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ON THE CURVATURE OF INVARIANT KROPINA METRICS

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Abstract. In the present article we compute the flag curvature of a special type of invariant Kropina metrics on homogeneous spaces.

1. Introduction

Let M be a smooth $n-$dimensional manifold and TM be its tangent bundle. A Finsler metric on M is a non-negative function $F : TM \longrightarrow \mathbb{R}$ which has the following properties:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$,
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$,
(3) the $n \times n$ Hessian matrix $[g_{ij}(x, y)] = [\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$ is positive definite at every point $(x, y) \in TM^0$.

For a smooth manifold M suppose that g and b are a Riemannian metric and a 1-form respectively as follows:

$$g = g_{ij}dx^i \otimes dx^j \tag{1.1}$$

$$b = b_i dx^i. \tag{1.2}$$

An important family of Finsler metrics is the family of $(\alpha, \beta)-$metrics which is introduced by M. Matsumoto (see [5]) and has been studied by many authors. An interesting and important example of such metrics is the Kropina metrics with the following form:

$$F(x, y) = \frac{\alpha(x, y)^2}{\beta(x, y)}, \tag{1.3}$$

where $\alpha(x, y) = \sqrt{g_{ij}(x)y^i y^j}$ and $\beta(x, y) = b_i(x)y^i$.

In a natural way, the Riemannian metric g induces an inner product on any cotangent space T^*_xM such that $< dx^i(x), dx^j(x) >= g^{ij}(x)$. The induced inner

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product on T_x^*M induces a linear isomorphism between T_x^*M and T_xM (for more details see [3].). Then the 1-form b corresponds to a vector field $\tilde{X}$ on M such that

$$g(y, \tilde{X}(x)) = \beta(x, y). \tag{1.4}$$

Therefore we can write the Kropina metric $F = \frac{\alpha^2}{\beta}$ as follows:

$$F(x, y) = \frac{\alpha(x, y)^2}{g(\tilde{X}(x), y)}. \tag{1.5}$$

Flag curvature, which is a generalization of the concept of sectional curvature in Riemannian geometry, is one of the fundamental quantities which associates with a Finsler space. Flag curvature is computed by the following formula:

$$K(P, Y) = \frac{g_Y(R(U, Y)Y, U)}{g_Y(Y, Y).g_Y(U, U) - g_Y^2(Y, U)}, \tag{1.6}$$

where $g_Y(U, V) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}(F^2(Y + sU + tV))|_{s=t=0}$, $P = span\{U, Y\}$, $R(U, Y)Y = \nabla_U \nabla_Y Y - \nabla_Y \nabla_U Y - \nabla_{[U,Y]} Y$ and ∇ is the Chern connection induced by F (see [1] and [9].).
In general, the computation of the flag curvature of Finsler metrics is very difficult, therefore it is important to find an explicit and applicable formula for the flag curvature. In [2], we have studied the flag curvature of invariant Randers metrics on naturally reductive homogeneous spaces and in [7] we generalized this study on a general homogeneous space. Also in [8] we considered $(\alpha, \beta)-$metrics of the form $\frac{(\alpha+\beta)^2}{\alpha}$ and gave the flag curvature of these metrics. In this paper we study the flag curvature of invariant Kropina metrics on homogeneous spaces.

2. Flag curvature of invariant Kropina metrics on homogeneous spaces

Let G be a compact Lie group, H a closed subgroup, and g_0 a bi-invariant Riemannian metric on G. Assume that $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively. The tangent space of the homogeneous space G/H is given by the orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to g_0. Each invariant metric g on G/H is determined by its restriction to $\mathfrak{m}$. The arising Ad_H-invariant inner product from g on $\mathfrak{m}$ can extend to an Ad_H-invariant inner product on $\mathfrak{g}$ by taking g_0 for the components in $\mathfrak{h}$. In this way the invariant metric g on G/H determines a unique left invariant metric on G that we also denote by g. The values of g_0 and g at the identity are inner products on $\mathfrak{g}$. We denote them by $< ., . >_0$ and $< ., . >$. The inner product $< ., . >$ determines a positive definite endomorphism ϕ of $\mathfrak{g}$ such that $< X, Y >=< \phi X, Y >_0$ for all $X, Y \in \mathfrak{g}$.
T. Püttmann has shown that the curvature tensor of the invariant metric $< ., . >$ on the compact homogeneous space G/H is given by

$$\begin{aligned} < R(X,Y)Z,W > &= -\{\frac{1}{2}(< B_-(X,Y),[Z,W] >_0 + < [X,Y],B_-(Z,W) >_0) \\ &+ \frac{1}{4}(< [X,W],[Y,Z]_{\mathfrak{m}} > - < [X,Z],[Y,W]_{\mathfrak{m}} > \\ &- 2 < [X,Y],[Z,W]_{\mathfrak{m}} >) + (< B_+(X,W),\phi^{-1}B_+(Y,Z) >_0 \\ &- < B_+(X,Z),\phi^{-1}B_+(Y,W) >_0)\}, \end{aligned} \tag{2.1}$$

where B_+ and B_- are defined by

$$\begin{aligned} B_+(X,Y) &= \frac{1}{2}([X,\phi Y]+[Y,\phi X]), \\ B_-(X,Y) &= \frac{1}{2}([\phi X,Y]+[X,\phi Y]), \end{aligned}$$

and $[.,.]_{\mathfrak{m}}$ is the projection of $[.,.]$ to $\mathfrak{m}$.(see [6].).

Notice. We added a minus to the Püttmann's formula because our definition of the curvature tensor R is different from the Püttmann's definition in a minus sign.

Theorem 2.1. *Let $G, H, \mathfrak{g}, \mathfrak{h}, g, g_0$ and ϕ be as above. Assume that $\tilde{X}$ is an invariant vector field on G/H and $X := \tilde{X}_H$. Suppose that $F = \frac{\alpha^2}{\beta}$ is the Kropina metric arising from g and $\tilde{X}$ such that its Chern connection coincides to the Levi-Civita connection of g. Suppose that (P,Y) is a flag in $T_H(G/H)$ such that $\{Y,U\}$ is an orthonormal basis of P with respect to $< .,. >$. Then the flag curvature of the flag (P,Y) in $T_H(G/H)$ is given by*

$$K(P,Y)=\frac{3<U,X><R(U,Y)Y,X>+2<Y,X><R(U,Y)Y,U>}{2(\frac{<U,X>}{<Y,X>})^2+2}, \tag{2.2}$$

where

$$\begin{aligned} <R(U,Y)Y,X> &= -\frac{1}{4}(<[\phi U,Y]+[U,\phi Y],[Y,X]>_0 \\ &\quad +<[U,Y],[\phi Y,X]+[Y,\phi X]>_0)-\frac{3}{4}<[Y,U],[Y,X]_{\mathfrak{m}}> \\ &\quad -\frac{1}{2}<[U,\phi X]+[X,\phi U],\phi^{-1}([Y,\phi Y])>_0 \\ &\quad +\frac{1}{4}<[U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi X]+[X,\phi Y])>_0, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} <R(U,Y)Y,U> &= -\frac{1}{2}<[\phi U,Y]+[U,\phi Y],[Y,U]>_0 \\ &\quad -\frac{3}{4}<[Y,U],[Y,U]_{\mathfrak{m}}>-<[U,\phi U],\phi^{-1}([Y,\phi Y])>_0 \\ &\quad +\frac{1}{4}<[U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi U]+[U,\phi Y])>_0 . \end{aligned} \tag{2.4}$$

Proof. The Chern connection of F coincides on the Levi-Civita connection of g. Therefore the Finsler metric F and the Riemannian metric g have the same curvature tensor. We denote it by R.
By using the definition of $g_Y(U,V)$ and some computations for F we have:

$$\begin{aligned} g_Y(U,V) &= \frac{1}{g^4(Y,X)}\{(2g(Y,U)g(Y,X)-g(U,X)g(Y,Y))(2g(Y,V)g(Y,X) \\ &\quad -g(V,X)g(Y,Y))+g(Y,Y)(g(Y,X)(2g(U,V)g(Y,X) \\ &\quad +2g(Y,V)g(U,X)-2g(V,X)g(Y,U))-2g(U,X)(2g(Y,V)g(Y,X) \\ &\quad -g(V,X)g(Y,Y)))\}. \end{aligned} \tag{2.5}$$

By attention to this consideration that $\{Y, U\}$ is an orthonormal basis for P with respect to g and (2.5) we have

$$\begin{aligned} g_Y(R(U,Y)Y,U) &= \frac{1}{<Y,X>^4}\{<U,X> \\ &(3<R(U,Y)Y,X> -2<Y,R(U,Y)Y><Y,X>) \\ &+2<Y,X>(<R(U,Y)Y,U><Y,X> \\ &-<U,X><Y,R(U,Y)Y>)\}, \end{aligned} \tag{2.6}$$

and

$$g_Y(Y,Y).g_Y(U,U) - g_Y^2(U,Y) = \frac{2<U,X>^2}{<Y,X>^6} + \frac{2}{<Y,X>^4}. \tag{2.7}$$

Now by using Püttmann's formula [6, eq. (2.1)] we have:

$$\begin{aligned} <X,R(U,Y)Y> &= -\frac{1}{4}(<[\phi U,Y]+[U,\phi Y],[Y,X]>_0 \\ &+<[U,Y],[\phi Y,X]+[Y,\phi X]>_0) - \frac{3}{4}<[Y,U],[Y,X]_{\mathfrak{m}}> \\ &-\frac{1}{2}<[U,\phi X]+[X,\phi U],\phi^{-1}([Y,\phi Y])>_0 \\ &+\frac{1}{4}<[U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi X]+[X,\phi Y])>_0, \end{aligned} \tag{2.8}$$

$$<R(U,Y)Y,Y>=0, \tag{2.9}$$

and

$$\begin{aligned} <R(U,Y)Y,U> &= -\frac{1}{2}<[\phi U,Y]+[U,\phi Y],[Y,U]>_0 \\ &-\frac{3}{4}<[Y,U],[Y,U]_{\mathfrak{m}}> - <[U,\phi U],\phi^{-1}([Y,\phi Y])>_0 \\ &+\frac{1}{4}<[U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi U]+[U,\phi Y])>_0 . \end{aligned} \tag{2.10}$$

Substituting the equations (2.6), (2.7), (2.8), (2.9) and (2.10) in the equation (1.6) completes the proof.

□

Now we continue our study with a special type of Riemannian homogeneous spaces which has been named naturally reductive. We remind that a homogeneous space $M = G/H$ with a $G-$invariant indefinite Riemannian metric g is said to be naturally reductive if it admits an $ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ satisfying the condition

$$B(X,[Z,Y]_{\mathfrak{m}}) + B([Z,X]_{\mathfrak{m}},Y) = 0 \qquad \text{for} \quad X,Y,Z \in \mathfrak{m}, \tag{2.11}$$

where B is the bilinear form on $\mathfrak{m}$ induced by $\mathfrak{g}$ and $[,]_{\mathfrak{m}}$ is the projection to $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (For more details see [4].).
In this case the above formula for the flag curvature reduces to a simpler equation.

Theorem 2.2. *In the previous theorem let G/H be a naturally reductive homogeneous space. Then the flag curvature of the flag (P,Y) in $T_H(G/H)$ is given by 2.2* *where,*

$$R(U,Y)Y = \frac{1}{4}[Y,[U,Y]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[U,Y]_{\mathfrak{h}}] \tag{2.12}$$

Proof. By using Proposition 3.4 in [4] (page 202) the claim clearly follows. □

If the invariant Kropina metric is defined by a bi-invariant Riemannian metric on a Lie group then there is a simpler formula for the flag curvature, we give this formula in the following theorem.

Theorem 2.3. *Let G be a Lie group, g be a bi-invariant Riemannian metric on G, and $\tilde{X}$ be a left invariant vector field on G. Suppose that $F = \frac{\alpha^2}{\beta}$ is the Kropina metric defined by g and $\tilde{X}$ on G such that the Chern connection of F coincides on the Levi-Civita connection of g. Then for the flag curvature of the flag $P = span\{Y,U\}$, where $\{Y,U\}$ is an orthonormal basis for P with respect to g, we have:*

$$K(P,Y) = \frac{-3 < U,X >< [[U,Y],Y],X > -2 < Y,X >< [[U,Y],Y],U >}{8(\frac{<U,X>}{<Y,X>})^2 + 8}, \tag{2.13}$$

Proof. g is bi-invariant. Therefore we have $R(U,Y)Y = -\frac{1}{4}[[U,Y],Y]$. Now by using Theorem 2.2 the proof is completed. □

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