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SCREEN SEMI INVARIANT LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. In this paper, we introduce a new class of ligtlike submanifold called screen semi- invariant (SSI) lightlike submanifolds of a semi Riemannian product manifold. We give examples of such submanifolds and study the geometry of leaves of distributions which are involved in the definition of SSI-lightlike submanifolds. We obtain, necessary and sufficient conditions for the SSI-lightlike submanifold to be locally product manifold. Finally, we give some characterizations for totally umbilical SSI-lightlike and screen anti-invariant lightlike submanifolds of semi-Riemannian product manifolds.

1. Introduction

The geometry of lightlike submanifolds of semi-Riemannian manifolds is developed by K.L. Duggal-A.Bejancu [8] and K.L. Duggal and B. Şahin [4]. The lightlike submanifolds have been studied in various manifolds by many authors, [2], [3], [5], [6], [7]. In [3], K.L. Duggal and B. Şahin introduced a new class of lightlike submanifolds which is called Screen Cauchy Riemannian (SCR) lightlike submanifolds of indefinite Kaehler manifolds. They have shown that, SCR-lightlike submanifolds include invariant (complex) and screen real subcases of lighlike submanifolds. The geometry of submanifolds of a Riemannian product manifold (Semi-Riemannian Product manifold) have been extensively studied by many geometers, [12], [11],[10]. In case Riemannian, the invariant submanifolds and semi invariant submanifolds are investigated by Ximin, L. and Shao, F.-M., [13]. As an analouge of CR-lightlike submanifolds, semi-invariant lightlike submanifolds were introduced by M. Atçeken and E. Kılıç [1]. Therefore, in [9], E.Kılıç and B. Şahin introduced radical anti-invariant lightlike submanifolds of semi-Riemannian product manifold. In this paper, we introduce a new class of lightlike submanifolds of semi-Riemannian product manifolds which is called screen semi invariant (SSI) lightlike manifold and investigate the geometry of such submanifolds.

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In Section 2 and Section 3, we give the basic concepts on lightlike submanifolds and product manifolds which will be used throughout this paper. In section 4, we introduce SSI-lightlike submanifolds and give examples. We investigate the integrability conditions of all the distributions. We also obtain that the SSI-lightlike submanifolds and its leave of the screen distribution are locally product manifolds under some conditions. In section 5, we study totally umbilical SSI-submanifolds and give a condition for its Ricci tensor to be symmetric. We prove that there exist no totally umbilical SSI-lightlike submanifolds in positively or negatively curved (or null sectional curved) semi-Riemannian product manifolds. Finally, in section 6, we study the geometry of screen anti-invariant lightlike submanifolds of semi-Riemannian Product manifolds.

2. Lightlike Submanifolds

In this paper, we use the same notations and terminologies as in [8].

Let $(\overline{M}, \overline{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold with index $q > 0$ and M be a submanifold of n-codimension of $\overline{M}$. If $\overline{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike (degenerate) submanifold of $\overline{M}$. We denote by g the induced metric of $\overline{g}$ on M and suppose that g is degenerate, then for each tangent space T_xM,

$$T_xM^{\perp} = \{U_x \in T_x\overline{M} : g_x(U_x, V_x) = 0, \quad \forall V_x \in T_xM\},$$

is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthonormal distributions. In this case, there exists a subspace

$$Rad(T_xM) = T_xM \cap T_xM^{\perp}$$

which is called Radical subspace. The mapping

$$Rad(TM) : x \in M \longrightarrow Rad(T_xM)$$

defines a smooth distribution on M of rank$(Rad(TM)) = r > 0$, then M is called r-lightlike submanifold and $Rad(TM)$ is called radical distribution on M.

There are four possible cases with respect to the dimension and codimension of M and rank of $Rad(TM)$. We recall that
Case 1) M is called r-lightlike submanifold, if $1 \leq r < min\{m, n\}$.
Case 2) M is called co-isotropic submanifold, if $1 \leq r = n < m$.
Case 3) M is called isotropic submanifold, if $1 \leq r = m < n$.
Case 4) M is called totally lightlike submanifold, if $1 \leq r = m = n$.

For Case 1, there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to $Rad(TM)$ in TM. Therefore, we can write

$$TM = Rad(TM) \perp S(TM). \tag{2.1}$$

As $S(TM)$ is non-degenerate vector subbundle of $T\overline{M}\mid_M$, we put

$$T\overline{M}\mid_M = S(TM) \perp S(TM)^{\perp}, \tag{2.2}$$

where $S(TM)^{\perp}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\overline{M}|_{M}$. If we use the fact that $S(TM)$ and $S(TM)^{\perp}$ are non-degenerate, we have the following orthogonal direct decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}. \tag{2.3}$$

Denote an r-lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$.

Theorem 2.1. [8] *Let* $\left(M, g, S(TM), S(TM^{\perp})\right)$ *be a* r*-lightlike submanifold of a semi-Riemannian manifold* $\left(\overline{M}, \overline{g}\right)$. *Then there exists a complementary vector bundle* $\ell tr(TM)$ *called a lightlike transversal bundle of* $Rad(TM)$ *in* $S(TM^{\perp})^{\perp}$ *and a basis of* $\Gamma\left(\ell tr(TM)|_{U}\right)$ *consists of smooth sections* $\{N_1, ..., N_r\}$ *of* $S(TM^{\perp})^{\perp}\,|_{U}$ *such that*

$$\overline{g}\left(N_i, \xi_j\right) = \delta_{ij}, \quad \overline{g}\left(N_i, N_j\right) = 0, \quad i, j = 1, ...r,$$

where $\{\xi_1, ..., \xi_r\}$ *is a basis of* $\Gamma(Rad(TM)|_{U})$.

Theorem 2.2. [8] *Let* M *be an* r*-lightlike submanifold of a semi-Riemannian manifold* $\overline{M}$. *Then the induced connection* ∇ *is a metric connection if and only if* $Rad(TM)$ *is a parallel distribution w.r.t.* ∇.

We consider the vector bundle

$$tr(TM) = \ell tr(TM) \perp S(TM^{\perp}). \tag{2.4}$$

Thus we have

$$T\overline{M} = TM \oplus tr(TM) = S(TM) \perp S(TM^{\perp}) \perp (Rad(TM) \oplus \ell tr(TM). \tag{2.5}$$

Now, let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ and ∇ be induced connection on M. Then the Gauss and Weingarten formulas are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)\ , \forall X, Y \in \Gamma\left(TM\right) \tag{2.6}$$

and

$$\overline{\nabla}_X V {=} {-}A_V X + \nabla^{\perp}_X V\ , \forall X \in \Gamma\left(TM\right) \tag{2.7}$$

for any $V \in \Gamma\left(tr(TM)\right)$, where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^{\perp}_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. It follows that $\nabla^{\perp}$ is linear connections on $tr(TM)$. Using the projections $L : tr(TM) \longrightarrow \ell tr(TM)$ and $S : tr(TM) \longrightarrow S(TM^{\perp})$, then we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^{\ell}(X, Y) + h^{s}(X, Y) \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\ell}_X N + D^{s}(X, N) \tag{2.9}$$

and

$$\overline{\nabla}_X W = -A_W X + \nabla^{s}_X W + D^{\ell}(X, W), \tag{2.10}$$

for any $X, Y \in \Gamma\left(TM\right), N \in \Gamma\left(ltr(TM)\right)$ and $W \in \Gamma\left(S(TM^{\perp})\right)$, where $h^{l}(X, Y) = Lh(X, Y)$, $h^{s}(X, Y) = Sh(X, Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma\left(TM\right)$, $\nabla^{\ell}_X N$, $D^{\ell}(X, W)$ $\in \Gamma\left(\ell tr(TM)\right)$ and $\nabla^{s}_X W, D^{s}(X, N) \in \Gamma\left(S(TM^{\perp})\right)$.

By using (2.8), (2.9) and (2.10) we obtain

$$\overline{g}\left(h^{s}(X, Y), W\right) + \overline{g}\left(Y, D^{\ell}(X, W)\right) = g(A_W X, Y). \tag{2.11}$$

We denote the projection morphism of TM to the screen distribution $S(TM)$ by P. According to (2.1) we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY) \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*^t}_X \xi \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X, PY), \nabla^{*^t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. It follows that ∇^* and ∇^{*^t} are linear connections on $S(TM)$ and $Rad(TM)$, respectively. Then we have the following equations

$$\overline{g}\left(h^l(X, PY), \xi\right) = g\left(A^*_\xi X, PY\right) \ , \overline{g}\left(h^*(X, PY), N\right) = g\left(A_N X, PY\right) \tag{2.14}$$

$$g\left(A^*_\xi PX, PY\right) = g\left(PX, A^*_\xi PY\right) \ , A^*_\xi \xi = 0 \tag{2.15}$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(\ell tr(TM))$.

In general, the induced connection on lightlike submanifold M is not metric connection. Since $\overline{\nabla}$ is metric connection, ∇g is obtained from (2.6) and (2.8) as

$$(\nabla_X g)(Y, Z) = \overline{g}(h^\ell(X, Y), Z) + \overline{g}(h^\ell(X, Z), Y) \tag{2.16}$$

for any $X, Y, Z \in \Gamma(TM)$.
If $\overline{M}$ is a real space form with constant sectional curvature c, then the Riemannian curvature tensor $\overline{R}$ of $\overline{M}$ is given by

$$\overline{R}(X, Y)Z = c\{\overline{g}(Y, Z)X - \overline{g}(X, Z)Y\}, \tag{2.17}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$.

Now, we recall that the equation of Gauss for the lightlike immersion of M in $\overline{M}$ is given by

$$\begin{aligned} \overline{R}(X, Y)Z &= R(X, Y)Z + A_{h^\ell(X,Z)}Y - A_{h^\ell(Y,Z)}X + (\nabla_X h^\ell)(Y, Z) \\ &- (\nabla_Y h^\ell)(X, Z) + A_{h^s(X,Z)}Y + D^\ell(X, h^s(Y, Z)) \\ &- A_{h^s(Y,Z)}X - D^\ell(Y, h^s(X, Z)) + (\nabla_X h^s)(Y, Z) \\ &- (\nabla_Y h^s(X, Z) + D^s(X, h^\ell(Y, Z)) - D^s(Y, h^\ell(X, Z)) \end{aligned} \tag{2.18}$$

for any $X, Y, Z \in \Gamma(TM)$.
We refer to [8] for the dependence of all the induced geometric objects of M on $\{S(TM), S(TM^\perp)\}$.

3. Semi-Riemannian Product Manifolds

Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2-dimensional semi-Riemannian manifolds with constant indexes $q_1 > 0$, $q_2 > 0$, respectively. Let $\pi : M_1 \times M_2 \longrightarrow M_1$ and $\sigma : M_1 \times M_2 \longrightarrow M_2$ be the projections which are given by $\pi(x, y) = x$

and $\sigma(x,y) = y$ for any $(x,y) \in M_1 \times M_2$. We denote the product manifold by $\overline{M} = (M_1 \times M_2, \overline{g})$, where

$$\overline{g}(X,Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y)$$

for any $X, Y \in \Gamma(T\overline{M})$ and $*$ means tangent mapping. Then we have $\pi_*^2 = \pi_*$, $\sigma_*^2 = \sigma_*$, $\pi_*\sigma_* = \sigma_*\pi_* = 0$ and $\pi_* + \sigma_* = I$, where I is identity transformation. Thus $(\overline{M}, \overline{g})$ is a $(m_1 + m_2)$-dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$ is characterized by M_1 and M_2 which are totally geodesic submanifolds of $\overline{M}$.

Now, if we put $F = \pi_* - \sigma_*$, then we can easily see that $F^2 = I$ and

$$\overline{g}(FX, Y) = \overline{g}(X, FY),$$

for any $X, Y \in \Gamma(T\overline{M})$. Then it can be seen that

$$(\overline{\nabla}_X F)Y = 0, \tag{3.1}$$

for any $X, Y \in \Gamma(T\overline{M})$, that is, F is parallel with respect to $\overline{\nabla}$ [12].

The Riemannian curvature tensor field of $M_1 \times M_2$ satisfied

$$\overline{R}(X,Y)FZ = F\overline{R}(X,Y)Z,$$

for any $X, Y, Z \in \Gamma(TM_1 \times TM_2)$.

Now, suppose that M_1 and M_2 are real space forms with constant sectional c_1 and c_2, respectively. Then the Riemannian curvature tensor $\bar{R}$ of $\bar{M} = M_1(c_1) \times M_2(c_2)$ is given by

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{1}{16}(c_1 + c_2)\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(FY,Z)FX - \bar{g}(FX,Z)FY\} \\ &+ \frac{1}{16}(c_1 - c_2)\{\bar{g}(FY,Z)X - \bar{g}(FX,Z)Y + \bar{g}(Y,Z)FX - \bar{g}(X,Z)FY\}, \end{aligned} \tag{3.2}$$

for any $X, Y, Z \in \Gamma(T\bar{M}))$ [14].

Let M be a submanifold of a Riemannian (or semi-Riemannian) product manifold $\overline{M} = M_1 \times M_2$. If $F(TM) = TM$, then M is called invariant submanifold, if $F(TM) \subset TM^\perp$, then M is called anti-invariant submanifold.

4. SCREEN SEMI INVARIANT LIGHTLIKE SUBMANIFOLDS OF A PRODUCT MANIFOLD

In this section, we introduce *Screen Semi-Invariant* (SSI) submanifolds of semi-Riemannian product manifolds, give examples and investigate the geometry of leaves of distributions.

Definition 4.1. Let $(\overline{M}, \overline{g})$ be a semi-Riemannian product manifold and M be a lightlike submanifold of $\overline{M}$. We say that M is SSI-lightlike submanifold of $\overline{M}$ if the following statements are satisfied:
1) There exists a non-null distribution $D \subseteq S(TM)$ such that

$$S(TM) = D \perp D^\perp, \quad FD = D, \quad FD^\perp \subseteq S(TM^\perp), \quad D \cap D^\perp = \{0\}, \tag{4.1}$$

where $D^\perp$ is orthogonal complementary to D in $S(TM)$.
2) $Rad\,TM$ is invariant with respect to F, that is $FRad(TM) = Rad(TM)$.

Then we have

$$F\ell tr(TM) = \ell tr(TM), \tag{4.2}$$

$$TM = D'\perp D^{\perp}, \quad D' = D\perp Rad(TM). \tag{4.3}$$

Hence it follows that D' is also invariant with respect to F. We denote the orthogonal complement to $FD^{\perp}$ in $S(TM^{\perp})$ by D_0. Then, we have

$$tr(TM) = ltr(TM)\perp FD^{\perp}\perp D_0. \tag{4.4}$$

If $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$, then we say that M is a proper SSI-lightlike submanifold of $\overline{M}$. Hence, for on proper M, we have $\dim(D) \geq 1$, $\dim(D^{\perp}) \geq 1$, $\dim(M) \geq 3$ and $\dim(\overline{M}) \geq 5$. Furthermore, there exists no proper SSI-lightlike hypersurface of a semi-Riemannian product manifold.

If $D = \{0\}$, that is $FS(TM) \subseteq S(TM^{\perp})$, then we say that M is screen anti-invariant lightlike submanifold.

Example 4.1. Let M_1 and M_2 be $\mathbb{R}^3_1$ and $\mathbb{R}^2$, respectively. Then $\overline{M} = M_1 \times M_2$ is a semi-Riemannian product manifold with metric tensor $\overline{g} = \pi^* g_1 + \sigma^* g_2$, where g_1 and g_2 are the standard metric tensors of $\mathbb{R}^3_1$ and $\mathbb{R}^2$ with $(-,+,+)$ and $(+,+)$, π_* and σ_* are the projections of $\Gamma(T\overline{M})$ to $\Gamma(TM_1)$ and $\Gamma(TM_2)$, respectively. Let M be a submanifold of $\overline{M}$ given by equations

$$x^1 = \sqrt{2}u_1 + u_3, \qquad x^2 = u_1 + u_3, \qquad x^3 = u_1 + (\sqrt{2}-1)u_3,$$
$$x^4 = u_2 + (\frac{\sqrt{2}-1}{\sqrt{2}})u_3, \qquad x^5 = u_2 - (\frac{\sqrt{2}-1}{\sqrt{2}})u_3,$$

where u_1, u_2, u_3 are real parameters. Then TM is spanned by $\{U_1, U_2, U_3\}$, where

$$U_1 = \sqrt{2}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \qquad U_2 = \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5},$$
$$U_3 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + (\sqrt{2}-1)\frac{\partial}{\partial x^3} + (\frac{\sqrt{2}-1}{\sqrt{2}})\frac{\partial}{\partial x^4} - (\frac{\sqrt{2}-1}{\sqrt{2}})\frac{\partial}{\partial x^5}.$$

Hence M is a 1-lightlike submanifold with $Rad(TM) = Span\{U_1\}$. $S(TM)$ and $S(TM)^{\perp}$ are spanned by $\{U_2, U_3\}$ and $\{H\}$, respectively, where

$$H = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + (\sqrt{2}-1)\frac{\partial}{\partial x^3} - (\frac{\sqrt{2}-1}{\sqrt{2}})\frac{\partial}{\partial x^4} + (\frac{\sqrt{2}-1}{\sqrt{2}})\frac{\partial}{\partial x^5}.$$

Then the lightlike transversal vector bundle $ltr(TM)$ is spanned by

$$N = -\frac{1}{2\sqrt{2}}\frac{\partial}{\partial x^1} + \frac{1}{4}\frac{\partial}{\partial x^2} + \frac{1}{4}\frac{\partial}{\partial x^3}$$

Therefore, $D = Span\{U_2\}$, $D^{\perp} = Span\{U_3\}$, $D_0 = \{0\}$ and $FRad(TM) = Rad(TM)$, $FD = D$, $FD^{\perp} = S(TM^{\perp})$, $Fltr(TM) = ltr(TM)$. Thus, M is a proper SSI-lightlike submanifold of $\overline{M}$ whit $D' = Span\{U_1, U_2\}$.

Proposition 4.1. *Let M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then M is an invariant lightlike submanifold of $\overline{M}$ if and only if $D^{\perp} = \{0\}$.*

Proof. If M is a invariant lightlike submanifold of $\overline{M}$, then $FTM = TM$ and $D^{\perp} = \{0\}$. Conversely, if $D^{\perp} = \{0\}$, then $FTM = TM$. $\square$

From this Proposition, we have the following Corollary.

Corollary 4.1. *Let M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. If M is a co-isotropic or isotropic or totally lightlike, then M is a invariant lightlike submanifold.*

Example 4.2. Let M_1 and M_2 be $\mathbb{R}^4_2$ and $\mathbb{R}^2_1$ with standard metrics g_1 and g_2, respectively. Consider a submanifold M in $M_1 \times M_2$ given by the equations

$$x_3 = x_1 \cos\alpha - x_5 \sin\alpha, \quad x_4 = -x_1 \sin\alpha - x_5 \cos\alpha, \quad x_6 = \sqrt{2}\, x_5,$$

where (x_1, x_2, x_3, x_4) and (x_5, x_6) are standard coordinate systems of $\mathbb{R}^4_2$ and $\mathbb{R}^2_1$, respectively. Then TM is spanned by

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_3} - \sin\alpha \frac{\partial}{\partial x_4}, \\ Z_2 &= \frac{\partial}{\partial x_2}, \\ Z_3 &= -\sin\alpha \frac{\partial}{\partial x_3} - \cos\alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \sqrt{2}\frac{\partial}{\partial x_6}. \end{aligned}$$

Thus M is a 1-lightlike submanifold with invariant $Rad(TM) = Span\{Z_1\}$. The screen distribution $S(TM) = Span\{Z_2, Z_3\}$ and $D = Span\{Z_2, \}$, $D^\perp = Span\{Z_3\}$. On the other hand $S(TM^\perp)$ is spanned by $W_1 = \sin\alpha \frac{\partial}{\partial x_3} + \cos\alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \sqrt{2}\frac{\partial}{\partial x_6}$ and $W_2 = \sqrt{2}\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}$ and the lightlike transversal bundle $\ell tr(TM)$ is spanned by $N = -\frac{1}{2}\frac{\partial}{\partial x_1} + \frac{1}{2}\cos\alpha \frac{\partial}{\partial x_3} - \frac{1}{2}\sin\alpha \frac{\partial}{\partial x_4}$. Hence, $FD = D$, $FD^\perp \subset S(TM^\perp)$ and M is a proper SSI-lightlike submanifold of $M_1 \times M_2$.

Let M be a lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then, for each $X \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, we put

$$FX = fX + \omega X, \quad FV = BV + CV \tag{4.5}$$

where fX, BV and ωX, CV are the tangent and the transversal parts of FX and FV. If M is a SSI-lightlike submanifold of $\overline{M}$, then $fX \in \Gamma(D')$ and $\omega X \in \Gamma(FD^\perp)$, respectively.

Theorem 4.1. *Let M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then the screen distribution of M is integrable if and only if the following three conditions are satisfied*

$$\overline{g}(A_N Y, FX) = \overline{g}(A_N X, FY), \quad X, Y \in \Gamma(D), \tag{4.6}$$

$$\overline{g}(A_N Y, FX) = -\overline{g}(D^s(X, N), FY), \quad X \in \Gamma(D),\ Y \in \Gamma(D^\perp), \tag{4.7}$$

$$\overline{g}(D^s(X, N), FY) = \overline{g}(D^s(Y, N), FX), \quad X, Y \in \Gamma(D^\perp). \tag{4.8}$$

Theorem 4.2. *Let M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then the distribution D' is integrable if and only if $h(X, FY) = h(FX, Y)$, for all $X, Y \in \Gamma(D')$.*

These last two theorems are similar to Theorem 3.3 and Theorem 3.4 given in [3], respectively.

Theorem 4.3. *Let M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then the distribution $D^\perp$ is integrable if and only if $A_{FZ}W = A_{FW}Z$, for any $W, Z \in \Gamma(D^\perp)$.*

Proof. Since F is parallel with respect to $\overline{\nabla}$, from (2.8), (2.10) and (4.5), we get

$$-A_{FW}Z + D^{\ell}(Z, FW) + \nabla^s_Z FW = f\nabla_Z W + \omega\nabla_Z W + Bh(Z, W) + Ch(Z, W)$$

for all $W, Z \in \Gamma(D^{\perp})$. Taking tangential part of this equation, we have

$$-A_{FW}Z = f\nabla_Z W + Bh(Z, W). \tag{4.9}$$

By replacing role of vector fields W and Z in (4.9), by a direct calculation, we obtain

$$A_{FZ}W - A_{FW}Z = f[Z, W].$$

Since $[Z, W] = f[Z, W] + \omega[Z, W]$, $D^{\perp}$ is integrable if and only if $f[Z, W] = 0$ and we complete the proof. □

Corollary 4.2. *Let M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. If the distribution $D^{\perp}$ is integrable, then the following statements holds.*
a) A_N is self-adjoint on $D^{\perp}$ with respect to g, for any $N \in \Gamma(\ell tr(TM))$.
b) $A_{FZ}W$ has no components in D, for any $Z, W \in \Gamma(D^{\perp})$.

Proof. Suppose that $D^{\perp}$ is integrable. Then, $A_{FZ}W = A_{FW}Z$, for any $Z, W \in \Gamma(D^{\perp})$. Since $\overline{g}(FW, FN) = \overline{g}(W, N) = 0$ and $\overline{\nabla}$ is a metric connection, we obtain

$$\overline{g}(A_{FW}Z, FN) = -g(W, A_N Z), \quad \overline{g}(A_{FZ}W, FN) = -g(Z, A_N W),$$

for any $N \in \Gamma(\ell tr(TM))$. From this last two equations, we have $g(Z, A_N W) = g(W, A_N Z)$.

Since $D^{\perp}$ is integrable, $\overline{g}([Z, W], FX) = 0$, for any $Z, W \in \Gamma(D^{\perp})$, $X \in \Gamma(D)$. From (2.11), we have

$$\overline{g}(h^s(Z, X), FW) = g(A_{FW}Z, X). \tag{4.10}$$

Using (2.8) and (2.10), we obtain

$$\overline{g}(h^s(X, Z), FW) = -g(A_{FZ}X, W). \tag{4.11}$$

From (4.10) and (4.11), we have

$$g(A_{FW}Z, X) = -g(A_{FZ}X, W). \tag{4.12}$$

Since $\overline{\nabla}$ is a metric connection and $\overline{g}(Z, FX) = 0$ and using to symmetric of h^s, we obtain

$$g(A_{FZ}W, X) = g(A_{FZ}X, W). \tag{4.13}$$

From (4.12) and (4.13), we have

$$g(A_{FW}Z, X) = -g(A_{FZ}W, X). \tag{4.14}$$

On the other hand, we get

$$\begin{aligned} \overline{g}([Z, W], FX) &= g(A_{FZ}W, X) - g(A_{FW}Z, X) \\ &= 2g(A_{FZ}W, X) = 0. \end{aligned}$$

Thus we have (b). □

Theorem 4.4. *Let M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then the distribution D is integrable if and only if the following statements holds:*
a) A_N is self adjoint on D, for any $N \in \Gamma(\ell tr(TM))$.
b) $g(FY, A_U X) = g(FX, A_U Y)$, $X, Y \in \Gamma(D)$ and $U \in \Gamma(FD^\perp)$.

Proof. Suppose that D is integrable. Then, $[X,Y] \in \Gamma(D)$, that is $\overline{g}([X,Y],N) = 0$ and $\overline{g}([X,Y],FU) = 0$, $X,Y \in \Gamma(D)$, $N \in \Gamma(\ell tr(TM))$ and $U \in \Gamma(FD^\perp)$. Thus we have

$$\overline{g}([X,Y],N) = g(Y, A_N X) - g(X, A_N Y), \tag{4.15}$$

$$\overline{g}([X,Y],FU) = g(FY, A_U X) - g(FX, A_U Y). \tag{4.16}$$

Hence, from (4.15) and (4.16), we obtain (a) and (b), respectively.
Conversely, (a) and (b) are satisfied. From (4.15) and (4.16), we have $[X,Y] \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$. □

Theorem 4.5. *Let M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then the following assertions are equivalent:*
a) $S(TM)$ is parallel.
b) A_{FZ} is $S(TM)$-valued for $Z \in \Gamma(D^\perp)$.
c) $D^s(X, FN)$ is D_0-valued, for $X \in \Gamma(TM)$, $N \in \Gamma(\ell tr(TM))$.

Proof. $S(TM)$ is parallel if and only if $\overline{g}(\overline{\nabla}_X Z, N) = 0$, for any $X, Z \in \Gamma(S(TM))$ and $N \in \Gamma(\ell tr(TM))$. Since $\overline{g}(\nabla_X Z, N) = \overline{g}(\overline{\nabla}_X Z, N)$ and F is parallel with respect to $\overline{\nabla}$, we obtain

$$\overline{g}(\overline{\nabla}_X Z, N) = \overline{g}(\overline{\nabla}_X FZ, FN). \tag{4.17}$$

If $Z \in \Gamma(D^\perp)$, then $\overline{g}(A_{FZ}X, N) = 0$, that implies (b). Since $\overline{\nabla}$ is a Levi-Civita connection, from (4.17), we get $\overline{g}(FZ, D^s(X, FN)) = 0$. Thus we have (c). □

Theorem 4.6. *Suppose that the secreen ditribution of M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$ is integrable. Then the following statements are equivalent.*
1) The distribution D defines a totally geodesic folation in $S(TM)$.
2) $Bh^s(X,Y) = 0$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.
3) $A_{FZ}X$ has no components in D, for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^\perp)$.

Proof. We assume that D is totally geodesic in $S(TM)$. Then $\nabla^*_X Y \in \Gamma(D)$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Thus we have $g(\nabla^*_X FY, Z) = 0$, for any $Z \in \Gamma(D^\perp)$. From (2.6) and (2.12), we get

$$g(\nabla^*_X FY, Z) = \overline{g}(\overline{\nabla}_X Y, FZ) = 0.$$

From (2.8), we have

$$\overline{g}(h^s(X,Y), FZ) = 0.$$

Hence we obtain (2). Since $\overline{\nabla}$ is a Levi-Civita connection, we get

$$\overline{g}(\overline{\nabla}_X Y, FZ) = g(A_{FZ}X, Y) = 0.$$

Thus we have (3). □

It is easy cheak that, D is totally geodesic in $S(TM)$ if and only if $D^\perp$ is totally geodesic in $S(TM)$. So we have following corollary.

Corollary 4.3. *Suppose that the secreen ditribution of M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$ is integrable. Then $S(TM)$ is a locally product manifold if and only if $A_{FZ}X$ has no components in D, for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$.*

Theorem 4.7. *Let M be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then M is a locally product manifold if and only if $\nabla f = 0$*

Proof. Let M be alocally product manifold. Then the leaves of distributions D' and $D^{\perp}$ are both totally geodesic in M. Since $\overline{\nabla} F = 0$ and from (2.6) and (2.7) we get

(4.18) $\nabla_X fY + h(X, fY) = f\nabla_X Y + \omega\nabla_X Y + Bh(X,Y) + Ch(X,Y),$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D')$. Since D' is totally geodesic in M, $\nabla_X Y \in \Gamma(D)$. Then, for any $U \in \Gamma(FD^{\perp})$, we have

$$\overline{g}(\overline{\nabla}_X FY, U) = \overline{g}(\overline{\nabla}_X Y, FU) = g(\nabla_X Y, FU) = 0.$$

Hence we get $Bh(X,Y) = 0$. Comparing the tangential and transversal parts with respect to D of equation (4.18), $\nabla_X fY = f\nabla_X Y$, that is $(\nabla_X f)Y = 0$.

Similarly,

(4.19) $-A_{FZ}X + \nabla^{\perp}_X FZ = f\nabla_X Z + \omega\nabla_X Z + Bh(X,Z) + Ch(X,Z)$

for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$. From (4.19), we have

$$-A_{FZ}X = f\nabla_X Z + Bh(X,Z).$$

For any $Y \in \Gamma(D')$, we get

$$g(f\nabla_X Z, Y) = -g(A_{FZ}X, Y) = -g(\nabla_X fY, Z) = 0,$$

that is $f\nabla_X Z = 0$, which implies that $(\nabla_X f)Z = 0$.

Conversely, we suppose that $\nabla f = 0$. Then we have $\nabla_X fY = f\nabla_X Y$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D')$. Thus $\nabla_X fY \in \Gamma(D)$ and the distribution D' is totally geodesic in M. Similarly, $\nabla_X fZ = f\nabla_X Z = 0$, for any $X \in \Gamma(TM)$, $Z \in \Gamma(D^{\perp})$ and $D^{\perp}$ is totally geodesic in M. □

5. Totally Umbilical SSI-Lightlike Submanifolds

In this section, we study totally umbilical SSI-Ligthlike submanifolds of a semi-Riemannian product manifold.

Definition 5.1. [7] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called totally umbilical in $\overline{M}$, if there is a smooth transversal vector field $\mathcal{H} \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that, for all $X, Y \in \Gamma(TM)$,

$$h(X,Y) = g(X,Y)\mathcal{H}. \tag{5.1}$$

It is known that M is totally umbilical if and only if on each coordinate neighborhood $\mathcal{U}$, there exist smooth vector fields $\mathcal{H}^{\ell} \in \Gamma(\ell tr(TM))$ and $\mathcal{H}^{s} \in \Gamma(S(TM^{\perp}))$ such that

$$h^{\ell}(X,Y) = g(X,Y)\mathcal{H}^{\ell}, \quad h^{s}(X,Y) = g(X,Y)\mathcal{H}^{s}, \tag{5.2}$$

for any $X, Y \in \Gamma(TM)$.

Corollary 5.1. *Let M be a totally umbilical SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then the distribution $D^{\perp}$ is totally geodesic in M.*

Proof. Let $X, Y \in \Gamma(D^{\perp})$. Then we have

$$\nabla_X Y = \widetilde{\nabla}_X Y + \widetilde{h}(X, Y),$$

where $\widetilde{\nabla}_X Y \in \Gamma(D^{\perp})$ and $\widetilde{h}(X, Y) \in \Gamma(D')$. Since D' is a invariant distribution, for any $Z \in \Gamma(D')$, we have $FZ = fZ \in \Gamma(D')$. Since $\overline{\nabla}$ is a Levi-Civita connection, it can be easily calculated

$$\begin{aligned} g(\widetilde{h}(X, Y), FZ) &= g(\nabla_X Y, FZ) \\ &= \overline{g}(\overline{\nabla}_X Y, FZ) \\ &= -\overline{g}(FY, h^s(X, Z)). \end{aligned}$$

Since $X \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D')$, from (5.2), we have

$$h^s(X, Z) = 0,$$

and we have assertion of corollary. □

Theorem 5.1. *Let M be a totally umbilical SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then the following assertions are equivalent:*
1) The distribution D' is totally geodesic in M.
2) A_{FZ} is $D^{\perp}$-valued, for any $Z \in \Gamma(D^{\perp})$.
3) $\mathcal{H}^s \in \Gamma(D_0)$.

Proof. Let $X, Y \in \Gamma(D')$. Then we have

$$\nabla_X Y = \nabla'_X Y + h'(X, Y),$$

where $\nabla'_X Y \in \Gamma(D')$ and $h'(X, Y) \in \Gamma(D^{\perp})$. Since $\overline{\nabla}$ is a Levi-Civita connection, it can be easily calculated

$$\begin{aligned} g(h'(X, FY), Z) &= \overline{g}(h^s(X, Y), Z) \\ &= g(FY, A_{FZ} X), \end{aligned}$$

for any $Z \in \Gamma(D^{\perp})$. Thus we have (1)-(3). □

From Corollary 5.1 and Theorem 5.1, we have the following theorem.

Theorem 5.2. *Let M be a totally umbilical SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then M is a locally product manifold if and only if $\mathcal{H}^s \in \Gamma(D_0)$.*

Theorem 5.3. *Let M be a totally umbilical SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1(c_1) \times M_2(c_2)$. Then, the Ricci tensor on M is symmetric if and only if $A_{\mathcal{H}^\ell}$ is self adjoint on M.*

Proof. The Ricci tensor of a lightlike submanifold is given by

$$Ric(X, Y) = \sum_{i=1}^{m} \varepsilon_i g(R(e_i, X)Y, e_i) + \sum_{j=1}^{r} \overline{g}(R(\xi_j, X)Y, N_j),$$

for any $X, Y \in \Gamma(TM)$, where $\{e_1, ..., e_m\}$ is a orthonormal basis of $\Gamma(S(TM))$, $\{\xi_1, ..., \xi_r\}$ and $\{N_1, ..., N_r\}$ are lightlike basis of $\Gamma(Rad\ TM)$ and $\Gamma(\ell tr(TM))$, respectively and $\overline{g}(N_i, \xi_j) = \delta_{ij}$, for any $i, j \in \{1, ..., r\}$. From (3.2) and (18), we obtain

$$\begin{aligned} Ric(X,Y) - Ric(Y,X) &= -g(A_{\mathcal{H}^\ell}X,Y) + g(A_{\mathcal{H}^\ell}Y,X) \\ &- g(A_{\mathcal{H}^s}X,Y) + g(A_{\mathcal{H}^s}Y,X). \end{aligned}$$

Suppose that Ricci tensor is symmetric on M. If $X, Y \in \Gamma(Rad\,TM)$, then we have

$$g(A_{\mathcal{H}^\ell}X,Y) = g(A_{\mathcal{H}^\ell}Y,X) = g(A_{\mathcal{H}^s}X,Y) = g(A_{\mathcal{H}^s}Y,X) = 0.$$

If $X \in \Gamma(Rad\,TM)$ and $Y \in \Gamma(S(TM))$, from (2.11) we have

$$g(A_{\mathcal{H}^s}X,Y) = g(A_{\mathcal{H}^s}Y,X) = 0.$$

If $X, Y \in \Gamma(S(TM))$, then from (2.11), we get

$$g(A_{\mathcal{H}^s}X,Y) = g(X,Y)g(\mathcal{H}^s,\mathcal{H}^s),$$

that is $-g(A_{\mathcal{H}^\ell}X,Y) + g(A_{\mathcal{H}^\ell}Y,X) = 0$. Thus we have our assertion. □

Theorem 5.4. *There exist no totally umbilical proper SSI-lightlike submanifold with $dim(D) \geq 2$ in any negatively or positively curved (and also null sectional curved) semi-Riemannian product manifold.*

Proof. Suppose that M is totally umbilical proper SSI-lightlike submanifold in semi-Riemannian product manifold $\overline{M}(c)$ with $c \neq 0$. From (2.19), for $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$, we have

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)X,Y) &= \overline{g}(\overline{R}(X,Y)FX,FY) \\ &= \overline{g}((\nabla_X h^s)(Y,FX),FY) - \overline{g}((\nabla_Y h^s)(X,FX),FY). \end{aligned}$$

From (5.2), we get

$$(\nabla_X h^s)(Y,FX) = -(g(\nabla_X Y,FX) + g(Y,\nabla_X FX))\mathcal{H}^s.$$

Since $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$, we have $\overline{g}(FX,Y) = 0$. Since $\overline{g}$ is parallel with respect to $\overline{\nabla}$, we get

$$0 = X\overline{g}(Y,FX) = g(\nabla_X Y,FX) + g(Y,\nabla_X FX).$$

Since $dim(D) \geq 2$, we chose $X \in \Gamma(D)$ such that $g(X,FX) = 0$. From (5.2), we obtain

$$(\nabla_Y h^s)(X,FX) = -2g(\nabla_Y X,FX)\mathcal{H}^s.$$

Therefore,

$$0 = Y\overline{g}(X,FX) = 2g(\nabla_Y X,FX).$$

Hence, $\overline{g}(\overline{R}(X,Y)X,Y) = 0$ which is a contradiction. Similarly, it can be proved for the null sectional curved case. □

6. Screen Anti-Invariant Lightlike Submanifolds

In this section, we will investigate the screen anti-invariant lightlike submanifolds of semi-Riemannian product manifolds.

Let M be a screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$. Then we have

$$S(TM^\perp) = FS(TM) \perp D_0.$$

We say that M is a proper screen anti-invariant lightlike submanifold, if $S(TM) \neq \{0\}$ and $D_0 \neq \{0\}$. Thus we have the following proposition.

Proposition 6.1. *There exist no proper screen anti-invariant co-isotropic, isotropic or totally lightlike submanifold of a semi-Riemannian product manifold* $\overline{M}$.

Example 6.1. Consider in $\mathbb{R}^3_1 \times \mathbb{R}^4_1$ the submanifold M given by

$$x^1 = u_1 + u_2, x^2 = u_1 + u_2, x^3 = u_3, \quad y^1 = u_1 - u_2, y_2 = u_1 - u_2, y^3 = u_3, y^4 = 0,$$

where (x^1, x^2, x^3) and (y^1, y^2, y^3, y^4) are standard coordinate systems of $\mathbb{R}^3_1$, respectively, and $\mathbb{R}^4_1$ and u_1, u_2, u_3 are real parameters. Then we have

$$\begin{aligned} TM \quad = \quad & Span\{U_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2},\ U_2 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}, \\ & U_3 = \frac{\partial}{\partial x^3} + \frac{\partial}{\partial y^3}\}. \end{aligned}$$

The radical distribution $Rad(TM)$ is spanned by $\{U_1, U_2\}$ and the screen distribution $S(TM)$ is spanned by U_3. Hence M is a 2-lightlike submanifold of $\mathbb{R}^3_1 \times \mathbb{R}^4_1$. Take

$$S(TM^{\perp}) = \{V_1 = \frac{\partial}{\partial x^3} - \frac{\partial}{\partial y^3},\ V_2 = \frac{\partial}{\partial y^4}\},$$

and by the direct calculations we get

$$\begin{aligned} \ell tr(TM) = Span\{N_1 = -\frac{1}{2}\{2\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}\}, \\ N_2 = -\frac{1}{2}\{2\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - 2\frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}\}\}. \end{aligned}$$

We easily check that, $Rad(TM)$ and $\ell tr(TM)$ are invariant distributions with respect to F and $FS(TM) \subset S(TM^{\perp})$, where $D_0 = Span\{V_1\}$. Thus M is a screen anti-invariant lightlike submanifold.

Let M be a screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$.Then, for any $X \in \Gamma(TM)$, we can write

$$FX = \overline{f}X + \varpi X, \tag{6.1}$$

where $\overline{f}X \in \Gamma(Rad(TM))$ and $\varpi X \in \Gamma(FS(TM))$. Similarly, for any $V \in \Gamma(tr(TM))$, we can write

$$FV = \overline{B}V + \overline{C}V, \tag{6.2}$$

where $\overline{B}V \in \Gamma(S(TM))$ and $\overline{C}V \in \Gamma(tr(TM))$.

Theorem 6.1. *Let M be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if $h^s(X, \xi') \in \Gamma(D_0)$, for any $\xi' \in \Gamma(Rad\, TM)$, $X \in \Gamma(TM)$.*

Proof. If $\xi \in \Gamma(Rad(TM))$, then there exists a $\xi' \in \Gamma(Rad\, TM)$ such that $\xi = F\xi'$. From (3.1) and Gauss formula, we get

$$\nabla_X \xi + h(X, \xi) = \overline{f}\nabla_X \xi' + \varpi \nabla_X \xi'^{\prime\ell}(X, \xi') + \overline{B}h^s(X, \xi') + \overline{C}h^s(X, \xi'),$$

for any $X \in \Gamma(TM)$. If we take tangential component of this equation, we have

$$\nabla_X \xi = \overline{f}\nabla_X \xi' + \overline{B}h^s(X, \xi').$$

Thus, the radical distribution $Rad(TM)$ is a parallel distribution if and only if $h^s(X, \xi') \in \Gamma(D_0)$. From Theorem 2.2 we have the assertion of the theorem. $\square$

Theorem 6.2. *Let M be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the following assertion equivalent:*
1) *$S(TM)$ is integrable.*
2) *For any $X, Y \in \Gamma(S(TM))$, $N \in \Gamma(\ell tr(TM))$, $\overline{g}(A_{FY}X, N) = \overline{g}(A_{FX}Y, N)$.*
3) *$\overline{g}(FY, D^s(X, N)) = \overline{g}(FX, D^s(Y, N))$.*

Proof. Suppose that $S(TM)$ is integrable. Then we have $\overline{g}([X, Y], FN) = 0$, for any $X, Y \in \Gamma(S(TM))$, $N \in \Gamma(\ell tr(TM))$. From (2.6) and (3.1) we have $\overline{g}(A_{FY}X, N) = \overline{g}(A_{FX}Y, N)$. Since $\overline{\nabla}$ is a metric connection, we get $\overline{g}(A_{FY}X, N) = \overline{g}(FY, D^s(X, N))$ and (3) is satisfied. Since $\overline{g}([X, Y], FN) = \overline{g}(FY, D^s(X, N)) - \overline{g}(FX, D^s(Y, N))$, (3)$\Rightarrow$(1). □

Theorem 6.3. *Let M be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the radical distribution integrable if and only if*

$$h^s(\xi, F\xi') = h^s(F\xi, \xi'),$$

for any $\xi, \xi' \in \Gamma(Rad(TM))$.

Proof. For any $\xi, \xi' \in \Gamma(Rad(TM))$ and $U \in \Gamma(FS(TM))$, from (2.8) and (3.1), we get

$$\overline{g}([\xi, \xi'], FU) = \overline{g}(h^s(\xi, F\xi') - h^s(F\xi, \xi'), U).$$

This the assertion of the theorem. □

Theorem 6.4. *Let M be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the following assertion equivalent:*
1) *The screen distribution $S(TM)$ defines a totally geodesic foliation in M.*
2) *A_{FY} is valued $S(TM)$, for all $Y \in \Gamma(S(TM))$.*
3) *For any $X \in \Gamma(TM)$ and $N \in \Gamma(\ell tr(TM))$, $D^s(X, N) \in \Gamma(D_0)$.*

Proof. Suppose that $S(TM)$ is totally geodesic. Then, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(S(TM))$, $\nabla_X Y \in \Gamma(S(TM))$. Thus we have $\overline{g}(\nabla_X Y, FN) = \overline{g}(\overline{\nabla}_X FY, N) = 0$ and (2) is satisfied. Since $\overline{\nabla}$ is a metric connection, we get $\overline{g}(\overline{\nabla}_X FY, N) = -\overline{g}(FY, D^s(X, N)) = 0$ and $D^s(X, N) \in \Gamma(D_0)$. This is complete of proof. □

Theorem 6.5. *Let M be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the screen distribution is a parallel distribution in M if and only if $A_{\varpi Y}$ is $S(TM)$ valued.*

Proof. $S(TM)$ is parallel if and only if $\overline{g}(\nabla_X Y, FN) = 0$, for any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(\ell tr(TM))$. Since $\overline{g}(\nabla_X Y, FN) = \overline{g}(\overline{\nabla}_X FY, N)$, we obtain $\overline{g}(\nabla_X Y, FN) = -g(A_{\varpi Y}X, N)$. Thus we have the assertion of the theorem. □

Theorem 6.6. *Let M be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is a locally product manifold if and only if $\overline{f}$ is parallel with respect to induced connection ∇, that is, $\nabla\overline{f} = 0$.*

Proof. We suppose that M is a locally product manifold. Then the leaves of the distributions of $Rad(TM)$ and $S(TM)$ are totally geodesic in M. Thus $\nabla_Z \overline{f}\xi \in$

$\Gamma(Rad(TM))$, for any $Z \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Since $F\xi = \overline{f}\xi$, from (3.1) we get

$$\begin{aligned} 0 &= \overline{\nabla}_Z F\xi - F(\overline{\nabla}_Z \xi) \\ &= \nabla_Z \overline{f}\xi - \overline{f}(\nabla_Z \xi) + h(Z, \overline{f}\xi) - Fh(Z, \xi). \end{aligned}$$

If we take tangential component of this equation, we get $(\nabla_Z \overline{f})\xi = 0$. For any $X \in \Gamma(S(TM))$, $\nabla_Z \overline{f}X = 0$ and $\overline{f}(\nabla_Z X) = 0$. Thus we have $\overline{f}(\nabla_Z X) = 0$ and $\overline{f}$ is parallel.

Now suppose that $\overline{f}$ is parallel with respect to ∇. Then

$$\nabla_Z \overline{f}X = \overline{f}(\nabla_Z X)$$

for any $X, Z \in \Gamma(TM)$. If $X \in \Gamma(Rad\ TM)$, then we have $\nabla_Z \overline{f}X \in \Gamma(Rad(TM))$ and $\Gamma(Rad(TM))$ is totally geodesic in M. If $X \in \Gamma(S(TM))$, then we have $\overline{f}X = 0$ and $\overline{f}(\nabla_Z X) = 0$, that is $\nabla_Z X \in \Gamma(S(TM))$. □

Now, let M be totally umbilical proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then, from (2.6) and (5.1) we have

$$\overline{\nabla}_X \xi = \nabla_X \xi,$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Since $Rad(TM)$ is invariant distribution w.r.t. F, there exists a $\xi' \in \Gamma(Rad(TM))$ such that $\xi = F\xi'$. From above equation and (3.1), we get

$$\overline{\nabla}_X \xi = \overline{f}\nabla_X \xi'. \tag{6.3}$$

Since $\overline{f}\nabla_X \xi' \in \Gamma(Rad(TM))$, then $\overline{\nabla}_X \xi \in \Gamma(Rad(TM))$, i.e. the radical distribution is a parallel distribution in M. From Theorem 2.2, we have following corollary.

Corollary 6.1. *Let M be totally umbilical screen proper anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the induced connection ∇ is a metric connection.*

Corollary 6.2. *Let M be totally umbilical proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the radical distribution defines a totally geodesic foliation in M.*

Proof. The radical distribution defines a totally geodesic foliation if and only if $\nabla_{\xi_1}\xi \in \Gamma(Rad(TM))$, for any $\xi_1, \xi \in \Gamma(Rad(TM))$. If we take ξ_1 for X in equation (6.3), then we have $\nabla_{\xi_1}\xi \in \Gamma(Rad(TM))$. □

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