International Electronic Journal of Geometry
Volume 4 No. 2 pp. 141–154 (2011) ©IEJG

SCREEN CAUCHY-RIEMANN LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN PRODUCT MANIFOLDS

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(Communicated by Ramesh SHARMA)

Abstract. We introduce Screen Cauchy-Riemann(SCR) lightlike submanifolds of a semi-Riemannian product manifold and give examples. We obtain integrability conditions for the distributions and investigate the geometry of leaves of distributions involved. We also obtain necessary and sufficient condition for SCR-lightlike submanifold to be locally lightlike Riemannian product manifold. Finally, we prove that there exists no totally umbilical and curvature invariant proper SCR-lightlike submanifold in any semi-Riemannian product real space form.

1. Introduction

A. Bejancu introduced the notion of CR-submainfolds which was emerged as a single setting to study holomorphic and totally real submanifolds of a Kaehler manifold. In [2], Duggal and Bejancu initiated the study of CR-lightlike submanifolds of an indefinite Kaehler manifold which excludes the complex and real lightlike submanifolds ([2],p.210). Later on, Duggal and Sahin[4] gave the notion of Screen Cauchy-Riemann(SCR)- lightlike submanifolds of an indefinite Kaehler manifold which contains complex and screen real subcases. However, there is no inclusion relation between screen Cauchy-Riemann and CR submanifolds. On the other hand, semi-invariant lightlike submanifolds of a semi-Riemannian product manifold were introduced by Atceken and Kilic in [1] and studied totally umbilical, curvature invariant lightlike submanifolds in real space forms as well as integrability conditions for the distributions involved in the definition of semi-invariant lightlike submanifolds.

In the present paper, we introduce and study screen CR-lightlike submanifold of a semi-Riemannian product manifold. The paper is arranged as follows. In sections 2 and 3, we summarize basic materials on lightlike submanifolds and semi-Riemannian product manifolds which will be useful throughout this paper. In section 4, we give examples of proper SCR-lightlike submanifolds, obtain integrability conditions for the distributions involved, investigate the geometry of leaves of

2000 *Mathematics Subject Classification.* 53C15, 53C42, 53C50.
Key words and phrases. Semi-Riemannian Product Manifold, Screen Cauchy-Riemannian Lightlike Submanifold and SCR-Lightlike Product.

distributions and give necessary and sufficient condition for SCR-lightlike submanifold to be locally lightlike Riemannian product. We also prove that there exists no totally umbilical and curvature invariant proper SCR-lightlike submanifold in any semi-Riemannian product of two real space forms.

2. **Preliminaries**

We follow [2] for the notation and fundamental equations for lightlike submanifolds used in this paper. A submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric g induced from $\overline{g}$ and the radical distribution $Rad\, TM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad\ TM$ in the tangent bundle TM of M, i.e.,

$$TM = Rad\, TM \perp S(TM)$$

Consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complementry vector bundle of $Rad\, TM$ in the normal bundle $TM^\perp$ of M. Since for any local basis $\{\xi_i\}$ of $Rad\, TM$, there exist a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $[S(TM)]^\perp$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exist a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [2]. Let $tr(TM)$ be complementary (but not orthogonal)vector bundle to TM in $T\overline{M}|_M$.Then

$$tr(TM) = ltr(TM) \perp S(TM^\perp),$$

$$T\overline{M}|_M = S(TM) \perp [(Rad\ \ TM) \oplus ltr(TM)] \perp S(TM^\perp).$$

Following are four subcases of a lightlike submanifold $(M, g, S(TM), S(TM^\perp)$.
Case 1: r-lightlike if $r < min\{m, n\}$.
Case 2: Co-isotropic if $r = n < m$; $S(TM^\perp) = \{0\}$.
Case 3: Isotropic if $r = m < n$; $S(TM) = \{0\}$
Csae 4: Totally lightlike if $r = m = n$; $S(TM) = \{0\} = S(TM^\perp)$.
The Gauss and Weingarten equations are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \forall\, X, Y \in \Gamma(TM) \tag{2.1}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \ \ \forall\, X \in \Gamma(TM), U \in \Gamma(tr(TM))$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively, ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y) \tag{2.2}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N) \tag{2.3}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W) \tag{2.4}$$

for each $X, Y \in \Gamma(TM), N \in \Gamma(ltr(TM))$ *and* $W \in \Gamma(S(TM^\perp))$. Denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ by P. Then, by using (2.1), (2.2), (2.3), (2.4) and a metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y) \tag{2.5}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X)$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY),$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.6}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad\,TM)$. By using above equations we obtain

$$\overline{g}(h^l(X, PY), \xi) = g(A^*_\xi X, PY)$$

$$(\overline{g}(h^*(X, PY), N) = g(A_N X, PY)$$

$$\overline{g}(h^l(X, \xi), \xi) = 0, A^*_\xi \xi = 0.$$

In general, the induced connection ∇ on M is not a metric connection whereas ∇^* is a metric connection on $S(TM)$. Using (2.2) and the fact that $\overline{\nabla}$ is a metric connection, we get

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y).$$

The Gauss equation for M is given by

$$\begin{aligned}\overline{R}(X, Y)Z = {} & R(X, Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ & + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) \\ & + (\nabla_X h^s)(Y, Z) - (\nabla_Y h^s)(X, Z) + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z))\end{aligned} \tag{2.7}$$

for each $X, Y, Z \in \Gamma(TM)$.

3. Semi-Riemannian Product Manifolds

Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2- dimensional semi-Riemannian manifolds with constant indices $q_1 > 0$ and $q_2 > 0$ respectively. Let $\pi : M_1 \times M_2 \longrightarrow M_1$, and $\sigma : M_1 \times M_2 \longrightarrow M_2$ be the projections which are given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for any $(x, y) \in M_1 \times M_2$. We denote the product manifold by $\overline{M} = (M_1 \times M_2, \overline{g})$ where

$$\overline{g}(X, Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y)$$

for any $X, Y \in \Gamma(T\overline{M})$, where $*$ denotes the differential mapping. Then we have

$$\pi_*^2 = \pi_*, \sigma_*^2 = \sigma_*, \pi_*\sigma_* = \sigma_*\pi_* = 0 \ \textit{and} \ \pi_* + \sigma_* = I,$$

where I is the identity map of $\Gamma(M_1 \times M_2)$. Thus $(\overline{M}, \overline{g})$ is a $(m_1 + m_2)$-dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The Riemannian product manifold $\overline{M} = (M_1 \times M_2, \overline{g})$ is characterized by M_1 and M_2 which are totally geodesic submanifold of $\overline{M}$.

Now, if we put $F = \pi_* - \sigma_*$, then we can easily see that $F^2 = I$ and

$$g(FX, Y) = g(X, FY) \tag{3.1}$$

for any $X, Y \in \Gamma(T\overline{M})$, where F is called almost Riemannian product structure on $M_1 \times M_2$. If we denote the Levi-Civita connection on $\overline{M}$ by $\overline{\nabla}$, then

$$(\overline{\nabla}_X F)Y = 0$$

for any $X, Y \in \Gamma(T\overline{M})$, that is, F is parallel with respect to $\overline{\nabla}$.

Now, let M_1 and M_2 be two real space forms with constant sectional curvatures c_1 and c_2, respectively. Then the Riemannian curvature tensor $\overline{R}$ of $\overline{M} = M_1(c_1) \times M_2(c_2)$ is given by

$$\overline{R}(X,Y)Z = \frac{1}{16}(c_1 + c_2)\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \overline{g}(FY,Z)FX \tag{3.2}$$

$-\overline{g}(FX,Z)FY\} + \frac{1}{16}(c_1 - c_2)\{\overline{g}(FY,Z)X - \overline{g}(FX,Z)Y + \overline{g}(Y,Z)FX - \overline{g}(X,Z)FY\}$ for any $X, Y, Z \in \Gamma(T\overline{M})$[8].

4. Screen Cauchy-Riemann Lightlike Submanifolds

In the present section, we define screen Cauchy-Riemann lightlike submanifolds of a semi-Riemannian product manifold and study the integrability condition of distributions involved in the definition of such submanifolds.

Definition 4.1. Let $(\overline{M}, \overline{g})$ be a semi-Riemannian product manifold and M be a r-lightlike submanifold of $\overline{M}$. We say that M is a SCR-lightlike submanifold of M, if the following conditions are satisfied:
(i) There exists real non-null distributions D and $D^{\perp}$ such that

$$S(TM) = D \oplus D^{\perp},\ \ F(D^{\perp}) \subset (S(TM^{\perp})),\ \ D \cap D^{\perp} = \{0\},$$

where $D^{\perp}$ is orthogonal complement to D in $S(TM)$.
(ii) The distribution D and $Rad\,TM$ are invariant with respect to F.

It follows that $ltr(TM)$ is also invariant with respect to F. Hence we have

$$TM = \overline{D} \oplus D^{\perp}, \tag{4.1}$$

where $\overline{D} = D \perp Rad\,TM$.

Denote the orthogonal complement to $F(D^{\perp})$ in $S(TM^{\perp})$ by μ. The submanifold M is said to be a proper SCR-lightlike submanifold of $\overline{M}$ if neither $D = \{0\}$ nor $D^{\perp} = \{0\}$.

Now, we give examples of proper SCR-lightlike submanifolds of semi-Riemannian product manifolds.

Example 4.1. Let $R^8_2 = R^4_1 \times R^4_1$ be a semi-Riemannian product manifold with semi-Riemannian product metric tensor $\overline{g} = \pi_* g_1 \otimes \sigma_* g_2,$, where $g_i (i = 1, 2)$ denote standard metric tensor of R^4_1. Consider a product structure F defined by

$$F(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = (\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_i}),$$

where (x^j, y^j) are the cartesian co-ordinates of R^8_2.

Let M be a submanifold of R^8_2 defined by

$$x^1 = s,\ x^2 = t,\ x^3 = w,\ x^4 = u\cos\alpha + v\sin\alpha$$

$$y^1 = t,\ y^2 = s,\ y^3 = 0,\ y^4 = u\sin\alpha + v\cos\alpha.$$

Then, a local frame of TM is given by

$$\xi_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2},\ \ \xi_2 = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2}$$

$$Z_1 = \cos\alpha\frac{\partial}{\partial x_4} + \sin\alpha\frac{\partial}{\partial y_4},\ \ Z_2 = \sin\alpha\frac{\partial}{\partial x_4} + \cos\alpha\frac{\partial}{\partial y_4},\ \ Z_3 = \frac{\partial}{\partial x_3}.$$

Hence M is a 2-lightlike submanifold with invariant $Rad\,TM = span\{\xi_1, \xi_2\}$. It is easy to see that lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{2}(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2}),\ \ N_2 = \frac{1}{2}(-\frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2})$$

and the screen transversal bundle $S(TM^{\perp})$ is spanned by

$$W = \frac{\partial}{\partial y_3}.$$

By direct calculations, we get

$$FZ_1 = Z_2,\ \ FZ_2 = Z_1,\ and\ \ FZ_3 = W.$$

If we take $D = span\{Z_1, Z_2\}$ and $D^{\perp} = span\{Z_3\}$, then D is invariant and $FD^{\perp} \subset (S(TM^{\perp}))$. Hence M is a proper screen Cauchy-Riemann lightlike submanifold of R^8_2.

Example 4.2. Let $R^8_2 = R^4_1 \times R^4_1$ be a semi-Riemannian product manifold with the product structure

$$F(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = (-\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}),$$

where (x^j, y^j) are the cartesian co-ordinates of R^8_2.

Suppose M is a submanifold of R^8_2 defined by

$$x^1 = s - t,\ x^2 = u_1,\ x^3 = (s - t),\ x^4 = (u_2 - u_3)\cos\alpha$$

$$y^1 = 0,\ y^2 = u_1,\ y^3 = (u_2 + u_3)\sin\alpha,\ y^4 = 0.$$

Then the tangent bundle TM is spanned by

$$\xi_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3},\ \ \xi_2 = -\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3},$$

$$Z_1 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2},\ \ Z_2 = \cos\alpha\frac{\partial}{\partial x_4} + \sin\alpha\frac{\partial}{\partial y_3},\ \ Z_3 = -\cos\alpha\frac{\partial}{\partial x_4} + \sin\alpha\frac{\partial}{\partial y_3}.$$

Thus M is a 2-lightlike submanifold with $Rad\,TM = span\{\xi_1, \xi_2\}$, which is invariant with respect to F. Moreover, the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{2}(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}),\ \ N_2 = \frac{1}{2}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3})$$

and the screen transversal bundle $S(TM^{\perp})$ is spanned by

$$W = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}.$$

It is easy to see that

$$FZ_1 = W,\ \ FZ_2 = Z_3,\ and\ \ FZ_3 = Z_2.$$

Let $D = span\{Z_2, Z_3\}$ and $D^{\perp} = span\{Z_1\}$. Then D is invariant and $FD^{\perp} \subset (S(TM^{\perp}))$. Hence M is a proper screen Cauchy-Riemann lightlike submanifold of R^8_2.

Example 4.3. Let M be a submanifold of a semi-Riemannian product manifold $R_4^{12} = R_2^6 \times R_2^6$ with the product structure

$$F(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = (\frac{\partial}{\partial x_i}, -\frac{\partial}{\partial y_i})$$

given by

$$x^1 = u_1 + u_2,\ x^2 = (u_1 + u_2)\sinh\theta,\ x^3 = u_4 + u_5,\ x^4 = u_3,\ x^5 = (u_4 + u_5)\cos\theta,$$

$$x^6 = (u_1+u_2)\cosh\theta,\ y^1 = 0,\ y^2 = 0,\ y^3 = 0,\ y^4 = u_3,\ y^5 = (u_4-u_5)\sin\theta,\ y^6 = 0.$$

Then, the tangent bundle is spanned by $\xi_1, \xi_2, Z_1, Z_2, Z_3$, where

$$\xi_1 = \frac{\partial}{\partial x_1} + \sinh\theta\frac{\partial}{\partial x_2} + \cosh\theta\frac{\partial}{\partial x_6},\ \ \xi_2 = \frac{\partial}{\partial x_1} + \sinh\theta\frac{\partial}{\partial x_2} + \cosh\theta\frac{\partial}{\partial x_6}$$

$$Z_1 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_4},\ \ Z_2 = \frac{\partial}{\partial x_3} + \cos\theta\frac{\partial}{\partial x_5} + \sin\theta\frac{\partial}{\partial y_5},\ \ Z_3 = \frac{\partial}{\partial x_3} + \cos\theta\frac{\partial}{\partial x_5} - \sin\theta\frac{\partial}{\partial y_5}.$$

Thus M is a 2-lightlike submanifold with invariant $Rad\ TM = span\{\xi_1, \xi_2\}$ and the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{2}(-\frac{\partial}{\partial x_1} + \sinh\theta\frac{\partial}{\partial x_2} + \cosh\theta\frac{\partial}{\partial x_6}),\ \ N_2 = \frac{1}{2}(-\frac{\partial}{\partial x_1} + \sinh\theta\frac{\partial}{\partial x_2} + \cosh\theta\frac{\partial}{\partial x_6}).$$

Furthermore, the screen transversal bundle $S(TM^\perp)$ is spanned by

$$W = \frac{\partial}{\partial x_4} - \frac{\partial}{\partial y_4}.$$

Then after simple calculations, we get

$$FZ_1 = W,\ \ FZ_2 = Z_3,\ and\ \ FZ_3 = Z_2.$$

Choose $D = span\{Z_2, Z_3\}$ and $D^\perp = span\{Z_1\}$. Clearly D is invariant with respect to F and $FD^\perp \subset (S(TM^\perp))$. Hence M is a proper screen Cauchy-Riemann lightlike submanifold of R_4^{12}.

For any vector field $X \in \Gamma(TM)$, we write

$$FX = fX + \omega X \tag{4.2}$$

where fX and ωX are tangential and normal parts of FX. Similarly, for the vector field $V \in \Gamma(tr(TM))$, we set

$$FV = BV + CV \tag{4.3}$$

where BV and CV are tangential and normal parts of FV.

Using (2.2),(2.4),(4.2),(4.3) and the fact that F is parallel on $\overline{M}$, we get

$$\overline{\nabla}_X FY = F\overline{\nabla}_X Y,$$

which implies

$$\nabla_X fY + h^l(X, fY) + h^s(X, fY) - A_{\omega Y}X + \nabla^s_X \omega Y + D^l(X, \omega Y) = f\nabla_X Y + \omega\nabla_X Y$$
$$+Fh^l(X, Y) + Bh^s(X, Y) + Ch^s(X, Y)$$

for any $X, Y \in \Gamma(TM)$. Taking into account the tangential, lightlike transversal and screen transversal components of the above equation, respectively, we get

$$(\nabla_X f)Y = A_{\omega Y}X + Bh^s(X, Y), \tag{4.4}$$

$$h^l(X, fY) + D^l(X, \omega Y) = Fh^l(X, Y), \tag{4.5}$$

$$(\nabla_X \omega)Y = Ch^s(X, Y) - h^s(X, fY). \tag{4.6}$$

For the integrability of the distributions $\overline{D}$ and $D^{\perp}$ involved in the definition of a screen CR-lightlike submanifold, we have:

Theorem 4.1. *Let M be a proper SCR-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the distribution $\overline{D}$ is integrable if and only if the second fundamental form of M satisfies*

$$h^s(X, fY) = h^s(Y, fX)$$

for any $X, Y \in \Gamma(\overline{D})$.

Proof. Using (4.6), for any $X, Y \in \Gamma(\overline{D})$, we obtain

$$\omega \nabla_X Y = Ch^s(X, Y) - h^s(X, fY). \tag{4.7}$$

By interchanging X and Y in (4.7), we get

$$\omega \nabla_Y X = Ch^s(Y, X) - h^s(Y, fX). \tag{4.8}$$

Using (4.7), (4.8) and the fact that h is symmetric, we arrive at

$$\omega[X, Y] = h^s(Y, fX) - h^s(X, fY),$$

which proves our assertion. □

Theorem 4.2. *Let M be a proper SCR-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the distribution $D^{\perp}$ is integrable if and only if the shape operator of M satisfies*

$$A_{\omega X} Y = A_{\omega Y} X$$

for any $X, Y \in \Gamma(D^{\perp})$.

Proof. From (4.4) and after calculation, we have

$$-f \nabla_X Y = A_{\omega Y} X + Bh^s(X, Y) \tag{4.9}$$

for $X, Y \in \Gamma(D^{\perp})$. By interchanging X and Y in (4.9), we obtain

$$-f \nabla_Y X = A_{\omega X} Y + Bh^s(X, Y). \tag{4.10}$$

From (4.9) and (4.10), we conclude that

$$f[X, Y] = A_{\omega X} Y - A_{\omega Y} X,$$

which proves our assertion. □

For the leaves of $\overline{D}$ to be totally geodesic foliation on M, we have:

Theorem 4.3. *Let M be a SCR-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the invariant distribution $\overline{D}$ defines a totally geodesic foliation on M if and only if $h^s(X, FY)$ has no component in $\Gamma(S(TM^{\perp}))$, for any $X, Y \in \Gamma(\overline{D})$.*

Proof. The distribution $\overline{D}$ defines a totally geodesic foliation on M if and only if $\nabla_X Y \in \Gamma(\overline{D})$. Using (4.1) and the fact that $F(D^{\perp}) \subset S(TM^{\perp})$, we conclude that $\nabla_X Y \in \Gamma(\overline{D})$ if and only if

$$\overline{g}(\nabla_X Y, Z) = 0$$

for any $X, Y \in \Gamma(\overline{D})$ and $Z \in \Gamma(D^{\perp})$.

On the other hand, using (2.2) and (3.1), we obtain

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X FY, FZ) - \overline{g}(Fh^s(X, Y), FZ)$$

$$= \overline{g}(h^s(X, FY), FZ),$$

which proves our assertion. □

Now, we define SCR-lightlike product submanifolds of a semi-Riemannian product manifold similar to the definition of SCR-lightlike product given by Duggal and Sahin [5] and prove two characterization theorems for the existence of such submanifolds in semi-Riemannian product manifolds.

Definition 4.2. A SCR-lightlike submanifold M of a semi-Riemannian product manifold $\overline{M}$ is a SCR-lightlike product if both distributions $\overline{D}$ and $D^{\perp}$ of M are integrable and their leaves are totally geodesic in M, i.e, M is locally a product manifold $(M_1 \otimes M_2, g)$ with $g = g_1 + g_2$, where g_1 is the degenerate metric tensor of the leaf M_1 of $\overline{D}$ and g_2 is the non-degenerate metric tensor of the leaf M_2 of $D^{\perp}$.

Lemma 4.1. *For a proper SCR-lightlike submanifold M of a semi-Riemannian product manifold $\overline{M}$, one has*

$$D^l(Z, FW) = D^l(W, FZ) \tag{4.11}$$

for all $Z, W \in \Gamma(D^{\perp})$.

Proof. Using (2.6), $\overline{\nabla} F = 0$ and (2.4), for $\xi \in \Gamma(Rad\,TM)$ and $Z, W \in \Gamma(D^{\perp})$ we obtain

$$g(A^*_{F\xi} Z, W) = g(\xi, D^l(Z, FW)). \tag{4.12}$$

Similarly

$$g(A^*_{F\xi} W, Z) = g(\xi, D^l(W, FZ)). \tag{4.13}$$

From (4.12), (4.13) and the fact that A^* is self-adjoint, we have

$$g(\xi, D^l(Z, FW) - D^l(W, FZ)) = 0,$$

which proves our assertion. □

Theorem 4.4. *Let M be a proper SCR-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is a SCR-lightlike product if and only if $\nabla f = 0$.*

Proof. Let M be a SCR-lightlike product submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the distributions $\overline{D}$ and $D^{\perp}$ are integrable and its leaves are totally geodesic in M. Using (2.2), (4.2), (4.3) and the fact that $\overline{\nabla}_U FX = F\overline{\nabla}_U X$, we arrive at

$$\nabla_U fX + h^l(U, fX) + h^s(U, fX) = f\nabla_U X + \omega\nabla_U X + Fh^l(U, X) + Bh^s(U, X) + Ch^s(U, X)$$

for any $U \in \Gamma(TM)$ and $X \in \Gamma(\overline{D})$. Comparing the components of $\overline{D}$ and $D^{\perp}$ on both sides of the above equation, respectively, we get

$$\nabla_U fX = f\nabla_U X \quad i.e, \quad (\nabla_U f)X = 0$$

and

$$Bh^s(U, X) = 0.$$

On the other hand, using $\overline{\nabla}_U FY = F\overline{\nabla}_U Y$, for any $U \in \Gamma(TM)$ and $Y \in \Gamma(D^{\perp})$ we have

$$-A_{FY}U + \nabla^s_U FY + D^l(U, FY) = f\nabla_U Y + \omega\nabla_U Y + Fh^l(U, Y) + Bh^s(U, Y) + Ch^s(U, Y) \tag{4.14}$$

Considering the tangential parts of (4.14), we obtain

$$-A_{FY}U = f\nabla_U Y + Bh^s(U,Y).$$

Since leaves of $D^\perp$ are totally geodesic in M, $\nabla_U Y \in D^\perp$ for all $Y \in \Gamma(D^\perp)$ and $f\nabla_U Y = 0$. Thus

$$A_{FY}U + Bh^s(U,Y) = 0. \tag{4.15}$$

Combining (4.4) with (4.15), we get $(\nabla_U f)Y = 0$.

Conversely, suppose $\nabla f = 0$. Then, using $\nabla f = 0$ and (2.2) for any $X, Z \in \Gamma(\overline{D})$, we obtain

$$\overline{\nabla}_Z fX - h^l(Z,fX) - h^s(Z,fX) - f\nabla_Z X = 0,$$

which implies

$$\overline{\nabla}_Z FX - h^l(Z,fX) - h^s(Z,fX) - F\overline{\nabla}_Z X + Fh^l(X,Z) + Fh^s(X,Z) + \omega\nabla_Z X = 0,$$

where we have used (2.2) and (4.2). Using $\overline{\nabla}F = 0$ in the above equation, we get

$$-h^l(Z,fX) - h^s(Z,fX) + Fh^l(X,Z) + Fh^s(X,Z) + \omega\nabla_Z X = 0. \tag{4.16}$$

Interchanging Z and X in (4.16) and substracting the resulting equation from (4.16), we arrive at

$$\omega[Z,X] - h^l(Z,fX) - h^s(Z,fX) + h^l(X,fZ) + h^s(X,fZ) = 0. \tag{4.17}$$

Taking inner product of (4.17) with FY, we obtain

$$g(\omega[Z,X],FY) = g(h^s(Z,fX) - h^s(X,fZ),FY),$$

from which we conclude that $\overline{D}$ is integrable in view of Theorem 4.1.
In respect of integrability of $D^\perp$, we have

$$F\overline{\nabla}_X Z = \overline{\nabla}_X \omega Z \tag{4.18}$$

for any $X, Z \in \Gamma(D^\perp)$. Using (2.2),(2.4) and (4.2) in (4.18), we get

$$f\nabla_X Z + \omega\nabla_X Z + Fh^l(X,Z) + Fh^s(X,Z) = -A_{\omega Z}X + \nabla^s_X \omega Z + D^l(X,\omega Z),$$

which implies

$$\omega\nabla_X Z + Fh^l(X,Z) + Fh^s(X,Z) = -A_{\omega Z}X + \nabla^s_X \omega Z + D^l(X,\omega Z), \tag{4.19}$$

where we have used $\nabla f = 0$, i.e, $f\nabla_Z X = \nabla_Z fX = 0$. Interchanging X and Z in (4.19) and substracting the resulting equation from (4.19), we get

$$\omega[X,Z] = -A_{\omega Z}X + A_{\omega X}Z + \nabla^s_X \omega Z - \nabla^s_Z \omega X + D^l(X,\omega Z) - D^l(Z,\omega X). \tag{4.20}$$

Taking inner product of (4.20) with $Y \in \Gamma(\overline{D})$ and using (4.11), we obtain

$$g([X,Z],FY) = g(-A_{\omega Z}X + A_{\omega X}Z, Y),$$

from which we conclude that $D^\perp$ is integrable in view of Theorem 4.2.
Now, we prove that leaves of $\overline{D}$ and $D^\perp$ are totally geodesic in M. Let $\nabla f = 0$. Then for any $Z \in \Gamma(TM)$ and $X \in \Gamma(\overline{D})$,

$$f\nabla_Z X = \nabla_Z fX \neq 0,$$

from which we infer that $\nabla_Z X \in \Gamma(\overline{D})$, i.e, the leaves of $\overline{D}$ are totally geodesic in M.
In the similar way, for any $Z \in \Gamma(TM)$ and $Y \in \Gamma(D^\perp)$, we have

$$f\nabla_Z Y = \nabla_Z fY = 0,$$

from which we infer that $\nabla_Z Y \in \Gamma(D^{\perp})$, that is, the leaves of $D^{\perp}$ are totally geodesic in M. This completes the proof of theorem. □

Theorem 4.5. *Let M be a proper SCR-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is a SCR-lightlike product if and only if*

$$Bh^s(Z, X) = 0 \tag{4.21}$$

for each $Z \in \Gamma(TM)$ and $X \in \Gamma(\overline{D})$.

Proof. Let M be a SCR-lightlike product. Then, from Theorem 4.4 and (4.4) we have

$$Bh^s(Z, X) = 0$$

for any $Z \in \Gamma(TM)$ and $X \in \Gamma(\overline{D})$.
Conversely, suppose that (4.21) is satisfied for each $Z \in \Gamma(TM)$ and $X \in \Gamma(\overline{D})$. Then, using (4.21) and (4.4) we get

$$(\nabla_X f)Z = 0.$$

Thus by Theorem 4.4, M is a SCR-lightlike product. □

As a consequence of Theorem 4.4 and Theorem 4.5, we have the following result:

Theorem 4.6. *Let M be a proper SCR-lightlike totally umbilical submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is a SCR-lightlike product if M is totally geodesic submanifold in $\overline{M}$.*

Definition 4.3. A SCR-lightlike submanifold M of a semi-Riemannian product manifold is said to be $\overline{D}$-totally geodesic(resp. $D^{\perp}$-totally geodesic) if its second fundamental form h satisfies $h(X, Y) = 0$ (resp. $h(Z, W) = 0$) for $X, Y \in \Gamma(\overline{D})(Z, W \in \Gamma(D^{\perp}))$.

For a proper screen CR-lightlike submanifold to be $\overline{D}$-totally geodesic, we have

Theorem 4.7. *Let M be a proper SCR-lightlike submanifold of a semi-Riemannian Product manifold $\overline{M}$. Then M is $\overline{D}$-totally geodesic submanifold if and only if $A_W X$ and $A^*_\xi X$ have no components in $\Gamma(D)$ for any $X \in \Gamma(\overline{D})$, $W \in \Gamma(S(TM^{\perp})$ and $\xi \in \Gamma(Rad\,TM)$.*

Proof. For any $X, Y \in \Gamma(\overline{D})$, $W \in \Gamma(S(TM^{\perp})$ and $\xi \in \Gamma(Rad\,TM)$, we have

$$\overline{g}(h^s(X, FY), W) = \overline{g}(\overline{\nabla}_X FY, W) = \overline{g}(A_W X, FY) \tag{4.22}$$

and

$$\overline{g}(h^l(X, FY), \xi) = \overline{g}(\overline{\nabla}_X FY, \xi) = \overline{g}(FY, A^*_\xi X). \tag{4.23}$$

Thus, our assertion follows from (4.22) and (4.23). □

Theorem 4.8. *Let M be a proper SCR-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is $D^{\perp}$- totally geodesic submanifold if and only if $A_W Y$ and $A^*_\xi X$ have no components in $\Gamma(D^{\perp})$ for any $Y, Z \in \Gamma(D^{\perp})$, $W \in \Gamma(S(TM^{\perp}))$ and $\xi \in \Gamma(Rad\,TM)$.*

Proof. Let $Y, Z \in \Gamma(D^{\perp})$, $W \in \Gamma(S(TM^{\perp}))$ and $\xi \in \Gamma(Rad\,TM)$. Then

$$\overline{g}(h^s(Z,Y),W) = \overline{g}(\overline{\nabla}_Y Z, W) = \overline{g}(A_W Y, Z) \tag{4.24}$$

and

$$\overline{g}(h^l(Z,Y),\xi) = \overline{g}(\overline{\nabla}_Z Y, \xi) = \overline{g}(Y, A^*_\xi Z). \tag{4.25}$$

Thus, proof follows from (4.24) and (4.25). □

A characterization of totally geodesic submanifolds is given by following:

Theorem 4.9. *Let M be a proper SCR-lightlike submanifold of a semi-Riemannian product $\overline{M}$.Then M is totally geodesic submanifold if and only if $S(TM^{\perp})$ and $Rad\,TM$ are Killing distributions on $\overline{M}$.*

The proof of the above theorem is similar to Theorem 3.1 of [7].

Definition 4.4. Let M be a proper SCR-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is mixed geodesic submanifold if the second fundamental form of M satisfies $h(X,Y) = 0$ for any $X \in \Gamma(\overline{D})$ and $Y \in \Gamma(D^{\perp})$.

For the proper screen CR-lightlike submanifold to be mixed geodesic, we have:

Theorem 4.10. *Let M be a proper SCR-lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then M is mixed-geodesic if and only if the shape operators $A_U X$ and $A^*_\xi X$ of M have no components in $\Gamma(D^{\perp})$ for any $X \in \Gamma(\overline{D})$, $U \in \Gamma(S(TM^{\perp}))$ and $\xi \in \Gamma(Rad\,TM)$.*

Proof. For any $X \in \Gamma(\overline{D})$, $Y \in \Gamma(D^{\perp})$, $\xi \in \Gamma(Rad\,TM)$ and $U \in \Gamma(S(TM^{\perp}))$, we have

$$\overline{g}(h^s(X,Y),U) = \overline{g}(\overline{\nabla}_X Y, U) = g(A_U X, Y) \tag{4.26}$$

and

$$\overline{g}(h^s(X,Y),\xi) = \overline{g}(\overline{\nabla}_X Y, \xi) = g(Y, A^*_\xi X). \tag{4.27}$$

Thus, our assertion follows from (4.26) and (4.27). □

Definition 4.5. [3] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be totally umbilical if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that

$$h(X,Y) = \overline{g}(X,Y)H \tag{4.28}$$

for any $X, Y \in \Gamma(TM)$.

Using (2.2) and (4.28), it is easy to see that M is totally umbilical if and only if on each coordinate neighbourhood u there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that

$$h^l(X,Y) = g(X,Y)H_l, \;\; h^s(X,Y) = g(X,Y)H_s$$

for any $X, Y \in \Gamma(TM)$. Thus, we have

Theorem 4.11. *There does not exist any totally umbilical proper SCR-lightlike submanifold in the semi-Riemannian product of two real space forms $M_1(c_1)$ and $M_2(c_2)$ with $c_1 + c_2 \neq 0$.*

Proof. Let, if possible, M be a totally umbilical proper SCR-lightlike submanifold of $M_1(c_1) \times M_2(c_2)$ with $c_1 + c_2 \neq 0$. By using (2.7), for any $X, Y \in \Gamma(TM)$, $Z \in \Gamma(D)$ and $FW \in \Gamma(S(TM^{\perp}))$, we obtain

$$\overline{g}(\overline{R}(X,Y)FZ, FW) = \overline{g}((\overline{\nabla}_X h)(Y, FZ), FW) - \overline{g}((\overline{\nabla}_Y h)(X, FZ), FW). \tag{4.29}$$

On the other hand, from (4.28), we have

$$(\overline{\nabla}_X h)(Y, Z) = g(Y, Z)\nabla^{\perp}_X H. \tag{4.30}$$

Using (4.29) and (4.30), we get

$$\overline{g}(\overline{R}(X,Y)FZ, FW) = \overline{g}(Y, FZ)\overline{g}(\nabla^{\perp}_X H, FW) - \overline{g}(X, FZ)\overline{g}(\nabla^{\perp}_Y H, FW).$$

Replacing X by Z and Y by W in the above equation, we arrive at

$$\overline{K}(Z, W, FZ, FW) = \overline{g}(W, FZ)\overline{g}(\nabla^{\perp}_Z H, FW) - \overline{g}(Z, FZ)\overline{g}(\nabla^{\perp}_W H, FW).$$

Choosing Z orthogonal to FZ in the above equation, we get

$$\overline{K}(Z, W, FZ, FW) = 0. \tag{4.31}$$

Also, from (3.2), we have

$$\overline{K}(Z, W, FZ, FW) = \frac{-1}{16}(c_1 + c_2). \tag{4.32}$$

Thus, our assertion follows from (4.31), (4.32) and the fact that $c_1 + c_2 \neq 0$. □

Theorem 4.12. *Let M be a proper SCR-lightlike totally umbilical submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the following statement are equivalent*
i) The distribution $\overline{D}$ is parallel in M.
ii) $g(Z, FY)H_s = g(Z, Y)CH_s$, for any $Y \in \Gamma(\overline{D})$ and $Z \in \Gamma(TM)$, i.e $BH_s = 0$.
iii) The transversal vector field H_s is invariant with respect to F.
iv) f is parallel in M.

Proof. Since $\overline{D}$ is an invariant distribution, we have $FX = fX$ for any $X \in \Gamma(\overline{D})$. If M is totally umbilical, from (4.28) and the fact that $\overline{\nabla}_X FY = F\overline{\nabla}_X Y$ we have

$$\nabla_X fY + g(X, FY)H_l + g(X, FY)H_s = f\nabla_X Y + \omega\nabla_X Y + g(X, Y)FH_l + g(X, Y)BH_s + g(X, Y)CH_s$$

for any $X, Y \in \Gamma(\overline{D})$. Taking the tangential, lightlike transversal and screen transversal component of both sides of the above equation, we obtain

$$\nabla_X fY = f\nabla_X Y + g(X, Y)BH_s,$$

$$g(X, FY)H_l = g(X, Y)FH_l,$$

$$\omega\nabla_X Y = g(X, Y)CH_s - g(X, FY)H_s, \tag{4.33}$$

from which we conclude that $\nabla_X Y \in \Gamma(\overline{D})$ if and only if $g(X, FY)H_s = g(X, Y)CH_s$. Thus proof is complete. □

Interchanging X and Y in (4.33) and then subtracting the resulting equation from (4.33), we have:

Corollary 4.1. *Let M be a proper SCR-lightlike submanifold of a semi-Riemannian product manifold M. The distribution D is always integrable if M is a totally umbilical proper SCR-lightlike submanifold.*

From (4.2) and (3.2), we have

$$\begin{aligned}\overline{R}(X,Y)Z = &\frac{1}{16}(c_1+c_2)\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \overline{g}(FY,Z)fX + \overline{g}(FY,Z)\omega X\\ &- \overline{g}(FX,Z)fY - \overline{g}(FX,Z)\omega Y\} + \frac{1}{16}(c_1-c_2)\{\overline{g}(FY,Z)X\\ &- \overline{g}(FX,Z)Y + \overline{g}(Y,Z)fX + \overline{g}(Y,Z)\omega X - \overline{g}(X,Z)fY - \overline{g}(X,Z)\omega Y\}\end{aligned} \tag{4.34}$$

for any $X,Y,Z \in \Gamma(T\overline{M})$. Using (2.7) and (4.34), the equations of Gauss and Codazzi for the submanifold M, respectively, can be written as

$$\begin{aligned}R(X,Y)Z = &\frac{1}{16}(c_1+c_2)\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \overline{g}(FY,Z)fX - \overline{g}(FX,Z)fY)\}\\ &+ \frac{1}{16}(c_1-c_2)\{\overline{g}(FY,Z)X - \overline{g}(FX,Z)Y + \overline{g}(Y,Z)fX - \overline{g}(X,Z)fY\}\\ &+ A_{h^l(Y,Z)}X + A_{h^s(Y,Z)}X - A_{h^l(X,Z)}Y - A_{h^s(X,Z)}Y\end{aligned}$$

and

$$\begin{aligned}(\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) = &\frac{1}{16}(c_1+c_2)\{\overline{g}(FY,Z)\omega X - g(FX,Z)\omega Y\}\\ &+ \frac{1}{16}(c_1-c_2)\{g(Y,Z)\omega X - g(X,Z)\omega Y\}\end{aligned} \tag{4.35}$$

for any $X,Y,Z \in \Gamma(TM)$.

Definition 4.6. Let M be a r-lightlike submanifold of any semi-Riemannian manifold $\overline{M}$. M is said to be curvature-invariant lightlike submanifold if the covariant derivative of the second fundamental form h of M is satisfies

$$(\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) = 0.$$

for any $X,Y,Z \in \Gamma(TM)$.

Theorem 4.13. *There does not exist any curvature-invariant proper SCR-lightlike submanifold in the semi-Riemannian product of two real space forms $M_1(c_1)$ and $M_2(c_2)$ with $c_1, c_2 \neq 0$.*

Proof. For $X,\ Z \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$, using (4.35) we get

$$(\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) = -\frac{1}{16}(c_1+c_2)\overline{g}(FX,Z)\omega Y - \frac{1}{16}(c_1-c_2)\overline{g}(X,Z)\omega Y,$$

from which our assertion follows. □

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