a *pre-Codazzi structure* (one may call it also *asymmetric Codazzi structure*), and a manifold M carrying a pre-Codazzi structure a *pre-Codazzi manifold* (Definition 2.1). Similarly, we may call a pre-Codazzi structure with flat connection a *pre-Hessian structure* (Definition 5.1). Dual pre-Codazzi structures can be defined by an analog of equation (1.5). The relation between non-degenerate half-dimensional immersions in para-Kähler manifolds and pre-Codazzi manifolds will turn out to be similar to the relation described above between Lagrangian immersions in para-Kähler manifolds and Codazzi manifolds (Theorem 2.1, Theorem 3.1).

This generalization extends also to the results concerning the specific para-Kähler manifolds $\mathbb{E}^{2n}_n$ and the cross-ratio manifold. Namely, every non-degenerate half-dimensional immersion in $\mathbb{E}^{2n}_n$ carries a natural pre-Hessian structure $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ defined as in the previous paragraph (Proposition 5.1), and every pre-Hessian structure can locally be represented in this way (Proposition 5.2). From the duality of the pre-Codazzi structures induced by the distributions $D^{\mathbf{X}}, D^{\mathbf{P}}$ it then follows that the dual of a pre-Hessian structure must also be a pre-Hessian structure (Theorem 5.1). This defines a duality relation on pre-Hessian structures (Definition 5.2).

On every non-degenerate half-dimensional immersion in the cross-ratio manifold, the pre-Codazzi structure $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ is such that the affine connection $\nabla^{\mathbf{X}}$ is projectively flat with Ricci tensor equal to $R_{ij} = n\hat{\mathbf{Q}}_{ij} - \hat{\mathbf{Q}}_{ji}$ (Theorem 4.2), and every projectively flat manifold can be locally obtained in this way (Theorem 4.3). This relation defines a natural pre-Codazzi structure on projectively flat manifolds (Definition 4.3). The duality relation between pre-Codazzi structures on a manifold then induces a duality relation between projectively flat connections on this manifold (Theorem 4.4, Definition 4.4).

Note that there is a conceptual difference between the cases described in the previous two paragraphs. A projectively flat connection determines the second member of its natural pre-Codazzi structure by its Ricci tensor, and thus contains all the information itself. A flat connection, on the contrary, only determines the local isomorphism of the manifold with the flat affine space, and the information content is in the second member of the pre-Hessian structure.

Throughout the paper, the structures we consider are supposed to be smooth. This assumption is introduced to facilitate the exposition and can be appropriately relaxed.

1.1. **Notation.** In Sections 2 and 3 $\mathbf{M}$ will denote a general $2n$-dimensional para-Kähler manifold with pseudo-Riemannian metric g, symplectic form ω and para-complex structure J. The eigendistributions of J with eigenvalues $+1, -1$ will be denoted by $D^{\mathbf{P}}$, $D^{\mathbf{X}}$, respectively. We study immersions $f : M \to \mathbf{M}$ of an n-dimensional manifold M into $\mathbf{M}$. The immersion f induces a pseudo-Riemannian metric $\hat{g}$ and a closed skew-symmetric 2-form $\hat{\omega}$ on M. We also introduce the second order covariant tensor fields $\mathbf{Q} = g + \omega$ and $\hat{\mathbf{Q}} = \hat{g} + \hat{\omega}$ on $\mathbf{M}$ and on M, respectively. If one or both of the distributions $D^{\mathbf{X}}$, $D^{\mathbf{P}}$ are transversal to the immersion f, then f can be viewed as an affine immersion with the corresponding transversal distribution. We denote the corresponding induced affine connections by $\nabla^{\mathbf{X}}, \nabla^{\mathbf{P}}$, respectively, and the corresponding affine fundamental forms by $\alpha^{\mathbf{P}}, \alpha^{\mathbf{X}}$. In Sections 4 and 5 we specialize $\mathbf{M}$ to the cross-ratio manifold and the flat para-Kähler space, respectively.

2. Immersions in para-Kähler manifolds

In this section we study half-dimensional immersions $f : M \to \mathbf{M}$ into para-Kähler manifolds. In Subsection 2.1 we introduce coordinate charts and the concept of the para-Kähler potential on $\mathbf{M}$. In Subsection 2.2 we define and consider pre-Codazzi structures. In Subsection 2.3 we show that if a certain non-degeneracy condition is satisfied, then f defines a pre-Codazzi structure on M. This is the main result of this section. Finally, in Subsection 2.4 we specialize the results of Subsection 2.3 to the case of Lagrangian immersions.

2.1. **The para-Kähler potential.** We will need an explicit description of the para-Kähler structure on $\mathbf{M}$. The para-complex structure J equips $\mathbf{M}$ with a local product structure. Namely, for every $\hat{z} \in \mathbf{M}$, there exists a neighbourhood $U \subset \mathbf{M}$ of $\hat{z}$ and a diffeomorphism $\varphi : U \to U_{\mathbf{X}} \times U_{\mathbf{P}}$ onto the product of simply connected open sets $U_{\mathbf{X}}, U_{\mathbf{P}} \subset \mathbb{R}^n$, with the following property. Let $\varphi^{\mathbf{X}} : U \to U_{\mathbf{X}}$, $\varphi^{\mathbf{P}} : U \to U_{\mathbf{P}}$ be the components of φ, then the distributions $D^{\mathbf{X}}, D^{\mathbf{P}}$ are the kernels of the differentials $\varphi^{\mathbf{X}}_*, \varphi^{\mathbf{P}}_*$, respectively. Introducing coordinates $x^1, \dots, x^n$ on $U_{\mathbf{X}}$ and $p^{n+1}, \dots, p^{2n}$ on $U_{\mathbf{P}}$, we obtain a coordinate chart on the set U. The coordinates of $z = \varphi^{-1}(x,p) \in U$ are then given by $(z^1, \dots, z^{2n}) = (x^1, \dots, x^n, p^{n+1}, \dots, p^{2n})$. We will call such charts *adapted* to the para-complex structure. In any such chart, the matrix of the para-complex structure J is given by

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}. \tag{2.1}$$

The metric g and the symplectic form ω of $\mathbf{M}$ can be recovered from their sum $\mathbf{Q} = g + \omega$ as the symmetric and skew-symmetric part of $\mathbf{Q}$, respectively. Note that the covariant second order tensor $\mathbf{Q}$ encodes the para-complex structure J as well, since the distributions $D^{\mathbf{P}}, D^{\mathbf{X}}$ are the right and left kernel of $\mathbf{Q}$, respectively, as will be shown in the following lemma.

Lemma 2.1. *Let $\mathbf{M}$ be a para-Kähler manifold with metric g, symplectic form ω, and para-complex structure J. Put further $\mathbf{Q} = g + \omega$. Then for every triple of vector fields X, Y, Z on $\mathbf{M}$ such that $JX = -X$ and $JY = Y$ we have $\mathbf{Q}[X,Z] = \mathbf{Q}[Z,Y] = 0$. On the other hand, if for some vector fields X, Y we have $\mathbf{Q}[X,Z] = \mathbf{Q}[Z,Y] = 0$ for all vector fields Z, then $JX = -X$, $JY = Y$.*

Proof. Assume the notations of the lemma. For every X, Y, Z we have by (1.1) that

$$\begin{aligned} \mathbf{Q}[X,Z] &= g[X,Z] + \omega[X,Z] = g[X,Z] + g[JX,Z] = g[X + JX, Z], \\ \mathbf{Q}[Z,Y] &= g[Y,Z] - \omega[Y,Z] = g[Y,Z] - g[JY,Z] = g[Y - JY, Z]. \end{aligned}$$

If now $JX = -X$ and $JY = Y$, then $X + JX = Y - JY = 0$ and by the above we obtain $\mathbf{Q}[X,Z] = \mathbf{Q}[Z,Y] = 0$. On the other hand, let X, Y be such that $\mathbf{Q}[X,Z] = \mathbf{Q}[Z,Y] = 0$ for all Z. Then $g[X + JX, Z] = g[Y - JY, Z] = 0$ for all Z, and hence $X + JX = Y - JY = 0$, because g is non-degenerate. □

In an adapted coordinate chart, the matrices of the tensors $\mathbf{Q}, g, \omega$ can hence be written as

$$\mathbf{Q} = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}, \qquad g = \frac{1}{2}\begin{pmatrix} 0 & Q \\ Q^T & 0 \end{pmatrix}, \qquad \omega = \frac{1}{2}\begin{pmatrix} 0 & Q \\ -Q^T & 0 \end{pmatrix}, \tag{2.2}$$

where Q is an invertible $n \times n$ matrix. The condition $\nabla\omega = 0$ leads to restrictions on the matrix Q as a function of $(x,p) \in U_{\mathbf{X}} \times U_{\mathbf{P}}$. We have the following result.

Proposition 2.1. [8, Section 2.2, Theorem 2 and its proof] *Let* $\mathbf{M}$ *be a para-Kähler manifold and let* $\varphi : U \to U_{\mathbf{X}} \times U_{\mathbf{P}}$ *be an adapted chart on* $\mathbf{M}$*, such that* $U_{\mathbf{X}}, U_{\mathbf{P}} \subset \mathbb{R}^n$ *are simply connected, with coordinates* x, p*, respectively, the para-complex structure on* U *is given by (2.1), and the para-Kähler structure on* U *is given by (2.2). Then there exists a real-valued function* q *on* U *such that*

$$Q(z) = \frac{\partial^2 q}{\partial x \partial p}. \tag{2.3}$$

The function q *is unique up to transformations of the form*

$$q(z) \mapsto q(z) + h(x) + h'(p) \tag{2.4}$$

for arbitrary smooth functions h, h' *on* $U_{\mathbf{X}}$ *and* $U_{\mathbf{P}}$*, respectively.*

Conversely, let $\mathbf{M}$ *be a* $2n$*-dimensional manifold equipped with a covariant second order tensor field* $\mathbf{Q}$ *that can locally be expressed by (2.3), (2.2) for some smooth scalar function* q *such that the matrix* Q *is everywhere invertible. Then (2.2), (2.1) define a para-Kähler structure on* $\mathbf{M}$*.*

The scalar field q is called the *para-Kähler potential* [8, Section 2.2].

We introduce the following index notation. Indices running from 1 to n will be denoted by lowercase Latin letters, indices running from $n+1$ to $2n$ by uppercase Latin letters, and indices running from 1 to $2n$ will be denoted by lower-case Greek letters. The use of the letters will be consistent, e.g., the indices denoted by α will consist of two groups of indices, denoted by a and A, respectively. A vector field X can then be written in an adapted coordinate chart as $X^\alpha = \begin{pmatrix} X^a \\ X^A \end{pmatrix}$. Likewise, we have $z^\alpha = \begin{pmatrix} z^a \\ z^A \end{pmatrix} = \begin{pmatrix} x^a \\ p^A \end{pmatrix}$ for the coordinates on $\mathbf{M}$. The Einstein summation convention will be applied to all three kinds of indices. For instance, the contraction of a 1-form w with a vector field X on $\mathbf{M}$ is given by $w[X] = w_\alpha X^\alpha = w_a X^a + w_A X^A$. The expression $w_a X^A$ is to be understood as the sum $\sum_{k=1}^n w_k X^{k+n}$. For convenience, we will index the rows of the matrix Q from 1 to n and the columns from $n+1$ to $2n$, such that $\mathbf{Q}_{aB} = Q_{aB} = \frac{\partial^2 q}{\partial x^a \partial p^B}$. Let Q^{Ab} denote the coefficients of the inverse matrix Q^{-1}, such that $Q_{aB}Q^{Bc} = \delta_a^c$ and $Q^{Ab}Q_{bC} = \delta_C^A$, δ denoting the Kronecker symbol.

The Christoffel symbols of the Levi-Civita connection ∇ of g are given by [1, eq. (6)]

$$\Gamma_{ab}^c = Q^{Dc}\frac{\partial Q_{aD}}{\partial z^b} = Q^{Dc}\frac{\partial^3 q}{\partial x^a \partial x^b \partial p^D}, \quad \Gamma_{AB}^C = Q^{Cd}\frac{\partial Q_{dA}}{\partial z^B} = Q^{Cd}\frac{\partial^3 q}{\partial p^A \partial p^B \partial x^d}, \tag{2.5}$$

while the other components of the Christoffel symbol vanish [1, Lemma 5.3].

2.2. Pre-Codazzi structures.

In this subsection we define pre-Codazzi structures and investigate some of their basic properties.

Definition 2.1. A *pre-Codazzi structure* on a manifold M is a pair (∇, P) of a torsion-free affine connection ∇ and a covariant second order tensor field P such that for every three vector fields X, Y, Z on M the Codazzi equation (1.4) holds. A manifold M equipped with a pre-Codazzi structure will be called a *pre-Codazzi manifold*.

It is a well-known fact in the theory of Fedosov manifolds that a skew-symmetric 2-form ω which is parallel with respect to some torsion-free affine connection ∇ must be closed [14, §19]. The following lemma can be viewed as a generalization of this result.

Lemma 2.2. *Let M be a pre-Codazzi manifold with pre-Codazzi structure (∇, P). Then the skew-symmetric part ω of P is closed, $d\omega = 0$.*

Proof. Let the Christoffel symbols of ∇ be given by Γ_{ij}^k in some coordinate chart on M with coordinates x^i. We then have

$$\nabla_k P_{ij} = \frac{\partial P_{ij}}{\partial x^k} - P_{lj}\Gamma_{ik}^l - P_{il}\Gamma_{jk}^l.$$

By the Codazzi equation $\nabla_k P_{ij} = \nabla_j P_{ik}$ and the symmetry of the Christoffel symbols in the lower indices we then get

$$\frac{\partial P_{ij}}{\partial x^k} - \frac{\partial P_{ik}}{\partial x^j} = P_{lj}\Gamma_{ik}^l - P_{lk}\Gamma_{ij}^l. \tag{2.6}$$

Taking the cyclic sum on both sides and noting that $\omega_{ij} = \frac{1}{2}(P_{ij} - P_{ji})$, we get again by the symmetry of the Christoffel symbols

$$2\left(\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j}\right) = P_{lj}\Gamma_{ik}^l - P_{lk}\Gamma_{ij}^l + P_{li}\Gamma_{kj}^l - P_{li}\Gamma_{jk}^l + P_{lk}\Gamma_{ji}^l - P_{lj}\Gamma_{ki}^l = 0.$$

But this is exactly the condition of closedness of the form ω. $\square$

The inverse assertion, namely that for every closed skew-symmetric 2-form ω there exists a torsion-free affine connection ∇ such that $\nabla\omega = 0$ [14, §19], has the following weaker pendant.

Proposition 2.2. *Let P be a non-degenerate covariant second order tensor field on a manifold M, such that the skew-symmetric part of P is closed. Then there exists a torsion-free affine connection ∇ on M such that (∇, P) is a pre-Codazzi structure.*

Proof. Assume the conditions of the proposition. Denote the skew-symmetric part of P by ω. Let $U \subset M$ be an open subset carrying a coordinate chart with coordinates x^i. In this chart, denote the left-hand side of (2.6) by L_{ijk}. By definition the object L_{ijk} is skew-symmetric in the indices j, k. Further, we have

$$L_{ijk} + L_{jki} + L_{kij} = 2\left(\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j}\right) = 0$$

for the cyclic sum of L_{ijk}, because ω is closed.

Let T be the linear space of covariant 3rd order tensors T_{ijk} over $\mathbb{R}^n$. Let $L \subset T$ be the linear subspace of tensors which are skew-symmetric in the last two indices and whose cyclic sum is zero. Let $S \subset T$ be the subspace of tensors which are symmetric in the first two indices, and consider the endomorphism $A : T_{ijk} \mapsto T'_{ijk} = -T_{ijk} + T_{ikj}$ of T. It is not hard to check that the image of S under A is exactly the subspace L, while the intersection of $\ker A$ with S consists of the subspace of totally symmetric tensors.

In view of the above, there exists an object T_{ijk} in S such that $L_{ijk} = -T_{ijk} + T_{ikj}$, and this object is determined by L_{ijk} up to a totally symmetric additive term. Since L_{ijk} is a smooth function of x, we can choose T_{ijk} to be a smooth function of x as well. Now since P is non-degenerate, there exists a unique Γ_{ij}^k such that

$T_{ijk} = P_{lk}\Gamma^l_{ij}$, and this Γ^k_{ij} also smoothly varies with x and is symmetric in the lower indices.

Then Γ^k_{ij} satisfies (2.6) and can be considered as the Christoffel symbol of a torsion-free affine connection ∇^U on U satisfying the Codazzi equation $\nabla^U_k P_{ij} = \nabla^U_j P_{ik}$. As in [13, Remark 1.4], the proof is completed by gluing the local connections ∇^U together to a global connection ∇ using a partition of unity. □

The condition that P is non-degenerate is essential. If, for instance, at some point $x \in U$ we have $P = 0$, but $L_{ijk} \neq 0$, then we are not able to find a torsion-free connection ∇^U satisfying the Codazzi equation at this point.

Definition 2.2. Let ∇ be an affine connection and P a non-degenerate covariant second order tensor field on M. We call the unique affine connection $\tilde{\nabla}$ on M such that equation (1.5) is satisfied for all vector fields X, Y, Z on M the *conjugate connection* of ∇ relative to P.

This definition simply extends the notion of conjugate connection to non-symmetric reference tensors P. Let us show that the conjugate connection is indeed well-defined. In index notation equation (1.5) can be written as

$$\frac{\partial P_{jk}}{\partial x^i} = P_{lk}\Gamma^l_{ij} + P_{jl}\tilde{\Gamma}^l_{ik}, \tag{2.7}$$

where $\Gamma^k_{ij}, \tilde{\Gamma}^k_{ij}$ are the Christoffel symbols of $\nabla, \tilde{\nabla}$, respectively. Since P is non-degenerate, this equation can be resolved for $\tilde{\Gamma}^l_{ik}$, which proves that the connection $\tilde{\nabla}$ exists and is unique.

Proposition 2.3. *Let ∇ be a torsion-free affine connection and P a non-degenerate second order tensor field on M, and let $\tilde{\nabla}$ be the conjugate connection of ∇ relative to P. Then $\tilde{\nabla}$ is torsion-free if and only if (∇, P) is a pre-Codazzi structure. In this case $(\tilde{\nabla}, P^T)$ is also a pre-Codazzi structure, where P^T is defined by $P^T[X,Y] = P[Y,X]$.*

Proof. Equation (2.7) can equivalently be written as $\nabla_i P_{jk} = P_{jl}(\tilde{\Gamma}^l_{ik} - \Gamma^l_{ik})$. Alternating i, k yields

$$\nabla_i P_{jk} - \nabla_k P_{ji} = P_{jl}(\tilde{\Gamma}^l_{ik} - \tilde{\Gamma}^l_{ki}).$$

Since P is non-degenerate, $\tilde{\nabla}$ is torsion-free if and only if the right-hand side of this relation vanishes. The left-hand side vanishes, however, if and only if (∇, P) is a pre-Codazzi structure, which proves the first part of the proposition.

In terms of P^T equation (2.7) can be rewritten as

$$\frac{\partial P^T_{kj}}{\partial x^i} = P^T_{lj}\tilde{\Gamma}^l_{ik} + P^T_{kl}\Gamma^l_{ij}.$$

Comparing this with (2.7) shows that ∇ is the conjugate connection of $\tilde{\nabla}$ relative to P^T. Applying the first part of the proposition, we obtain the second part. □

Definition 2.3. Let (∇, P) be a pre-Codazzi structure on M, such that P is non-degenerate. Denote by $\tilde{\nabla}$ the conjugate connection of ∇ relative to P, and define the tensor field P^T by $P^T[X,Y] = P[Y,X]$. We then call $(\tilde{\nabla}, P^T)$ the *dual pre-Codazzi structure* of (∇, P).

2.3. **Half-dimensional immersions.** In this subsection we construct a pre-Codazzi structure on non-degenerate n-dimensional immersions in $2n$-dimensional para-Kähler manifolds. Assume the notations of Subsection 1.1.

Lemma 2.3. *The tensor field $\hat{\mathbf{Q}}$ is non-degenerate if and only if the distributions $D^{\mathbf{X}}, D^{\mathbf{P}}$ are transversal to the immersion f.*

Proof. Pass to an adapted chart on $\mathbf{M}$ and let $y^1, \dots, y^n$ be coordinates on M. Then the tensor $\hat{\mathbf{Q}}$ is given by

$$\hat{\mathbf{Q}}_{ab} = \mathbf{Q}_{\gamma\delta}\frac{\partial z^\gamma}{\partial y^a}\frac{\partial z^\delta}{\partial y^b} = \frac{\partial x^c}{\partial y^a}Q_{cD}\frac{\partial p^D}{\partial y^b}. \tag{2.8}$$

The second equality holds in view of (2.2). Hence $\hat{\mathbf{Q}}$ is non-degenerate if and only if both derivatives $\frac{\partial x}{\partial y}, \frac{\partial p}{\partial y}$ are non-degenerate. But $\frac{\partial x}{\partial y}$ is non-degenerate if and only if $D^{\mathbf{X}}$ is transversal to f, and $\frac{\partial p}{\partial y}$ is non-degenerate if and only if $D^{\mathbf{P}}$ is transversal to f. This completes the proof. □

Suppose now that the distribution $D^{\mathbf{X}}$ is transversal to the immersion f. Consider an adapted coordinate chart $\varphi : U \to U^{\mathbf{X}} \times U^{\mathbf{P}}$ on $\mathbf{M}$, with components $\varphi^{\mathbf{X}} : U \to U_{\mathbf{X}}$, $\varphi^{\mathbf{P}} : U \to U_{\mathbf{P}}$. Let $U_M \subset M$ be an open subset such that $f[U_M] \subset U$. Then the composition $\varphi^{\mathbf{X}} \circ f : U_M \to U^{\mathbf{X}}$ is a local diffeomorphism. By possibly shrinking U_M, we can assume without restriction of generality that $\varphi^{\mathbf{X}} \circ f$ is injective, thereby introducing the coordinates $x^1, \dots, x^n$ on U_M. We shall call such a chart on M an *adapted chart*. The adapted charts form an atlas on M. In a similar manner, we can introduce the coordinates $p^{n+1}, \dots, p^{2n}$ on M if the distribution $D^{\mathbf{P}}$ is transversal to the immersion f.

The immersion f, considered as an affine immersion with transversal distribution $D^{\mathbf{X}}$, induces an affine connection $\nabla^{\mathbf{X}}$ on M. Let us compute this connection explicitly.

Lemma 2.4. *Let $f : M \to \mathbf{M}$ be an n-dimensional immersion into a $2n$-dimensional para-Kähler manifold. Suppose that the distribution $D^{\mathbf{X}}$ is transversal to the immersion f. Let $\nabla^{\mathbf{X}}$ be the affine connection induced on M by the immersion f with transversal distribution $D^{\mathbf{X}}$. Then in an adapted chart on M, the Christoffel symbols Γ^c_{ab} of the connection $\nabla^{\mathbf{X}}$ are given by the Christoffel symbols (2.5) of the Levi-Civita connection ∇ of g.*

Proof. Assume the conditions of the lemma and denote the Christoffel symbols of ∇ by $\Gamma^\gamma_{\alpha\beta}$. Let X, Y be vector fields on M and $\tilde{X}, \tilde{Y}$ (locally) extensions of their images $f_*(X), f_*(Y)$ under the differential of f. The vector field $\tilde{Z} = \nabla_{\tilde{X}}\tilde{Y}$ is given by

$$\tilde{Z}^\alpha = \frac{\partial \tilde{Y}^\alpha}{\partial z^\beta}\tilde{X}^\beta + \tilde{Y}^\delta\Gamma^\alpha_{\delta\beta}\tilde{X}^\beta.$$

Since the coordinate chart on M is adapted, we have $X^a = \tilde{X}^a$ and $Y^a = \tilde{Y}^a$. Taking into account the block-diagonal structure of the Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$, we get

$$\tilde{Z}^a = \frac{\partial \tilde{Y}^a}{\partial z^\beta}\tilde{X}^\beta + \tilde{Y}^\delta\Gamma^a_{\delta\beta}\tilde{X}^\beta = \frac{\partial Y^a}{\partial x^b}X^b + Y^d\Gamma^a_{db}X^b.$$

If we decompose $\tilde{Z}$ on $f[M]$ into a tangential component $f_*(Z)$ and a component α lying in the transversal distribution $D^{\mathbf{X}}$, then $\alpha^a = 0$ and $Z^a = \tilde{Z}^a$ again because the chart on M is adapted. Hence

$$Z^a = (\nabla^{\mathbf{X}}_X Y)^a = \frac{\partial Y^a}{\partial x^b} X^b + Y^d \Gamma^a_{db} X^b,$$

which shows the assertion of the lemma. □

Likewise, suppose that the distribution $D^{\mathbf{P}}$ is transversal to f. Then f, considered as an affine immersion with transversal distribution $D^{\mathbf{P}}$, induces an affine connection $\nabla^{\mathbf{P}}$ on M. In the same way as above it then follows that in the coordinates $p^{n+1}, \dots, p^{2n}$ on M, the Christoffel symbols Γ^C_{AB} of $\nabla^{\mathbf{P}}$ are given by (2.5).

By (1.3) we have

$$\nabla_{\tilde{X}} \tilde{Y} = f_*(\nabla^{\mathbf{X}}_X Y) + \alpha^{\mathbf{X}}[X,Y] = f_*(\nabla^{\mathbf{P}}_X Y) + \alpha^{\mathbf{P}}[X,Y] \tag{2.9}$$

for every two vector fields X, Y on M, where $\tilde{X}, \tilde{Y}$ are (locally) extensions of their images $f_*(X), f_*(Y)$ under the differential of f. The affine fundamental forms $\alpha^{\mathbf{X}}, \alpha^{\mathbf{P}}$ are symmetric and with values in $D^{\mathbf{X}}, D^{\mathbf{P}}$, respectively.

We are now in a position to prove the main result of this section.

Theorem 2.1. *Let $f : M \to \mathbf{M}$ be an n-dimensional immersion into a $2n$-dimensional para-Kähler manifold with metric g and symplectic form ω. Let further $\hat{\mathbf{Q}} = f^*(\mathbf{Q})$ be the pullback of the tensor $\mathbf{Q} = g + \omega$ on M. Then the following assertions hold.*

i) Suppose that the distribution $D^{\mathbf{X}}$ is transversal to the immersion f, and let $\nabla^{\mathbf{X}}$ be the affine connection induced on M by the immersion f with transversal distribution $D^{\mathbf{X}}$. Then $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ is a pre-Codazzi structure on M.

ii) Suppose that the distribution $D^{\mathbf{P}}$ is transversal to the immersion f, and let $\nabla^{\mathbf{P}}$ be the affine connection induced by the immersion f with transversal distribution $D^{\mathbf{P}}$. Then $(\nabla^{\mathbf{P}}, \hat{\mathbf{Q}}^T)$ is a pre-Codazzi structure on M, where $\hat{\mathbf{Q}}^T$ is defined by $\hat{\mathbf{Q}}^T[X,Y] = \hat{\mathbf{Q}}[Y,X]$.

iii) Assume that the conditions of both parts i) and ii) of the theorem are satisfied. Then the pre-Codazzi structures $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$, $(\nabla^{\mathbf{P}}, \hat{\mathbf{Q}}^T)$ are dual to each other.

Proof. Assume the conditions of i) and pass to an adapted chart on M. Locally the immersion f can be expressed by a vector-valued function $p = p(x)$, such that $z = f(x) = \begin{pmatrix} x \\ p(x) \end{pmatrix}$. By (2.8), $\hat{\mathbf{Q}}$ is given by $\hat{\mathbf{Q}}_{ab} = Q_{aD} \frac{\partial p^D}{\partial x^b}$. By Lemma 2.4 we then get

$$\begin{aligned}
\nabla^{\mathbf{X}}_c \hat{\mathbf{Q}}_{ab} &= \frac{\partial \hat{\mathbf{Q}}_{ab}}{\partial x^c} - \hat{\mathbf{Q}}_{ad}\Gamma^d_{cb} - \hat{\mathbf{Q}}_{db}\Gamma^d_{ca} \\
&= \left(\frac{\partial Q_{aD}}{\partial x^c} + \frac{\partial Q_{aD}}{\partial p^E}\frac{\partial p^E}{\partial x^c} \right) \frac{\partial p^D}{\partial x^b} + Q_{aD} \frac{\partial^2 p^D}{\partial x^b \partial x^c} \\
&\quad - Q_{aE} \frac{\partial p^E}{\partial x^d} Q^{Fd} \frac{\partial Q_{cF}}{\partial x^b} - Q_{dE} \frac{\partial p^E}{\partial x^b} Q^{Fd} \frac{\partial Q_{cF}}{\partial x^a} \\
&= \frac{\partial^3 q}{\partial x^a \partial p^D \partial p^E} \frac{\partial p^D}{\partial x^b} \frac{\partial p^E}{\partial x^c} + Q_{aD} \frac{\partial^2 p^D}{\partial x^b \partial x^c} - Q_{aE} \frac{\partial p^E}{\partial x^d} Q^{Fd} \frac{\partial^3 q}{\partial x^b \partial x^c \partial p^F}.
\end{aligned}$$

This expression is symmetric in the indices b, c, which shows that $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ is a pre-Codazzi structure. This proves the first part of the theorem.

In order to prove the second part, note that the isomorphism $J \mapsto -J$, $g \mapsto g$, $\omega \mapsto -\omega$ defines another para-Kähler structure on $\mathbf{M}$. Under this isomorphism, the distributions $D^{\mathbf{X}}$, $D^{\mathbf{P}}$ are exchanged and $\hat{\mathbf{Q}}$ is taken to $\hat{\mathbf{Q}}^T$. Hence the second part of the theorem follows from the first part, applied to the transformed para-Kähler structure.

Now let us prove the third part. Assume that both distributions $D^{\mathbf{X}}, D^{\mathbf{P}}$ are transversal to the immersion f. By Lemma 2.3, $\hat{\mathbf{Q}}$ is non-degenerate and hence the conjugate connection of $\nabla^{\mathbf{X}}$ relative to $\hat{\mathbf{Q}}$ is well-defined. Let us show that this is the connection $\nabla^{\mathbf{P}}$. Let X, Y, Z be vector fields on M, and let $\tilde{X}, \tilde{Y}, \tilde{Z}$ be local extensions of their images under f_* to a neighbourhood of the immersion f in $\mathbf{M}$. Then we have

$$\begin{aligned} X(\hat{\mathbf{Q}}[Y, Z]) &= \tilde{X}(\mathbf{Q}[\tilde{Y}, \tilde{Z}]) = \nabla_{\tilde{X}}\left(\mathbf{Q}[\tilde{Y}, \tilde{Z}]\right) = \mathbf{Q}[\nabla_{\tilde{X}}\tilde{Y}, \tilde{Z}] + \mathbf{Q}[\tilde{Y}, \nabla_{\tilde{X}}\tilde{Z}] \\ &= \mathbf{Q}[f_*(\nabla^{\mathbf{X}}_X Y) + \alpha^{\mathbf{X}}[X, Y], f_*(Z)] + \mathbf{Q}[f_*(Y), f_*(\nabla^{\mathbf{P}}_X Z) + \alpha^{\mathbf{P}}[X, Z]] \\ &= \mathbf{Q}[f_*(\nabla^{\mathbf{X}}_X Y), f_*(Z)] + \mathbf{Q}[f_*(Y), f_*(\nabla^{\mathbf{P}}_X Z)] = \hat{\mathbf{Q}}[\nabla^{\mathbf{X}}_X Y, Z] + \hat{\mathbf{Q}}[Y, \nabla^{\mathbf{P}}_X Z]. \end{aligned}$$

Here the third equality holds because $\nabla \mathbf{Q} = 0$, for the fourth equality we used (2.9), and the fifth equality holds because $D^{\mathbf{P}}, D^{\mathbf{X}}$ are the right and left kernels of $\hat{\mathbf{Q}}$, respectively. We thus indeed recover formula (1.5). This completes the proof of the third part. □

Finally, we shall give an explicit local expression of the closed form $\hat{\omega}$ as the differential of a 1-form.

Lemma 2.5. *Assume the conditions of i), Theorem 2.1 and pass to an adapted chart on M, such that the para-Kähler structure is given by a para-Kähler potential $q(x, p)$. Then the 1-form w on M given by $w_a(y) = -\left.\frac{\partial q}{\partial x^a}\right|_{(x,p)=f(y)}$ is a potential of $\hat{\omega}$, $\hat{\omega} = dw$.*

Proof. Assume the conditions of the lemma. The form $\hat{\omega}$ is the skew-symmetric part of the tensor $\hat{\mathbf{Q}}_{ab} = Q_{aC}\frac{\partial p^C}{\partial x^b}$, hence

$$\begin{aligned} \hat{\omega}_{ab} &= \frac{1}{2}\left(\frac{\partial^2 q}{\partial x^a \partial x^b} + \frac{\partial^2 q}{\partial x^a \partial p^C}\frac{\partial p^C}{\partial x^b}\right) - \frac{1}{2}\left(\frac{\partial^2 q}{\partial x^b \partial x^a} + \frac{\partial^2 q}{\partial x^b \partial p^C}\frac{\partial p^C}{\partial x^a}\right) \\ &= \frac{1}{2}\left(\frac{d}{dx^b}\frac{\partial q}{\partial x^a} - \frac{d}{dx^a}\frac{\partial q}{\partial x^b}\right), \end{aligned}$$

where $\frac{d}{dx}$ denotes the gradient on M. The assertion of the lemma now becomes evident. □

2.4. **Lagrangian immersions.** An n-dimensional immersion $f : M \to \mathbf{M}$ into a $2n$-dimensional para-Kähler manifold is Lagrangian if and only if the pullback $\hat{\omega}$ on M of the symplectic form ω vanishes, or equivalently, if the tensor field $\hat{\mathbf{Q}}$ on M is symmetric. In this case $\hat{\mathbf{Q}}$ equals the metric $\hat{g}$ on M, and Lemma 2.3 and Theorem 2.1 specialize to the following results.

Lemma 2.6. *Let $f : M \to \mathbf{M}$ be a Lagrangian immersion into a para-Kähler manifold with metric g, and let $\hat{g}$ be the induced metric on M. Then $\hat{g}$ is non-degenerate if and only if the distributions $D^{\mathbf{P}}, D^{\mathbf{X}}$ are transversal to the immersion f.*

Theorem 2.2. *Let $f : M \to \mathbf{M}$ be a Lagrangian immersion into a para-Kähler manifold with metric g, and let $\hat{g}$ be the induced metric on M. Then the following assertions hold.*

i) Suppose that the distribution $D^{\mathbf{X}}$ is transversal to the immersion f, and let $\nabla^{\mathbf{X}}$ be the affine connection induced on M by the immersion f with transversal distribution $D^{\mathbf{X}}$. Then $(\nabla^{\mathbf{X}}, \hat{g})$ is a Codazzi structure on M.

ii) Suppose that the distribution $D^{\mathbf{P}}$ is transversal to the immersion f, and let $\nabla^{\mathbf{P}}$ be the affine connection induced by the immersion f with transversal distribution $D^{\mathbf{P}}$. Then $(\nabla^{\mathbf{P}}, \hat{g})$ is a Codazzi structure on M.

iii) Assume that the conditions of both parts i) and ii) of the theorem are satisfied. Then the Codazzi structures $(\nabla^{\mathbf{X}}, \hat{g})$, $(\nabla^{\mathbf{P}}, \hat{g})$ are dual to each other.

Corollary 2.1. *Let $f : M \to \mathbf{M}$ be a Lagrangian immersion into a para-Kähler manifold with metric g. Suppose that the induced metric $\hat{g}$ on M is non-degenerate. Let $\nabla^{\mathbf{X}}, \nabla^{\mathbf{P}}$ be the affine connections induced by the affine immersion f with transversal distributions $D^{\mathbf{X}}, D^{\mathbf{P}}$, respectively, and let $\hat{\nabla}$ be the Levi-Civita connection of $\hat{g}$. Then $\hat{\nabla} = \frac{1}{2}\left(\nabla^{\mathbf{X}} + \nabla^{\mathbf{P}}\right)$.*

Proof. The Corollary is an immediate consequence of iii), Theorem 2.2 and of [17, Cor.4.4, p.21] or [19, Lemma 2.3]. □

Finally, we will establish a connection between the Codazzi structures in iii), Theorem 2.2 and the second fundamental form II_f induced on M by a non-degenerate Lagrangian immersion $f : M \to \mathbf{M}$ into a para-Kähler manifold. Recall the definition of the second fundamental form. For vector fields X, Y on M, let $\tilde{X}, \tilde{Y}$ be (locally) extensions of the images $f_*(X)$, $f_*(Y)$ to a neighbourhood of the immersion in $\mathbf{M}$. Then the covariant derivative $\nabla_{\tilde{X}}\tilde{Y}$ decomposes on the immersion as

$$\nabla_{\tilde{X}}\tilde{Y} = f_*(\hat{\nabla}_X Y) + \mathrm{II}_f[X, Y]. \tag{2.10}$$

Here $f_*(\hat{\nabla}_X Y)$ is the tangent, and $\mathrm{II}_f[X, Y]$ is the normal component of $\nabla_{\tilde{X}}\tilde{Y}$. Thus if f is considered as an affine immersion with transversal distribution equal to the normal bundle, then the Levi-Civita connection $\hat{\nabla}$ on M will be the induced affine connection, while the second fundamental form II_f will be the affine fundamental form. Equating the arithmetic mean of the second and third expression in (2.9) with the right-hand side of (2.10) and taking into account that $\hat{\nabla} = \frac{1}{2}\left(\nabla^{\mathbf{X}} + \nabla^{\mathbf{P}}\right)$, we obtain

$$\mathrm{II}_f[X, Y] = \frac{1}{2}\left(\alpha^{\mathbf{X}}[X, Y] + \alpha^{\mathbf{P}}[X, Y]\right). \tag{2.11}$$

In [5, Lemma 3.2, (iii)] Chen defined the totally symmetric 3-form

$$\sigma[X, Y, Z] = g[\mathrm{II}_f[X, Y], Jf_*(Z)] = -\omega[\mathrm{II}_f[X, Y], f_*(Z)] \tag{2.12}$$

on M. The following result relates this 3-form to the cubic form $C = \nabla^{\mathbf{X}}\hat{g}$ of the Codazzi structure $(\nabla^{\mathbf{X}}, \hat{g})$.

Theorem 2.3. *Let $f : M \to \mathbf{M}$ be a Lagrangian immersion into a para-Kähler manifold with metric g. Suppose that the induced metric $\hat{g}$ on M is non-degenerate. Then the 3-form σ defined by (2.12) and the cubic form $C = \nabla^{\mathbf{X}}\hat{g}$ of the Codazzi structure $(\nabla^{\mathbf{X}}, \hat{g})$ from i), Theorem 2.2 are related by $C = 2\sigma$.*

Proof. Assume above notations. Denote the type (1,2) difference tensor $\nabla^{\mathbf{X}} - \hat{\nabla}$ by Δ. For vector fields X, Y, $\Delta[X, Y]$ will then be another vector field on M. For vector fields X, Y, Z on M, we have

$$\begin{aligned}\sigma[X,Y,Z] &= -g[J\mathrm{II}_f[X,Y], f_*(Z)] = -\frac{1}{2}g[J\alpha^{\mathbf{X}}[X,Y] + J\alpha^{\mathbf{P}}[X,Y], f_*(Z)] \\ &= -\frac{1}{2}g[-\alpha^{\mathbf{X}}[X,Y] + \alpha^{\mathbf{P}}[X,Y], f_*(Z)] = -\frac{1}{2}g[f_*(\nabla^{\mathbf{X}}_X Y) - f_*(\nabla^{\mathbf{P}}_X Y), f_*(Z)] \\ &= -\frac{1}{2}\hat{g}[\nabla^{\mathbf{X}}_X Y - \nabla^{\mathbf{P}}_X Y, Z] = -\hat{g}[\nabla^{\mathbf{X}}_X Y - \hat{\nabla}_X Y, Z] = -\hat{g}[\Delta[X,Y], Z].\end{aligned}$$

Here in the second equality we used (2.11), in the third equality we used that $\alpha^{\mathbf{X}}, \alpha^{\mathbf{P}}$ have values in $D^{\mathbf{X}}, D^{\mathbf{P}}$, respectively, in the fourth equality we used the second equality in (2.9), and in the sixth equality we used Corollary 2.1. Since σ is totally symmetric, we also have $\hat{g}[\Delta[X,Y], Z] = \hat{g}[\Delta[X,Z], Y]$.

On the other hand, we have $C = \nabla^{\mathbf{X}}\hat{g} = (\nabla^{\mathbf{X}} - \hat{\nabla})\hat{g}$, and hence

$$C[X,Y,Z] = (\nabla^{\mathbf{X}}_X\hat{g} - \hat{\nabla}_X\hat{g})[Y,Z] = -\hat{g}[\Delta[X,Y], Z] - \hat{g}[\Delta[X,Z], Y].$$

The assertion of the theorem now easily follows. ☐

3. Representation of pre-Codazzi structures

In this section we show that every pre-Codazzi structure $(\bar{\nabla}, P)$ on a manifold M with non-degenerated tensor P can be generated by a half-dimensional embedding of M into some para-Kähler manifold $\mathbf{M}$. We shall construct $\mathbf{M}$ as a subset of the product $M \times M$, such that the embedding $f : M \to \mathbf{M}$ is the diagonal map. We suppose that the manifold M is second-countable, which is often a standard assumption.

Denote the symmetric and skew-symmetric part of P by ρ and ψ, respectively. For every subset $U \subset M$, denote the set $\{(y,y) \mid y \in U\} \subset M \times M$ by Δ_U, the product $U \times U$ by U^2, and the diagonal map $y \mapsto (y,y)$ on U by δ_U. Then $\delta_U : U \to U^2$ is an embedding which maps U diffeomorphically to Δ_U. Let $U \subset M$ be an open set carrying a coordinate chart with coordinates y^a. Then U^2 carries a chart on $M \times M$. We denote the coordinates on the first copy of U by x^a, and the coordinates on the second copy by p^A. Let us further introduce coordinates $u^a = \frac{x^a + p^A}{2}$, $v^a = \frac{x^a - p^A}{2}$ on U^2. Then Δ_U is given by the relation $v = 0$ and is parameterized by the coordinates u, and δ_U is given by the relation $u = \delta_U(y) = y$.

Let $\mathbf{U}$ be an open cover of M, such that for every finite subset $\Sigma \subset \mathbf{U}$, the intersection $\bigcap_{U \in \Sigma} U$ is either empty or simply connected, and every $U \in \mathbf{U}$ carries a coordinate chart. Such a cover exists by virtue of Lemma A.2. By Lemma 2.2, we have $d\psi = 0$ on M. Since every $U \in \mathbf{U}$ is simply connected, the restriction on U of the 2-form ψ is exact and we find a smooth 1-form w^U on U such that $dw^U = \psi|_U$, or in index notation

$$\psi_{ab} = \frac{1}{2}\left(\frac{\partial w^U_b}{\partial u^a} - \frac{\partial w^U_a}{\partial u^b}\right). \tag{3.1}$$

For every $U, U' \in \mathbf{U}$ such that $U \cap U' \neq \emptyset$, we have $dw^U|_{U\cap U'} = dw^{U'}|_{U\cap U'} = \psi|_{U\cap U'}$, which implies that the difference $w^U - w^{U'}$ is closed on $U \cap U'$. Since $U \cap U'$ is simply connected, $w^U - w^{U'}$ is exact, and there exists a scalar function $h^{U,U'} : U \cap U' \to \mathbb{R}$ such that $w^{U'} - w^U = dh^{U',U}$ on $U \cap U'$. Note that for every $U, U', U'' \in \mathbf{U}$ we have $dh^{U',U} - dh^{U'',U} = dh^{U',U''}$ on the simply connected intersection $U \cap U' \cap U''$, and hence

$$h^{U',U} - h^{U'',U} = h^{U',U''} + c_{U,U',U''}, \tag{3.2}$$

where $c_{U,U',U''}$ is a constant.

Let us now construct a local para-Kähler potential q_U on a subset of U^2 for every $U \in \mathbf{U}$. Fix a set $U \in \mathbf{U}$. By virtue of the diffeomorphism $\delta_U : U \to \Delta_U$, the quantities P, w^U, ρ and the Christoffel symbols $\bar\Gamma^c_{ab}$ of the connection $\bar\nabla$ on U can be considered as objects defined on Δ_U and given by functions of u. Using these functions, define on U^2 the scalar

$$q'_U(u,v) = -2w^U_a v^a - 2\rho_{ab}v^a v^b + \frac{2}{3}\left(\frac{\partial \rho_{ac}}{\partial u^b} - 2P_{db}\bar\Gamma^d_{ac} - \frac{1}{2}\frac{\partial^2 w^U_b}{\partial u^a \partial u^c}\right)v^a v^b v^c. \tag{3.3}$$

Lemma 3.1. *Assume above conditions and notations. Let a para-complex structure J on U^2 be given by the natural product structure, such that*

$$J\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial p}\right) = \left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial p}\right). \tag{3.4}$$

Then the para-Kähler potential $q_U(x,p) = q'_U\left(\frac{x+p}{2}, \frac{x-p}{2}\right)$, where q'_U is given by (3.3), defines a para-Kähler structure with para-complex structure J on a neighbourhood W_U of Δ_U in U^2, and the pre-Codazzi structure $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ generated on U by the embedding δ_U as in i), Theorem 2.1 coincides with the pre-Codazzi structure $(\bar\nabla, P)$.

Proof. Clearly the embedding δ_U is non-degenerate, both eigendistributions $D^{\mathbf{P}}, D^{\mathbf{X}}$ of J being transversal to Δ_U. Let us compute the matrix $Q = \frac{\partial^2 q_U}{\partial x \partial p}$. We have

$$Q = \left(\frac{\partial(u,v)}{\partial x}\right)^T \frac{\partial^2 q'_U}{\partial(u,v)^2}\frac{\partial(u,v)}{\partial p} = \frac{1}{4}\left(\frac{\partial^2 q'_U}{\partial u^2} + \frac{\partial^2 q'_U}{\partial v \partial u} - \frac{\partial^2 q'_U}{\partial u \partial v} - \frac{\partial^2 q'_U}{\partial v^2}\right).$$

Inserting the value (3.3) for the function $q'_U(u,v)$ and keeping only terms up to first order in v, we get

$$\begin{aligned} Q_{eF} \;=\; & -\frac{1}{2}\frac{\partial w^U_e}{\partial u^f} + \frac{1}{2}\frac{\partial w^U_f}{\partial u^e} + \rho_{ef} + \frac{1}{3}\Bigg(-4\frac{\partial \rho_{ae}}{\partial u^f} - \frac{\partial \rho_{ef}}{\partial u^a} + 2\frac{\partial \rho_{af}}{\partial u^e} + 2P_{de}\bar\Gamma^d_{af} + \\ & + 2P_{da}\bar\Gamma^d_{ef} + 2P_{df}\bar\Gamma^d_{ae} + \frac{1}{2}\frac{\partial^2 w^U_e}{\partial u^a \partial u^f} + \frac{1}{2}\frac{\partial^2 w^U_f}{\partial u^a \partial u^e} - \frac{\partial^2 w^U_a}{\partial u^e \partial u^f}\Bigg)v^a + O(||v||^2). \end{aligned} \tag{3.5}$$

But by (3.1) we have $-\frac{1}{2}\frac{\partial w^U_e}{\partial u^f} + \frac{1}{2}\frac{\partial w^U_f}{\partial u^e} + \rho_{ef} = P_{ef}$. By virtue of (2.8) and the fact that $\frac{\partial x}{\partial y} = \frac{\partial p}{\partial y} = I_n$ we then get $\hat{\mathbf{Q}} = Q = P$ on Δ_U. In particular, Q is non-degenerate in a neighbourhood W_U of the diagonal submanifold, because P is non-degenerate. On W_U the potential q_U then indeed defines a para-Kähler structure.

Let us now compute the derivative

$$\begin{aligned}
\frac{\partial Q_{eF}}{\partial x^a} &= \frac{1}{2}\left(\frac{\partial Q_{eF}}{\partial u^a} + \frac{\partial Q_{eF}}{\partial v^a}\right) \\
&= \frac{1}{3}\left(\frac{\partial \rho_{ef}}{\partial u^a} - 2\frac{\partial \rho_{ae}}{\partial u^f} + \frac{\partial \rho_{af}}{\partial u^e} + P_{de}\bar\Gamma^d_{af} + P_{da}\bar\Gamma^d_{ef} + P_{df}\bar\Gamma^d_{ae} - \frac{1}{2}\frac{\partial^2 w^U_e}{\partial u^a \partial u^f} + \right. \\
&\quad \left. + \frac{\partial^2 w^U_f}{\partial u^a \partial u^e} - \frac{1}{2}\frac{\partial^2 w^U_a}{\partial u^e \partial u^f}\right) + O(||v||) \qquad (3.6) \\
&= \frac{1}{3}\left(\frac{\partial P_{ef}}{\partial u^a} - \frac{\partial P_{ae}}{\partial u^f} - \frac{\partial P_{ea}}{\partial u^f} + \frac{\partial P_{af}}{\partial u^e} + P_{de}\bar\Gamma^d_{af} + P_{da}\bar\Gamma^d_{ef} + P_{df}\bar\Gamma^d_{ae}\right) + O(||v||) \\
&= P_{df}\bar\Gamma^d_{ae} + O(||v||).
\end{aligned}$$

Here the last relation is due to the Codazzi equation $\bar\nabla_k P_{ij} = \bar\nabla_j P_{ik}$ (compare (2.6) for the index notation). By virtue of $Q = P$ it follows that on Δ_U we have $\bar\Gamma^d_{ae} = Q^{Fd}\frac{\partial Q_{eF}}{\partial x^a}$. By Lemma 2.4 the Christoffel symbols of $\bar\nabla$ hence coincide with those of the connection $\nabla^{\mathbf{X}}$ on U, which completes the proof. □

Note also that by virtue of (3.3) on Δ_U we have $q_U = 0$ and

$$\frac{\partial q_U}{\partial x} = \frac{\partial q'_U}{\partial(u,v)}\frac{\partial(u,v)}{\partial x} = \frac{1}{2}\left(\frac{\partial q'_U}{\partial u} + \frac{\partial q'_U}{\partial v}\right) = -w^U. \qquad (3.7)$$

By virtue of (3.5),(3.6) we also have for all $y \in U$ that

$$\left.\frac{\partial^2 q_U}{\partial x \partial p}\right|_{(x,p)=(y,y)} = P(y), \qquad \left.\frac{\partial^3 q_U}{\partial x^a \partial x^b \partial p^C}\right|_{(x,p)=(y,y)} = P_{dc}(y)\bar\Gamma^d_{ab}(y). \qquad (3.8)$$

It rests to glue the different local para-Kähler structures together. As in the proof of Proposition 2.2, we will use a partition of unity to combine the local para-Kähler potentials. There is, however, a technical difficulty to overcome. The function (3.3) depends on the potential w^U of the exact 2-form ψ, and there is a freedom of choice of this 1-form w^U. If there is a mismatch in the choice of w^U for functions (3.3) in overlapping sets $U, U' \in \mathbf{U}$, then the procedure of combining the local para-Kähler potentials will not preserve the derivatives of (3.3) on Δ_M, and the resulting para-Kähler structure will not reproduce the given pre-Codazzi structure $(\bar\nabla, P)$. However, there might not exist a potential of ψ that is defined globally on M. This problem requires a somewhat more complicated construction.

Let $\mathbf{V}$ be an open strong star refinement of $\mathbf{U}$. Such a refinement exists by Corollary A.1. For every $V \in \mathbf{V}$, choose $U_V \in \mathbf{U}$ such that $\bigcup_{W\in\mathbf{V}:W\cap V\neq\emptyset} W \subset U_V$. For every $U \in \mathbf{U}$ and $V \in \mathbf{V}$ such that $V \subset U$, define a function q^V_U on $(U_V \cap U)^2$ by

$$q^V_U(x,p) = q_{U_V}(x,p) - h^{U,U_V}(x) + h^{U,U_V}(p). \qquad (3.9)$$

We then have for all $y \in U_V \cap U$ that $q^V_U(y,y) = q_{U_V}(y,y) = 0$ and by virtue of (3.7)

$$\left.\frac{\partial q^V_U}{\partial x}\right|_{(x,p)=(y,y)} = \left.\frac{\partial q_{U_V}}{\partial x}\right|_{(x,p)=(y,y)} - \frac{\partial h^{U,U_V}}{\partial y} = -w^{U_V}(y) - (w^U(y) - w^{U_V}(y)) = -w^U(y). \qquad (3.10)$$

By definition of q_U^V, the mixed derivatives of q_U^V and q_{U_V} coincide on $(U_V \cap U)^2$, and from (3.8) we obtain

$$\left.\frac{\partial^2 q_U^V}{\partial x \partial p}\right|_{(x,p)=(y,y)} = P(y), \qquad \left.\frac{\partial^3 q_U^V}{\partial x^a \partial x^b \partial p^C}\right|_{(x,p)=(y,y)} = P_{dc}(y)\bar{\Gamma}^d_{ab}(y) \tag{3.11}$$

for all $y \in U_V \cap U$.

We now construct a global para-Kähler structure in a neighbourhood $\mathbf{M}$ of $\Delta_M \subset M \times M$. Set $\mathbf{M}' = \bigcup_{V\in\mathbf{V}} V^2$. Then $\mathbf{M}'$ is a neighbourhood of Δ_M in $M \times M$. Consider a smooth partition of unity $\{\mu_V : \mathbf{M}' \to [0,1] \,|\, V \in \mathbf{V}\}$, subordinate to the cover $\{V^2 \,|\, V \in \mathbf{V}\}$ of $\mathbf{M}'$.

Fix a set $W \in \mathbf{V}$ and define a scalar function q^W on W^2 by

$$q^W = \sum_{V\in\mathbf{V}} \mu_V q^V_{U_W}. \tag{3.12}$$

Let us first show that q^W is well-defined. Let $V \in \mathbf{V}$ be such that $\mu_V(z) > 0$ for some $z \in W^2$. Then $z \in V^2$, hence $V \cap W \neq \emptyset$ and therefore $V \subset U_W$, $W \subset U_V$. It follows that the function $q^V_{U_W}$ is defined on $(U_V \cap U_W)^2$, and hence also on W^2. On the other hand, if $V \in \mathbf{V}$ is such that $\mu_V \equiv 0$ on W^2, then we set the contribution of the product $\mu_V q^V_{U_W}$ in the sum (3.12) equal to zero. Hence q^W is well-defined.

We shall now investigate how q^W at a given point $z \in \mathbf{M}'$ depends on W. Let $W, W' \in \mathbf{V}$. On $(W \cap W')^2$ we then have for $z = (x,p)$

$$\begin{aligned} q^W(z) - q^{W'}(z) &= \sum_{V\in\mathbf{V}} \mu_V(z)\left(q^V_{U_W}(z) - q^V_{U_{W'}}(z)\right) \\ &= \sum_{V\in\mathbf{V}} \mu_V(z)\left(-h^{U_W,U_V}(x) + h^{U_W,U_V}(p) + h^{U_{W'},U_V}(x) - h^{U_{W'},U_V}(p)\right) \\ &= \sum_{V\in\mathbf{V}} \mu_V(z)\left(-h^{U_W,U_{W'}}(x) - c_{U_V,U_W,U_{W'}} + h^{U_W,U_{W'}}(p) + c_{U_V,U_W,U_{W'}}\right) \\ &= -h^{U_W,U_{W'}}(x) + h^{U_W,U_{W'}}(p). \end{aligned}$$

Here for the third relation we used (3.2), and the sums are efficiently running over those $V \in \mathbf{V}$ which satisfy $\mu_V \not\equiv 0$ on $(W \cap W')^2$. For these V we have $V \cap W \cap W' \neq \emptyset$ and hence $W, W' \subset U_V \cap U_W \cap U_{W'}$. It follows that the scalar functions q^W are related by a transformation of type (2.4) for different W, and hence by virtue of (2.3),(2.2) define a global tensor field $\mathbf{Q}$ on $\mathbf{M}'$.

We now show that this tensor field defines the sought para-Kähler structure on a neighbourhood $\mathbf{M} \subset \mathbf{M}'$ of Δ_M in $M \times M$. Let us again consider the sum (3.12). For every $V \in \mathbf{V}$ such that $\mu_V \not\equiv 0$ on W^2, we have $q^V_{U_W} = 0$ on $\Delta_{U_V \cap U_W}$, and hence on Δ_W. Moreover, by virtue of (3.10) for every such V we have $\left.\frac{\partial q^V_{U_W}}{\partial x}\right|_{z=(y,y)} = -w^{U_W}(y)$ for all $y \in W$. By (3.11) the derivatives $\frac{\partial^2 q^V_{U_W}}{\partial x \partial p}$ and $\frac{\partial^3 q^V_{U_W}}{\partial x^2 \partial p}$ are also independent of V on Δ_W.

Lemma 3.2. *Let $W \subset \mathbb{R}^n$ be an open set with coordinates y^a. Denote the coordinates on the first copy of W in the product $W \times W$ by x^a, and those in the second copy by p^A. Let $\{q^i\}_{i\in I}$ be a family of smooth functions on $W \times W$, such that for all $i \in I$ and all $y \in W$ we have $q^i(y,y) = 0$, $\left.\frac{\partial q^i}{\partial x}\right|_{(x,p)=(y,y)} = -w(y)$,*

$\frac{\partial^2 q^i}{\partial x \partial p}\Big|_{(x,p)=(y,y)} = P(y)$, $\frac{\partial^3 q^i}{\partial x^2 \partial p}\Big|_{(x,p)=(y,y)} = T(y)$ *for some objects* w, P, T *on* W *of according dimensions.*

Let $\{\lambda^i : W^2 \to \mathbb{R} \,|\, i \in I\}$ *be a partition of unity on* W^2*, and set* $q = \sum_{i \in I} \lambda^i q^i$*. Then for all* $y \in W$ *we have* $q(y,y) = 0$, $\frac{\partial q}{\partial x}\Big|_{(x,p)=(y,y)} = -w(y)$, $\frac{\partial^2 q}{\partial x \partial p}\Big|_{(x,p)=(y,y)} = P(y)$, $\frac{\partial^3 q}{\partial x^2 \partial p}\Big|_{(x,p)=(y,y)} = T(y)$.

Proof. Introduce coordinates $u = \frac{x+p}{2}$, $v = \frac{x-p}{2}$ on $W \times W$. Then on Δ_W we have $\frac{\partial q^i}{\partial u} = \frac{\partial q^i}{\partial x} + \frac{\partial q^i}{\partial p} = 0$, because $q^i \equiv 0$ on Δ_W. Hence $\frac{\partial q^i}{\partial v} = \frac{\partial q^i}{\partial x} - \frac{\partial q^i}{\partial p} = 2\frac{\partial q^i}{\partial x}$ equals $-2w$ for all $i \in I$.

Further, we have on Δ_W

$$\begin{aligned}
\frac{\partial^2 q^i}{\partial u^2} &= \frac{\partial^2 q^i}{\partial x^2} + \frac{\partial^2 q^i}{\partial x \partial p} + \frac{\partial^2 q^i}{\partial p \partial x} + \frac{\partial^2 q^i}{\partial p^2} = 0, \\
\frac{\partial^2 q^i}{\partial v^2} &= \frac{\partial^2 q^i}{\partial x^2} - \frac{\partial^2 q^i}{\partial x \partial p} - \frac{\partial^2 q^i}{\partial p \partial x} + \frac{\partial^2 q^i}{\partial p^2} = -2\frac{\partial^2 q^i}{\partial x \partial p} - 2\frac{\partial^2 q^i}{\partial p \partial x} = -2(P + P^T).
\end{aligned}$$

Hence also the second derivative $\frac{\partial^2 q^i}{\partial v^2}$ coincides on Δ_W for all $i \in I$.

Moreover,

$$\frac{\partial^3 q^i}{\partial x \partial x^a \partial p} + \frac{\partial^3 q^i}{\partial x \partial p^A \partial p} = \frac{\partial}{\partial u^a} \frac{\partial^2 q^i}{\partial x \partial p} = \frac{\partial P}{\partial y^a} \tag{3.13}$$

does not depend on i. Since $\frac{\partial^3 q^i}{\partial x^2 \partial p} = T$ does not depend on i, the second summand on the left-hand side of (3.13) and hence the derivative $\frac{\partial^3 q^i}{\partial x \partial p^2}$ cannot depend on i neither.

We have

$$\begin{aligned}
\frac{\partial^3 q^i}{\partial u^2 \partial v} &= \frac{\partial^3 q^i}{\partial x^3} - \frac{\partial^3 q^i}{\partial x^2 \partial p} + \frac{\partial^3 q^i}{\partial x \partial p \partial x} - \frac{\partial^3 q^i}{\partial x \partial p^2} + \frac{\partial^3 q^i}{\partial p \partial x^2} - \frac{\partial^3 q^i}{\partial p \partial x \partial p} + \frac{\partial^3 q^i}{\partial p^2 \partial x} - \frac{\partial^3 q^i}{\partial p^3}, \\
\frac{\partial^3 q^i}{\partial v^3} &= \frac{\partial^3 q^i}{\partial x^3} - \frac{\partial^3 q^i}{\partial x^2 \partial p} - \frac{\partial^3 q^i}{\partial x \partial p \partial x} + \frac{\partial^3 q^i}{\partial x \partial p^2} - \frac{\partial^3 q^i}{\partial p \partial x^2} + \frac{\partial^3 q^i}{\partial p \partial x \partial p} + \frac{\partial^3 q^i}{\partial p^2 \partial x} - \frac{\partial^3 q^i}{\partial p^3} \\
&= \frac{\partial^3 q^i}{\partial u^2 \partial v} + 2\left(-\frac{\partial^3 q^i}{\partial x \partial p \partial x} + \frac{\partial^3 q^i}{\partial x \partial p^2} - \frac{\partial^3 q^i}{\partial p \partial x^2} + \frac{\partial^3 q^i}{\partial p \partial x \partial p}\right).
\end{aligned}$$

Hence $\frac{\partial^3 q^i}{\partial v^3}$ can be expressed as the sum of $\frac{\partial^3 q^i}{\partial u^2 \partial v}$, which on Δ_W is independent of i by virtue of the independence of $\frac{\partial q^i}{\partial v}$, and a linear combination of the mixed derivatives $\frac{\partial^3 q^i}{\partial x^2 \partial p}$, $\frac{\partial^3 q^i}{\partial x \partial p^2}$, which by the assumption of the lemma and by the preceding paragraph are also independent of i.

Hence the partial derivatives of q^i up to third order with respect to v are independent of i on Δ_W. Since this holds identically on Δ_W, the mixed derivatives of q^i up to third order are also independent of i on Δ_W. Thus at any given point $z \in \Delta_W$, all functions q^i belong to the same 3-jet. By Lemma B.1 (see Appendix), q has then to belong to the same 3-jet too. The assertion of the lemma now readily follows. □

By the preceding lemma, the values of $Q = \frac{\partial q^W}{\partial x \partial p}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^3 q^W}{\partial x^2 \partial p}$ on Δ_W are the same as those of the corresponding derivatives of the local para-Kähler potentials q_U defined in Lemma 3.1. Thus on Δ_M we have $Q = P$, and we find a neighbourhood

$\mathbf{M} \subset \mathbf{M}'$ of Δ_M such that Q is non-degenerate on $\mathbf{M}$. Then $\mathbf{Q}$ defines a para-Kähler structure on $\mathbf{M}$. By Lemma 2.4 and (2.5), this para-Kähler structure then defines on M the same pre-Codazzi structure as the local para-Kähler structures defined by the para-Kähler potentials q_U, namely the pre-Codazzi structure $(\bar{\nabla}, P)$.

We have proven the following theorem.

Theorem 3.1. *Let M be an n-dimensional manifold with pre-Codazzi structure $(\bar{\nabla}, P)$, such that P is non-degenerate everywhere on M. Then there exists a $2n$-dimensional para-Kähler manifold $\mathbf{M}$ with metric g and symplectic form ω and an embedding $f : M \to \mathbf{M}$ such that the distributions $D^{\mathbf{P}}, D^{\mathbf{X}}$ are transversal to f, and the pre-Codazzi structure $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ from i), Theorem 2.1 coincides with $(\bar{\nabla}, P)$.*

Corollary 3.1. *Let M be a manifold with Codazzi structure $(\bar{\nabla}, h)$, such that h is non-degenerate everywhere on M. Then there exists a para-Kähler manifold $\mathbf{M}$ with metric g and symplectic form ω and a Lagrangian embedding $f : M \to \mathbf{M}$ such that the distributions $D^{\mathbf{P}}, D^{\mathbf{X}}$ are transversal to f, and the Codazzi structure $(\nabla^{\mathbf{X}}, \hat{g})$ from i), Theorem 2.2 coincides with $(\bar{\nabla}, h)$.*

4. Projectively flat manifolds and the cross-ratio manifold

In Subsection 4.1 we introduce a special homogeneous para-Kähler manifold, the *cross-ratio manifold*. The term comes from the fact that the para-Kähler structure on this manifold is the infinitesimal limit of a finite structure defined by the projective cross-ratio. We will provide this construction in a companion paper, here we just give the definition of the para-Kähler structure. In Subsection 4.2 we introduce projectively flat manifolds and show that they carry a natural pre-Codazzi structure. In Subsection 4.3 we show that the affine connections $\nabla^{\mathbf{X}}, \nabla^{\mathbf{P}}$ induced on half-dimensional immersions in the cross-ratio manifold are projectively flat, and the pre-Codazzi structures $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$, $(\nabla^{\mathbf{P}}, \hat{\mathbf{Q}}^T)$ induced by these immersions coincide with the natural pre-Codazzi structures defined by the projectively flat connections $\nabla^{\mathbf{X}}, \nabla^{\mathbf{P}}$, respectively. In Subsection 4.4 we show that the natural pre-Codazzi structure of an arbitrary projectively flat manifold M can, at least locally, be realized by a half-dimensional immersion of M in the cross-ratio manifold. This will lead to a natural duality relation between projectively flat connections on M.

4.1. **The cross-ratio manifold.** Let $\mathbf{X} = \mathbb{R}P^n$ be the n-dimensional real projective space, and let $\mathbf{P} = \mathbb{R}P_n$ be its dual space. The space $\mathbf{X}$ is the set of 1-dimensional subspaces of the real vector space $\mathbb{R}^{n+1}$, while $\mathbf{P}$ is the set of 1-dimensional subspaces of the dual vector space $\mathbb{R}_{n+1}$. For $x \in \mathbf{X}$, $p \in \mathbf{P}$, we will call points $\tilde{x} \in \mathbb{R}^{n+1} \setminus \{0\}$, $\tilde{p} \in \mathbb{R}_{n+1} \setminus \{0\}$ *representatives* of x, p, respectively, if $\tilde{x} \in x$, $\tilde{p} \in p$. We say that x is *orthogonal* to p, $x \perp p$, if $\langle \tilde{x}, \tilde{p} \rangle = 0$ for any representatives $\tilde{x}, \tilde{p}$ of x, p, respectively.

Define the set

$$\mathbf{M} = \{z = (x, p) \in \mathbf{X} \times \mathbf{P} \,|\, x \not\perp p\}. \tag{4.1}$$

Then $\mathbf{M}$ is an open dense subset of $\mathbf{X} \times \mathbf{P}$, and hence a $2n$-dimensional manifold. This manifold carries a natural para-complex structure given by (3.4). Let a basis $e_0, \dots, e_n$ of $\mathbb{R}^{n+1}$ and a corresponding dual basis $e^0, \dots, e^n$ of $\mathbb{R}_{n+1}$ be given. These bases define coordinates $\tilde{x}^0, \dots, \tilde{x}^n$ and $\tilde{p}_0, \dots, \tilde{p}_n$ on $\mathbb{R}^{n+1}$ and $\mathbb{R}_{n+1}$, respectively. These coordinates, in turn, define affine charts on the projective spaces $\mathbf{X}, \mathbf{P}$ by $x^a = \frac{\tilde{x}^a}{\tilde{x}^0}$, $p_A = \frac{\tilde{p}_a}{\tilde{p}_0}$, respectively. We then say that the coordinates $z^\alpha = (x^a, p_A)$

form an *affine chart* on $\mathbf{M}$. A point $z = (x, p)$ belongs to this chart if and only if $1 + p^T x \neq 0$, because $1 + p^T x = 0$ if and only if $x \perp p$.

Different bases of $\mathbb{R}^{n+1}$ define different affine charts on $\mathbf{M}$. Clearly the affine charts form an atlas on $\mathbf{M}$. A linear coordinate change with transformation matrix $\mathbf{A} = \begin{pmatrix} a & a_h \\ a_v & A \end{pmatrix}$ in $\mathbb{R}^{n+1}$ induces a linear coordinate change with transformation matrix $\mathbf{A}^{-T} = \mathbf{B} = \begin{pmatrix} b & b_h \\ b_v & B \end{pmatrix}$ in $\mathbb{R}_{n+1}$, where a, b are scalars, a_v, b_v are column vectors, a_h, b_h are row vectors, and A, B are $n \times n$ matrices. These in turn induce the projective transformations

$$x \mapsto x' = \frac{Ax + a_v}{a_h x + a}, \qquad p \mapsto p' = \frac{Bp + b_v}{b_h p + b} \tag{4.2}$$

in $\mathbf{X}$ and $\mathbf{P}$, respectively.

Let us now pass to the definition of the para-Kähler structure. Given an affine chart, we determine the para-Kähler structure on this chart by the para-Kähler potential $q(z) = \log|1 + p^T x|$, leading to the matrix

$$Q(z) = \frac{(1 + p^T x)I_n - px^T}{(1 + p^T x)^2}. \tag{4.3}$$

We then have $\det Q = (1 + p^T x)^{-(n+1)}$, and Q is non-degenerated everywhere on this chart. Let us show that the para-Kähler structure is well-defined, i.e., it does not depend on the choice of the chart.

Lemma 4.1. *Let $z = (x, p)$ and $z' = (x', p')$ be linked by a transformation of type (4.2). Then the functions $q(z)$ and $q'(z) = q(z'(z))$ are linked by a transformation of type (2.4).*

Proof. We have

$$\begin{aligned} q'(z) = \log|1 + {p'}^T x'| &= \log\left|1 + \frac{(Bp + b_v)^T (Ax + a_v)}{(b_h p + b)(a_h x + a)}\right| \\ = \quad & \log\left|(b_h p + b)(a_h x + a) + (Bp + b_v)^T (Ax + a_v)\right| - \log|b_h p + b| - \log|a_h x + a| \\ = \quad & \log\left|\begin{pmatrix} 1 & p^T \end{pmatrix} \mathbf{B}^T \mathbf{A} \begin{pmatrix} 1 \\ x \end{pmatrix}\right| - \log|b_h p + b| - \log|a_h x + a| \\ = \quad & \log|1 + p^T x| - \log|b_h p + b| - \log|a_h x + a| = q(z) + h(x) + h'(p) \end{aligned}$$

with $h(y) = -\log|a_h y + a|$, $h'(y) = -\log|b_h y + b|$. $\square$

Thus (4.3) indeed defines an invariant para-Kähler structure on $\mathbf{M}$.

Definition 4.1. We call the manifold $\mathbf{M}$ defined by (4.1) and endowed with the para-Kähler structure defined by (4.3),(2.1),(2.2) the *cross-ratio manifold*.

In [12] Gadea and Montesinos Amilibia introduced a one-parametric family of para-Kähler manifolds $P_n(B)/\mathbb{Z}_2$, the *reduced para-complex projective spaces*, and an isomorphic family of spaces $P(\mathbb{R}^{n+1} \oplus \mathbb{R}_{n+1})/\mathbb{Z}_2$. A comparison of (4.3) with [12, eq. (1.3)] and of (2.1) with [12, eq. (1.4)] yields the following result.

Theorem 4.1. *The cross-ratio manifold $\mathbf{M}$ is canonically isomorphic to the space $P(\mathbb{R}^{n+1} \oplus \mathbb{R}_{n+1})/\mathbb{Z}_2$ with parameter value $c = 4$, which in turn is isomorphic to the reduced para-complex projective space $P_n(B)/\mathbb{Z}_2$ with parameter value $c = 4$.*

Inserting (4.3) into (2.5), we obtain

$$\Gamma^c_{ab} = -\frac{p_B\delta^c_a + p_A\delta^c_b}{1 + p_D x^d}, \qquad \Gamma^C_{AB} = -\frac{x^b\delta^C_A + x^a\delta^C_B}{1 + p_D x^d}, \tag{4.4}$$

the other components of $\Gamma^\gamma_{\alpha\beta}$ being zero.

4.2. **Projectively flat manifolds.**

Definition 4.2. [18, p.386] A torsion-free affine connection ∇ on a manifold M is called *projectively flat*[1] if there exists a flat affine connection ∇' on M such that the pregeodesics (i.e., the geodesics as curves without a distinguished parametrization) of ∇ and ∇' coincide.

By [20, eq. (3p), p.100] an affine connection is projectively flat if and only if its Christoffel symbols can locally be written as

$$\Gamma^k_{ij} = \rho_i\delta^k_j + \rho_j\delta^k_i \tag{4.5}$$

for some 1-form ρ in some coordinate system. In this coordinate system, the Ricci tensor of a projectively flat connection ∇ with Christoffel symbols (4.5) is given by [20, p.100], [18, p.485]

$$R_{ab} = nP_{ab} - P_{ba}$$

with

$$P_{ab} = -\nabla_b\rho_a - \rho_a\rho_b = -\frac{\partial\rho_a}{\partial x^b} + \rho_a\rho_b \tag{4.6}$$

and n the dimension of the manifold. In terms of the Ricci tensor, the tensor P is given by [20, p.100], [18, p.486]

$$P_{ab} = (n^2-1)^{-1}(nR_{ab} + R_{ba}), \tag{4.7}$$

which provides a description which is independent of the particular coordinate system.

Note that the Ricci tensor is symmetric if and only if the tensor P is symmetric. Then (4.7) simplifies to $P_{ab} = (n-1)^{-1}R_{ab}$ and P is just the normalized Ricci tensor [17, p.17]. Equivalently, there exists a nonzero parallel volume element on M [17, Proposition 3.1, p.14], in which case the manifold M is called *equiaffine*.

The projectively flat connection ∇ and the tensor P satisfy the Codazzi equation (1.4) [20, eq. (IIIp), p.104], [18, eq. (109.17), p.487][2]. Thus (∇, P) is a pre-Codazzi structure on M. It is a Codazzi structure if and only if the manifold M is equiaffine. Every projectively flat manifold is hence in a natural way a pre-Codazzi manifold.

Definition 4.3. For a manifold M with projectively flat connection ∇, define a tensor field P by (4.7), where R_{ab} is the Ricci tensor of ∇. We call (∇, P) the *canonical pre-Codazzi structure* of the projectively flat manifold M.

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[1] In this reference, the term *projectively Euclidean* is used.

[2] Note that both P and the Ricci tensor are defined in a different manner in this reference. The relation is given by $P_{ab} \mapsto P_{ba}$, $R_{ab} \mapsto -R_{ba}$.

4.3. **Half-dimensional immersions.** In this subsection we consider immersions $f : M \to \mathbf{M}$ of an n-dimensional manifold M in the cross-ratio manifold, such that the distribution $D^{\mathbf{X}}$ is transversal to f.

Lemma 4.2. *Let $f : M \to \mathbf{M}$ be an immersion of an n-dimensional manifold M into the cross-ratio manifold (4.1). If the distribution $D^{\mathbf{X}}$ is transversal to f, then the affine connection $\nabla^{\mathbf{X}}$ induced on M by f, viewed as affine immersion with transversal distribution $D^{\mathbf{X}}$, is projectively flat. If the distribution $D^{\mathbf{P}}$ is transversal to f, then the affine connection $\nabla^{\mathbf{P}}$ induced on M by f, viewed as affine immersion with transversal distribution $D^{\mathbf{P}}$, is projectively flat.*

Proof. Pass to an affine chart on $\mathbf{M}$ and a corresponding adapted chart on M with coordinates x^a. By Lemma 2.4 and (4.4) the Christoffel symbols of $\nabla^{\mathbf{X}}$ have the form (4.5) with

$$\rho_a = -\frac{p_A}{1 + p_B x^b}, \tag{4.8}$$

and $\nabla^{\mathbf{X}}$ is projectively flat. The second part of the lemma is proven in a similar manner. □

We will now establish a relation between the curvature of the connection $\nabla^{\mathbf{X}}$ and the tensor field $\hat{\mathbf{Q}}$ of the pre-Codazzi structure from i), Theorem 2.1.

Lemma 4.3. *Let $f : M \to \mathbf{M}$ be an immersion of an n-dimensional manifold M into the cross-ratio manifold (4.1), such that the distribution $D^{\mathbf{X}}$ is transversal to f. Then the tensor $\hat{\mathbf{Q}}$ coincides with the tensor P defined by (4.7), where R_{ab} is the Ricci tensor of the connection $\nabla^{\mathbf{X}}$.*

Proof. Assume the conditions of the lemma. Let $z^\alpha = (x^a, p_A)$ be the coordinates of an affine chart on the cross-ratio manifold $\mathbf{M}$, and let x^a be the corresponding coordinates of an adapted chart on M. Inserting (4.8), in vector form $\rho = -\frac{p}{1+p^T x}$, into (4.6), in matrix form $P = -\frac{\partial \rho}{\partial x} + \rho\rho^T$, we obtain for the matrix of P

$$\begin{aligned} P &= \frac{d}{dx}\frac{p}{1+p^T x} + \frac{pp^T}{(1+p^T x)^2} = \frac{(1+p^T x)\frac{\partial p}{\partial x} - p\left(x^T\frac{\partial p}{\partial x} + p^T\right)}{(1+p^T x)^2} + \frac{pp^T}{(1+p^T x)^2} \\ &= \frac{(1+p^T x)I_n - px^T}{(1+p^T x)^2}\frac{\partial p}{\partial x} = Q\frac{\partial p}{\partial x} = \hat{\mathbf{Q}}. \end{aligned}$$

Here by $\frac{d}{dx}$ we denote the derivative on M, and the last two relations are due to (4.3) and (2.8), respectively. □

Thus we get the following result.

Theorem 4.2. *Let $f : M \to \mathbf{M}$ be an immersion of an n-dimensional manifold M into the cross-ratio manifold (4.1). If the distribution $D^{\mathbf{X}}$ is transversal to f, then the pre-Codazzi structure $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ from i), Theorem 2.1 coincides with the canonical pre-Codazzi structure on M, viewed as a projectively flat manifold with connection $\nabla^{\mathbf{X}}$. If the distribution $D^{\mathbf{P}}$ is transversal to f, then the pre-Codazzi structure $(\nabla^{\mathbf{P}}, \hat{\mathbf{Q}}^T)$ from ii), Theorem 2.1 coincides with the canonical pre-Codazzi structure on M, viewed as a projectively flat manifold with connection $\nabla^{\mathbf{P}}$.*

Corollary 4.1. *Let $f : M \to \mathbf{M}$ be a Lagrangian immersion of an n-dimensional manifold in the cross-ratio manifold. If the distribution $D^{\mathbf{X}}$ is transversal to f, then the Codazzi structure $(\nabla^{\mathbf{X}}, \hat{g})$ from i), Theorem 2.2 has a projectively flat equiaffine connection $\nabla^{\mathbf{X}}$ with Ricci tensor equal to $(n-1)\hat{g}$. If the distribution $D^{\mathbf{P}}$ is transversal to f, then the Codazzi structure $(\nabla^{\mathbf{P}}, \hat{g})$ from ii), Theorem 2.2 has a projectively flat equiaffine connection $\nabla^{\mathbf{P}}$ with Ricci tensor equal to $(n-1)\hat{g}$.*

4.4. **Representation of projectively flat manifolds.** In this subsection we show that the canonical pre-Codazzi structure of every projectively flat manifold can be represented, at least locally, as in Theorem 4.2.

Theorem 4.3. *Let M be an n-dimensional projectively flat manifold with connection $\bar{\nabla}$. For every $y \in M$, there exists a neighbourhood U of y and an immersion $f : U \to \mathbf{M}$ in the cross-ratio manifold (4.1) such that the distribution $D^{\mathbf{X}}$ is transversal to f, and the affine connection $\nabla^{\mathbf{X}}$ induced on U by f, viewed as affine immersion with transversal distribution $D^{\mathbf{X}}$, coincides with the connection $\bar{\nabla}$. In particular, the pre-Codazzi structure $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ defined on U as in i), Theorem 2.1 coincides with the canonical pre-Codazzi structure defined by $\bar{\nabla}$ on U.*

Proof. Assume the conditions of the theorem. Since $\bar{\nabla}$ is projectively flat, there exists a chart on a neighbourhood of y such that the Christoffel symbols of $\bar{\nabla}$ are given by (4.5) for some 1-form ρ. Denote the coordinates of the chart by $x^1, \dots, x^n$ and assume without restriction of generality that the point y is given by the origin of the chart, $x = 0$. Let a vector-valued function $p(x)$ be given by

$$p_A(x) = -\frac{\rho_a}{1 + \rho_b x^b}.$$

Then $1 + p^T x \neq 0$ in a neighbourhood $U \subset M$ of $x = 0$. Define the immersion $f : U \to \mathbf{M}$ by $f : x \mapsto (x, p(x))$. Clearly the distribution $D^{\mathbf{X}}$ is transversal to f. Moreover, (4.8) holds and thus by Lemma 2.4 and (4.5) the connections $\nabla^{\mathbf{X}}, \bar{\nabla}$ coincide. By Theorem 4.2, $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ is then the canonical pre-Codazzi structure on M generated by $\bar{\nabla}$. □

Corollary 4.2. *Let M be an n-dimensional projectively flat equiaffine manifold with connection $\bar{\nabla}$. For every $y \in M$, there exists a neighbourhood U of y and a Lagrangian immersion $f : U \to \mathbf{M}$ in the cross-ratio manifold (4.1) such that the distribution $D^{\mathbf{X}}$ is transversal to f, and the Codazzi structure $(\nabla^{\mathbf{X}}, \hat{g})$ induced on U by f as in i), Theorem 2.2 is composed of $\bar{\nabla}$ and the normalized Ricci tensor $\frac{R_{ab}}{n-1}$ of $\bar{\nabla}$.*

Finally, let us deduce a duality relation between projectively flat connections on a manifold M.

Theorem 4.4. *Let M be a projectively flat manifold with connection ∇, and let (∇, P) be the canonical pre-Codazzi structure on M. Suppose that P is non-degenerate and let $(\tilde{\nabla}, P^T)$ be the dual pre-Codazzi structure. Then $\tilde{\nabla}$ is also projectively flat, and $(\tilde{\nabla}, P^T)$ is the corresponding canonical pre-Codazzi structure.*

Proof. Assume the conditions of the theorem. By Theorem 4.3, M can locally be immersed into the cross-ratio manifold $\mathbf{M}$ such that $D^{\mathbf{X}}$ is transversal to the immersion, ∇ coincides with the induced affine connection $\nabla^{\mathbf{X}}$, and $\hat{\mathbf{Q}} = P$. Since P is non-degenerate, the distribution $D^{\mathbf{P}}$ is by Lemma 2.3 also transversal to f. In particular, the immersion induces a connection $\nabla^{\mathbf{P}}$ on M with transversal distribution $D^{\mathbf{P}}$. By iii), Theorem 2.1 $\nabla^{\mathbf{P}}$ is conjugate to $\nabla^{\mathbf{X}}$ relative to $\hat{\mathbf{Q}}$, and hence coincides with the connection $\tilde{\nabla}$.

But $\nabla^{\mathbf{P}}$ is projectively flat by Lemma 4.2, and by Theorem 4.2 the pre-Codazzi structure $(\nabla^{\mathbf{P}}, \hat{\mathbf{Q}}^T)$ is the canonical pre-Codazzi structure corresponding to $\nabla^{\mathbf{P}}$. This completes the proof. □

Thus the duality between pre-Codazzi structures induces a duality relation between projectively flat connections.

Definition 4.4. Let M be a projectively flat manifold with connection ∇, and let (∇, P) be the canonical pre-Codazzi structure on M. We call the connection ∇ *non-degenerate* if the tensor field P is everywhere non-degenerate. For non-degenerate ∇, let $\tilde{\nabla}$ be the conjugate connection of ∇ relative to P. We call the connection $\tilde{\nabla}$ the *dual connection* of ∇.

For equiaffine projectively flat connections with non-degenerate Ricci tensor, this duality relation reduces to the duality relation induced by the conormal map on the local representation of the connection as centro-affine hypersurface immersion.

5. Hessian manifolds and the flat para-Kähler space

In this section we show that the affine connections $\nabla^{\mathbf{X}}, \nabla^{\mathbf{P}}$ induced on half-dimensional immersions in the flat para-Kähler space $\mathbb{E}^{2n}_n$ are flat, and the pre-Codazzi structures $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$, $(\nabla^{\mathbf{P}}, \hat{\mathbf{Q}}^T)$ induced by these immersions are hence pre-Hessian structures. Moreover, we show that an arbitrary pre-Hessian structure on a manifold M can, at least locally, be realized by the pre-Hessian structure $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ of a half-dimensional immersion $f : M \to \mathbb{E}^{2n}_n$. As an application, we show that the dual of a pre-Hessian structure is also a pre-Hessian structure.

We work in the canonical coordinate system on $\mathbb{E}^{2n}_n$, such that the para-Kähler structure is given by (1.2). The coordinates will be denoted by $z^\alpha = (x^1, \dots, x^n, p_{n+1}, \dots, p_{2n})$. In these coordinates we have $Q = I_n$ and $\Gamma^\gamma_{\alpha\beta} = 0$ for the Christoffel symbols of the Levi-Civita connection of g on $\mathbb{E}^{2n}_n$.

Definition 5.1. A *pre-Hessian structure* on an n-dimensional manifold M is a pair (∇, P) of a torsion-free flat affine connection ∇ and a covariant second order tensor field P such that for every three vector fields X, Y, Z on M the Codazzi equation (1.4) holds. A manifold M equipped with a pre-Hessian structure will be called a *pre-Hessian manifold.*

Proposition 5.1. *Let $f : M \to \mathbf{M}$ be an immersion of an n-dimensional manifold M into the flat para-Kähler space $\mathbb{E}^{2n}_n$. If the distribution $D^{\mathbf{X}}$ is transversal to f, then the pre-Codazzi structure $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ induced on M by f as in i), Theorem 2.1 is a pre-Hessian structure. If the distribution $D^{\mathbf{P}}$ is transversal to f, then the pre-Codazzi structure $(\nabla^{\mathbf{P}}, \hat{\mathbf{Q}}^T)$ induced on M by f as in ii), Theorem 2.1 is a pre-Hessian structure.*

Proof. Pass to an adapted chart on M with coordinates x^a. By Lemma 2.4 the Christoffel symbols of $\nabla^{\mathbf{X}}$ vanish, and hence $\nabla^{\mathbf{X}}$ is flat. Thus $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ is a pre-Hessian structure. The second part of the proposition is proven in a similar manner. □

Corollary 5.1. *Let $f : M \to \mathbf{M}$ be a Lagrangian immersion in the flat para-Kähler space $\mathbb{E}^{2n}_n$. If the distribution $D^{\mathbf{X}}$ is transversal to f, then the Codazzi structure $(\nabla^{\mathbf{X}}, \hat{g})$ induced on M by f as in i), Theorem 2.2 is a Hessian structure. If the distribution $D^{\mathbf{P}}$ is transversal to f, then the Codazzi structure $(\nabla^{\mathbf{P}}, \hat{g})$ induced on M by f as in ii), Theorem 2.2 is a Hessian structure.*

Proposition 5.2. *Let M be an n-dimensional manifold with pre-Hessian structure $(\bar{\nabla}, P)$. For every $y \in M$, there exists a neighbourhood U of y and an immersion $f : U \to \mathbb{E}^{2n}_n$ such that the distribution $D^{\mathbf{X}}$ is transversal to f, and the pre-Hessian structure $(\nabla^{\mathbf{X}}, \hat{\mathbf{Q}})$ induced on U by f as in Proposition 5.1 coincides with the pre-Hessian structure $(\bar{\nabla}, P)$.*

Proof. Assume the conditions of the proposition. Since $\bar{\nabla}$ is flat, there exists a chart with coordinates $x^1, \dots, x^n$ on a simply connected neighbourhood U of y such that the Christoffel symbols of $\bar{\nabla}$ vanish. Then the Codazzi equation (2.6) simplifies to the relation $\frac{\partial P_{ab}}{\partial x^c} = \frac{\partial P_{ac}}{\partial x^b}$ on U. It follows that $P_{ab} = \frac{\partial p_A}{\partial x^b}$ for some vector-valued function $p(x)$ on U. Define the immersion $f : U \to \mathbb{E}^{2n}_n$ by $f : x \mapsto (x, p(x))$. Clearly the distribution $D^{\mathbf{X}}$ is transversal to f. Moreover, $\hat{\mathbf{Q}} = Q\frac{\partial p}{\partial x} = \frac{\partial p}{\partial x} = P$ by (2.8) and the definition of p. The Christoffel symbols of the induced connection $\nabla^{\mathbf{X}}$ on M vanish by virtue of Lemma 2.4 and thus equal those of the connection $\bar{\nabla}$. The assertion of the proposition now follows. □

Corollary 5.2. *Let M be an n-dimensional manifold with Hessian structure $(\bar{\nabla}, P)$. For every $y \in M$, there exists a neighbourhood U of y and a Lagrangian immersion $f : U \to \mathbb{E}^{2n}_n$ such that the distribution $D^{\mathbf{X}}$ is transversal to f, and the Hessian structure $(\nabla^{\mathbf{X}}, \hat{g})$ induced on U by f as in Corollary 5.1 coincides with the Hessian structure $(\bar{\nabla}, P)$.*

Finally, we establish a duality relation between pre-Hessian structures on a manifold M.

Theorem 5.1. *Let M be a manifold with a pre-Hessian structure (∇, P). Suppose that P is non-degenerate and let $(\tilde{\nabla}, P^T)$ be the dual pre-Codazzi structure. Then $\tilde{\nabla}$ is also flat, and $(\tilde{\nabla}, P^T)$ is a pre-Hessian structure.*

Proof. Assume the conditions of the theorem. By Proposition 5.2, M can locally be immersed in $\mathbb{E}^{2n}_n$ such that $D^{\mathbf{X}}$ is transversal to the immersion, ∇ coincides with the induced affine connection $\nabla^{\mathbf{X}}$, and $\hat{\mathbf{Q}} = P$. Since P is non-degenerate, the distribution $D^{\mathbf{P}}$ is by Lemma 2.3 also transversal to f. In particular, this distribution induces the connection $\nabla^{\mathbf{P}}$ on M. By iii), Theorem 2.1 $\nabla^{\mathbf{P}}$ is conjugate to $\nabla^{\mathbf{X}}$ relative to $\hat{\mathbf{Q}}$, and hence coincides with the connection $\tilde{\nabla}$. But $\nabla^{\mathbf{P}}$ is flat by Proposition 5.1, and hence $\tilde{\nabla}$ is flat. □

Thus the duality between pre-Codazzi structures introduces a duality relation between pre-Hessian structures.

Definition 5.2. Let M be a manifold with pre-Hessian structure (∇, P), and suppose the tensor P is non-degenerate. Let $\tilde{\nabla}$ be the conjugate connection of ∇ relative to P. Then we call $(\tilde{\nabla}, P^T)$ the *dual pre-Hessian structure* of (∇, P).

For Hessian structures, this duality relation reduces to the well-known duality relation given in [19, Def. 2.6, p.25].

Appendix A. Topological lemmas

The purpose of this section is to provide two auxiliary topological results, namely Lemma A.2 and Corollary A.1.

Definition A.1. [3, pp.5–6] Let M be a topological space and let $\mathbf{U}$ be a cover of M. A cover $\mathbf{V}$ of M is called a *refinement* of $\mathbf{U}$ if for every $V \in \mathbf{V}$ there exists $U \in \mathbf{U}$ such that $V \subset U$. It is called a *star refinement* of $\mathbf{U}$ if for every $x \in M$, there exists $U \in \mathbf{U}$ such that $\bigcup_{V \in \mathbf{V}: x \in V} V \subset U$. It is called a *strong star refinement* of $\mathbf{U}$ if for every $W \in \mathbf{V}$, there exists $U \in \mathbf{U}$ such that $\bigcup_{V \in \mathbf{V}: W \cap V \neq \emptyset} V \subset U$.

Proposition A.1. [3, Proposition 4, p.6] *Let $\mathbf{U}$ be a cover of a topological space M, let $\mathbf{V}$ be a star refinement of $\mathbf{U}$, and let $\mathbf{W}$ be a star refinement of $\mathbf{V}$. Then $\mathbf{W}$ is a strong star refinement of $\mathbf{U}$.*

Definition A.2. [3, p.16] A topological space M is *star-normal* if every open cover $\mathbf{U}$ of M has an open star refinement.

Proposition A.2. [3, Theorem 26, p.16] *Every paracompact Hausdorff space is star-normal.*

From now on, we assume that a differentiable manifold is by definition second-countable, and hence also paracompact [16, p.4].

Corollary A.1. *Let M be a differentiable manifold and let $\mathbf{U}$ be an open cover of M. Then there exists an open strong star refinement $\mathbf{V}$ of $\mathbf{U}$.*

Proposition A.3. [16, Theorem 10.6] *A smooth differentiable manifold M possesses a smooth triangulation.*

From the proof of this result (see [16, Lemma 2.7 and p.103]) we can actually deduce a slightly stronger version.

Theorem A.1. *Let M be a manifold and let $\mathbf{U}$ be an open cover of M. Then there exists a smooth triangulation K of M which is subordinated to $\mathbf{U}$, i.e., such that for every simplex $\sigma \in K$, there exists $U \in \mathbf{U}$ such that $\sigma \subset U$.*

Definition A.3. [16, p.70] Let M be a smooth differentiable manifold with a given triangulation K. For $x \in M$, we call $\bigcup_{\sigma \in K: x \in \sigma} \sigma^o$ the *star* $St(x, K)$ of x. Here σ^o denotes the interior of the simplex σ.

The star $St(x, K)$ is a neighbourhood of x in M [16, p.70]. Clearly it is contractible onto x and hence simply connected.

Lemma A.1. *Let M be a smooth differentiable manifold with a given triangulation K. For every $x, y \in M$, either $St(x, K) \cap St(y, K) = \emptyset$ or there exists $z \in M$ such that $St(x, K) \cap St(y, K) = St(z, K)$.*

Proof. We have $St(x, K) = \bigcup_{\sigma \in K: x \in \sigma} \sigma^o$, $St(y, K) = \bigcup_{\sigma \in K: y \in \sigma} \sigma^o$, hence

$$St(x, K) \cap St(y, K) = \bigcup_{\sigma \in K: x \in \sigma, y \in \sigma} \sigma^o.$$

Since the intersection of two simplices in K is either empty or again a simplex in K, we have that either $St(x, K) \cap St(y, K) = \emptyset$ or there exists a minimal simplex σ^* containing both x and y. In the latter case we thus obtain

$$St(x, K) \cap St(y, K) = \bigcup_{\sigma \in K: \sigma^* \subset \sigma} \sigma^o.$$

In this case we can choose z to be any interior point of σ^*. ☐

Lemma A.2. *Let M be a smooth differentiable manifold and let $\mathbf{U}$ be an open cover of M. Then there exists an open refinement $\mathbf{V}$ of $\mathbf{U}$ such that for every finite subset $\Sigma \subset \mathbf{V}$ the intersection $\bigcap_{V\in\Sigma} V$ is either empty or simply connected.*

Proof. Let $\mathbf{W}$ be an open star refinement of $\mathbf{U}$. By Theorem A.1, there exists a triangulation K of M that is subordinated to $\mathbf{W}$. We now define $\mathbf{V} = \{St(x,K) \,|\, x \in M\}$. Then $\mathbf{V}$ is an open cover of M. By the preceding lemma, for every $V, V' \in \mathbf{V}$ we have either $V \cap V' = \emptyset$ or $V \cap V' \in \mathbf{V}$. Thus $\bigcap_{V\in\Sigma} V$ is either empty or again an element of $\mathbf{V}$ for every finite $\Sigma \subset \mathbf{V}$. But the elements of $\mathbf{V}$ are simply connected.

It rests to prove that $\mathbf{V}$ is a refinement of $\mathbf{U}$. For every $x \in M$ we have

$$St(x,K) = \bigcup_{\sigma\in K: x\in\sigma} \sigma^o \subset \bigcup_{\sigma\in K: x\in\sigma} \sigma \subset \bigcup_{W\in\mathbf{W}: x\in W} W.$$

The last inclusion comes from the fact that K is subordinated to $\mathbf{W}$. The assertion now follows from the supposition that $\mathbf{W}$ is a star refinement of $\mathbf{U}$. □

Appendix B. Lemma B.1

Lemma B.1. *Let $p \in \mathbb{R}^n$ be a point and let $\{q^i\}_{i\in I}$ be a finite family of smooth real-valued functions defined in a neighbourhood U of p. Suppose that all q^i belong to the same k-jet at p. Let further $\{\lambda^i\}_{i\in I}$ be a family of smooth functions defined in U such that $\sum_{i\in I} \lambda^i \equiv 1$ on U. Then $q = \sum_{i\in I} \lambda^i q^i$ belongs to the same k-jet at p as the functions q^i.*

Proof. Let P be the k-th order Taylor polynomial of q^i at p. Then the derivatives of the functions $\tilde{q}^i = q^i - P$ at p vanish up to order k, and $q = \sum_{i\in I} \lambda^i(\tilde{q}^i + P) = P + \tilde{q}$ with $\tilde{q} = \sum_{i\in I} \lambda^i \tilde{q}^i$. Computing the derivatives of the product $\lambda^i \tilde{q}^i$ at p explicitly using the product rule, it is easily seen that they also vanish up to order k. Hence the derivatives of $\tilde{q}$ at p vanish up to order k, and q belongs to the same k-jet as P, and thus as the functions q^i. □

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