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GENERALIZED NULLITY DISTRIBUTIONS ON ALMOST KENMOTSU MANIFOLDS

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Abstract. We consider almost Kenmotsu manifolds whose characteristic vector field belongs to two types of generalized nullity distributions. We prove that, in dimensions greater or equal to 5, the functions involved in the definition of such distributions can vary only in direction of ξ and the Riemannian curvature is completely determined. Furthermore, we provide examples of almost Kenmotsu manifolds satisfying generalized nullity conditions with non-constant smooth functions.

1. Introduction

One of the recent topics in the theory of almost contact metric manifolds is the study of the so-called nullity distributions. Historically, a first notion of k-nullity distribution, $k \in \mathbb{R}$, appeared in the Riemannian geometry framework, [12, 20]. More recently, a more general notion of nullity distribution, called (k, μ)-nullity distribution, with $k, \mu \in \mathbb{R}$, was introduced, in the context of contact geometry, by D. E. Blair, T. Koufogiorgos and B. J. Papantoniou in [2]. Such a problematic has been widely studied ([3, 11]) and by P. Dacko and Z. Olszak in the setting of almost cosymplectic manifolds as well ([4, 5, 6]). For an almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ the k-nullity distribution, $k \in \mathbb{R}$, is defined by putting for each $p \in M^{2n+1}$

$$N_p(k) = \{Z \in T_pM^{2n+1} \mid R_{XY}Z = k(g(Y,Z)X - g(X,Z)Y)\},$$

where X and Y are arbitrary vectors in T_pM^{2n+1}. Given $k, \mu \in \mathbb{R}$, the (k, μ)-nullity distribution, is the distribution given by putting

$$\begin{aligned} N_p(k,\mu) = \{Z \in T_pM^{2n+1} \mid R_{XY}Z &= k(g(Y,Z)X - g(X,Z)Y) \\ &+\mu(g(Y,Z)hX - g(X,Z)hY)\}, \end{aligned}$$

where $h = \frac{1}{2}\mathcal{L}_\xi\varphi$, $\mathcal{L}$ denoting the Lie differentiation.

In [8] the authors consider almost Kenmotsu manifolds $(M^{2n+1}, \varphi, \xi, \eta, g)$ satisfying the (k, μ)-*nullity condition*, which is obtained by requiring that ξ belongs to

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the (k,μ)-nullity distribution, that is, for all vector fields X,Y,

$$R_{XY}\xi = k\big(\eta(Y)X - \eta(X)Y\big) + \mu\big(\eta(Y)hX - \eta(X)hY\big)\,, \tag{1.1}$$

and they prove that $h = 0$ and $k = -1$. For this reason, in the same paper, they introduce and study a modified nullity condition involving the tensor field $h' = h \circ \varphi$, requiring the vector field ξ to belong to the so-called $(k,\mu)'$-nullity distribution, $k,\mu \in \mathbb{R}$, that is

$$R_{XY}\xi = k\big(\eta(Y)X - \eta(X)Y\big) + \mu\big(\eta(Y)h'X - \eta(X)h'Y\big), \tag{1.2}$$

for all vector fields X,Y.
In [19], we studied almost Kenmotsu manifolds with conformal Reeb foliation, that is with $h = 0$, proving in particular that ξ belongs to the k-nullity distribution if and only if $k = -1$.
The above conditions are called *generalized nullity conditions* when one allows k,μ to be smooth functions. In [17] the authors consider contact metric manifolds satisfying the generalized (k,μ)-nullity condition and they prove that k,μ are constant in dimension $2n+1 > 3$. In this paper, for both the generalized nullity conditions, we discuss the case $h \neq 0$ showing that almost Kenmotsu manifolds have a different behavior with respect to contact metric manifolds.

After some basic data concerning almost Kenmotsu manifolds, in section 3, we state some preliminary results on almost Kenmotsu manifolds satisfying (1.1) or (1.2) with k,μ functions. In particular, if $dim\, M^{2n+1} \geq 5$, we prove that k,μ can depend only on the direction of ξ and h and h' are η-parallel.

In sections 4 and 5, it is proved that, if such generalized nullity conditions are satisfied by an almost Kenmotsu manifold M^{2n+1} with $2n+1 \geq 5$, the Riemannian curvature is completely determined and the leaves of the distribution $\ker(\eta)$ are flat Kähler manifolds. Finally, in the last section, we show, through explicit examples, the existence of such manifolds with non-constant k,μ in any dimension.

As usually manifolds are assumed connected. For the curvature we adopt the notation in [16].

2. Preliminaries

An almost contact metric manifold is a $(2n+1)$-dimensional smooth manifold M endowed with a structure (φ,ξ,η,g) given by a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η satisfying $\varphi^2 = -I + \eta\otimes\xi, \eta(\xi) = 1$, and a Riemannian metric g such that $g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, for any vector fields X and Y. With this structure it can be associate a 2-form Φ, defined by $\Phi(X,Y) = g(X,\varphi Y)$ for any vector fields X and Y. The normality of an almost contact metric manifold is expressed by the vanishing of the tensor field $N = [\varphi,\varphi] + 2d\eta\otimes\xi$, where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. For more details, we refer to Blair's book [1].

It is well known that Kenmotsu manifolds can be characterized, through their Levi-Civita connection, by $(\nabla_X\varphi)(Y) = g(\varphi X,Y)\xi - \eta(Y)\varphi(X)$, for any vector fields X,Y. Equivalently, such manifolds can be defined as normal almost contact metric manifolds such that $d\eta = 0$ and $d\Phi = 2\eta\wedge\Phi$ (see [13]). Moreover, in [14] Kenmotsu proved that such a manifold M^{2n+1} is locally a warped product $]-\varepsilon,\varepsilon[\times_f N^{2n}$, N^{2n} being a Kähler manifold and $f = ce^t$, c a positive constant. More recently in [18, 15, 7], almost contact metric manifolds such that $d\eta = 0$ and $d\Phi = 2\eta\wedge\Phi$ are studied and they are called almost Kenmotsu. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold.

In an almost Kenmotsu manifold, since $d\eta = 0$, the distribution $\mathcal{D} = \ker(\eta)$, orthogonal to ξ, is integrable, and its integral submanifolds, endowed with the induced almost Hermitian structure, are almost Kähler manifolds; furthermore, we have $\mathcal{L}_\xi\eta = 0$ and $[\xi, X] \in \mathcal{D}$ for any $X \in \mathcal{D}$. Then, using the expression of the Levi-Civita connection for an almost contact metric manifold, ([1]), we have:

$$2g\big((\nabla_X\varphi)Y, Z\big) = 2\eta(Z)g(\varphi X, Y) - 2\eta(Y)g(\varphi X, Z) + g(N(Y,Z), \varphi X), \tag{2.1}$$

for any vector fields X, Y, Z, from which we deduce that $\nabla_\xi\varphi = 0$, so that $\nabla_\xi\xi = 0$, $\nabla_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$ and, for any vector field X,

$$\nabla_X\xi = X - \eta(X)\xi - \varphi h(X)\,. \tag{2.2}$$

Another important consequence of (2.1) is that the almost CR-structure $(\mathcal{D}, J)$ canonically associated with the given almost contact metric structure, where J is the restriction of φ to $\mathcal{D}$, is CR-integrable if and only if the tensor φ is η-parallel. Indeed, taking $Y, Z \in \mathcal{D}$, we notice that, being $\mathcal{D}$ integrable, $N(Y,Z)$ is orthogonal to ξ; therefore, $N(Y,Z) = 0$ is equivalent to the condition $g(N(Y,Z), \varphi X) = 0$ for all vector fields $X \in \mathcal{D}$. But, by (2.1), this is true if and only if $g\big((\nabla_X\varphi)Y, Z\big) = 0$.

The tensor fields h and $h' = h \circ \varphi$ are symmetric operators, anticommute with φ and verify $h(\xi) = h'(\xi) = 0$, $\eta \circ h = \eta \circ h' = 0$, $\mathrm{tr}(h) = \mathrm{tr}(h') = 0$. Obviously $h = 0$ if and only if $h' = 0$ and they admit the same eigenvalues. For an eigenvalue λ, we will denote the eigenspaces associated with h and h' by $[\lambda]$ and $[\lambda]'$, respectively. Furthermore, if $\lambda \neq 0$, then $[\lambda] \neq [\lambda]'$, since it is easy to check that $X \in [\lambda]$ implies $-X + \varphi X \in [\lambda]'$.

It is proved in [15] that the integral submanifolds of $\mathcal{D}$ are totally umbilical submanifolds of M^{2n+1} if and only if $h = 0$. In this case the manifold is locally a warped product $M' \times_f N^{2n}$, where N^{2n} is an almost Kähler manifold, M' is an open interval with coordinate t, and $f(t) = ce^t$ for some positive constant c ([7]). If, in addition, the integral submanifolds of $\mathcal{D}$ are Kähler, then M^{2n+1} is a Kenmotsu manifold. Hence, a 3-dimensional almost Kenmotsu manifold with $h = 0$ is a Kenmotsu manifold.

In [8] the authors study almost Kenmotsu manifolds satisfying the $(k,\mu)'$-nullity condition, proving that $k \leq -1$. Moreover if $k = -1$, then $h' = 0$ and M^{2n+1} is locally a warped product of an almost Kähler manifold and an open interval. If $k < -1$, then $\mu = -2$, h' admits three eigenvalues $\lambda, -\lambda, 0$, with 0 as simple eigenvalue and $\lambda = \sqrt{-1-k}$. The following classification theorem is established.

Theorem 2.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold satisfying the $(k,-2)'$-nullity condition and $h' \neq 0$. Then, M^{2n+1} is locally isometric to the warped product $\mathbb{H}^{n+1}(k-2\lambda) \times_f \mathbb{R}^n$ or $B^{n+1}(k+2\lambda) \times_{f'} \mathbb{R}^n$, where $f = ce^{(1-\lambda)t}$ and $f' = c'e^{(1+\lambda)t}$, with c, c' positive constants.*

We recall also some results stated in [9] for almost Kenmotsu manifolds with η-parallel h'. More precisely it is proved that, for any eigenvalue λ, the distribution $[\lambda]'$ is integrable and denoting by $[0]'$ the space of eigenvectors of 0 which are orthogonal to ξ, the following theorem is given.

Theorem 2.2. [[9],Theorem 2] *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that h' is η-parallel. Let $\{0, \lambda_1, -\lambda_1, \ldots, \lambda_r, -\lambda_r\}$ be the spectrum of h',* *with* $\lambda_i \neq 0$. *Then an integral manifold* $\widetilde{M}$ *of* $\mathcal{D}$ *is locally the Riemannian product*

$$M_0 \times M_{\lambda_1} \times M_{-\lambda_1} \times \ldots \times M_{\lambda_r} \times M_{-\lambda_r}, \tag{2.3}$$

where M_0, M_{λ_i} *and* $M_{-\lambda_i}$ *are integral manifolds of* $[0]'$, $[\lambda_i]'$ *and* $[-\lambda_i]'$ *respectively. Moreover,* M_0 *is an almost Kähler manifold and each* $M_{\lambda_i}\times M_{-\lambda_i}$ *is a bi-Lagrangian Kähler manifold. Denoting by* $2m_0+1$ *the multiplicity of* 0, *if* $m_0 > 0$ *then* M^{2n+1} *is CR-integrable if and only if* M_0 *is a Kähler manifold.*

Finally, we recall that the curvature of an almost Kenmotsu manifold satisfies (see [7, 8]):

$$\begin{aligned} R_{XY}\xi &= \eta(X)(Y-\varphi hY)-\eta(Y)(X-\varphi hX) \\ &\quad +(\nabla_Y\varphi h)X-(\nabla_X\varphi h)Y \end{aligned} \tag{2.4}$$

$$\varphi l \varphi - l = 2(-\varphi^2 + h^2) \tag{2.5}$$

$$\begin{aligned} &g(R_{\xi X}Y,Z)-g(R_{\xi X}\varphi Y,\varphi Z)+g(R_{\xi\varphi X}Y,\varphi Z)+g(R_{\xi\varphi X}\varphi Y,Z) \\ &= 2(\nabla_{hX}\Phi)(Y,Z)+2\eta(Y)g(Z,X-\varphi hX)-2\eta(Z)g(Y,X-\varphi hX), \end{aligned} \tag{2.6}$$

where l is the symmetric operator defined by $l(X) := R_{X\xi}\xi$, for any vector field X. We observe that the above relations can be rewritten in terms of h' since $\varphi\circ h = -h'$ and $h = -h'\circ\varphi$.

3. Properties of the generalized nullity conditions with $h \neq 0$.

Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost Kenmotsu manifold and $h = 0$. Obviously, the nullity distributions both reduce to a k-nullity distribution and if ξ belongs to the generalized k-nullity distribution, $k \in \mathfrak{F}(M^{2n+1})$, then $k = -1$. Indeed, comparing (2.4) and the nullity condition, we have $(k+1)(\eta(Y)X-\eta(X)Y) = 0$. Thus, considering $Y = \xi$, we obtain $(k+1)(\varphi^2(X)) = 0$, from which $k = -1$ follows.

Therefore, from now on, we assume $h \neq 0$, everywhere.

Proposition 3.1. *Let* $(M^{2n+1},\varphi,\xi,\eta,g)$ *be an almost Kenmotsu manifold satisfying either the generalized* (k,μ)*-nullity condition or the generalized* $(k,\mu)'$*-nullity condition, with* $h \neq 0$. *Then, one has*

$$h^2 = (k+1)\varphi^2 \quad \textit{or equivalently} \quad {h'}^2 = (k+1)\varphi^2 \tag{3.1}$$

$$Q(\xi) = 2nk\xi, \qquad Q \text{ being the Ricci operator} \tag{3.2}$$

Furthermore, in the case of generalized (k,μ)*-nullity condition, one has*

$$\nabla_\xi h = -2h - \mu\varphi h \tag{3.3}$$

$$R_{\xi X}Y = k(g(X,Y)\xi - \eta(Y)X) + \mu(g(hX,Y)\xi - \eta(Y)hX) \tag{3.4}$$

and in the case of generalized $(k,\mu)'$*-nullity condition, one has*

$$\nabla_\xi h' = -(\mu+2)h' \tag{3.5}$$

$$R_{\xi X}Y = k(g(X,Y)\xi - \eta(Y)X) + \mu(g(h'X,Y)\xi - \eta(Y)h'X) \tag{3.6}$$

Proof. From the (k,μ)-nullity condition we soon obtain

$$l(X) = R_{X\xi}\xi = -k\varphi^2(X) + \mu h(X) \tag{3.7}$$

for any X, so that (2.5) implies (3.1). In the case of the $(k,\mu)'$-nullity condition, we use (2.5) referred to h', namely $\varphi l\varphi - l = 2(-\varphi^2 + {h'}^2)$, since ${h'}^2 = h^2$. As for the second equation, we first remark that, if X is an eigenvector of h with eigenvalue

λ, (3.1) gives $\lambda^2 X = -(k+1)X$ and hence $\lambda^2 = -(k+1)$. It follows that $k < -1$ and h and h' both have the eigenvalues: 0 as simple eigenvalue, $\lambda = \sqrt{-1-k}$ and $-\lambda$. Moreover, for any X orthogonal to ξ, we have $Ric(X,\xi) = 0$, since $R_{XY}\xi = 0$, for any $X, Y \in \mathcal{D}$. Now, we choose an orthonormal local basis of eigenvectors of h (or of h') of the form $\{\xi, e_i, \varphi e_i\}_{1\le i\le n}$ with $e_i \in [\lambda]$ (or $e_i \in [\lambda]'$) and we find

$$\begin{aligned} g(Q(\xi),\xi) &= \textstyle\sum_{i=1}^n g(R_{e_i\xi}\xi, e_i) + \sum_{i=1}^n g(R_{\varphi e_i\xi}\xi, \varphi e_i) \\ &= n(k+\lambda\mu) + n(k-\lambda\mu) = 2nk, \end{aligned}$$

obtaining (3.2).

In the case of generalized (k,μ)-nullity condition, applying φ to (2.5) and taking into account that $\eta((\nabla_\xi h)X) = 0$ for any vector field X, we get

$$(\nabla_\xi h)X = -\varphi X - 2hX - \varphi h^2 X - \varphi l X,$$

which, using (3.7) and (3.1), gives (3.3). Finally, the last relation is a consequence of the symmetries of the curvature tensor field and of (1.1).

In the case of the generalized $(k,\mu)'$-nullity condition, using (2.4), we obtain that $l = \varphi^2 - 2h' - h'^2 - \nabla_\xi h'$, hence $-(k+1)\varphi^2 = -(\mu+2)h' - h'^2 - \nabla_\xi h'$ and (3.5) follows, applying (3.1). Again, the last relation follows from the symmetries of the curvature tensor field and from (1.2). □

Remark 3.1. If $h \neq 0$ each of the two nullity conditions implies that the almost Kenmotsu structure is CR-integrable. Indeed, for an almost Kenmotsu manifold, the CR-integrability is equivalent to the fact that the integral submanifolds of $\mathcal{D}$ are Kähler. This happens if and only if $(\nabla_X\varphi)Y = g(\varphi X + hX, Y)\xi$, for all $X, Y \in \mathcal{D}$ ([10]). In our case, this condition is satisfied. Indeed, if $X, Y \in \mathcal{D}$, by (3.4) or (3.6), we have $R_{\xi X}Y \in [\xi]$. Then, if we take Z in $\mathcal{D}$, being $h \neq 0$, (2.6) reduces to $(\nabla_{hX}\Phi)(Y,Z) = 0$, the structure is CR-integrable and, for any $X, Y \in \mathcal{D}$,

$$(\nabla_X\varphi)Y \in [\xi]. \tag{3.8}$$

Proposition 3.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h \neq 0$. If the generalized (k,μ)-nullity condition holds, then:*

$$\xi(\lambda) = -2\lambda, \qquad \xi(k) = -4(k+1).$$

If the generalized $(k,\mu)'$-nullity condition holds, then:

$$\xi(\lambda) = -\lambda(\mu+2), \qquad \xi(k) = -2(k+1)(\mu+2).$$

Moreover, in both cases, if $2n+1 \ge 5$, then for any $X \in \mathcal{D}$:

$$X(\lambda) = 0, \;\; X(k) = 0, \;\; X(\mu) = 0.$$

Proof. Let X be a unit eigenvector of h corresponding to the eigenvalue λ. Applying (3.3), we obtain $\xi(\lambda)X + \lambda\nabla_\xi X - h(\nabla_\xi X) = -2\lambda X - \mu\lambda\varphi X$ and the first formula, by taking the scalar product with X. Since $k = -1-\lambda^2$ the second formula follows. Analogously, let X be a unit eigenvector of h' corresponding to the eigenvalue λ. Applying (3.5), we obtain $\xi(\lambda)X + \lambda\nabla_\xi X - h'(\nabla_\xi X) = -\lambda(\mu+2)X$, and hence the first formula, by taking the scalar product with X. Then, since $\lambda^2 = -1-k$, we have $\xi(k) = -2(k+1)(\mu+2)$.

Finally, being $k < -1$, the non-zero eigenvalues of h are $\lambda = \sqrt{-1-k}$ and $-\lambda$ and both the eigendistributions $[\lambda]$ and $[-\lambda]$ have constant dimension n. Next, comparing the nullity conditions with (2.4), for any $X, Y \in \mathcal{D}$, we obtain

$$(\nabla_X h')Y - (\nabla_Y h')X = 0, \tag{3.9}$$

which implies that h' acts as a Codazzi tensor on each integral submanifold M^{2n} of the distribution $\mathcal{D}$ and, being $n > 1$, the eigenfunctions λ, $-\lambda$ are constant along $\mathcal{D}$ since $\mathcal{D} = [\lambda]' \oplus [-\lambda]' = [\lambda] \oplus [-\lambda]$. Thus, for any $X \in \mathcal{D}$ we have $X(\lambda) = 0$ and $X(k) = 0$ follows from $\lambda^2 = -1 - k$.

In the case of generalized $(k, \mu)'$-nullity condition, from $\xi(\lambda) = -\lambda(\mu+2)$, we get $X(\mu) = 0$ since, for $X \in \mathcal{D}$, $[X, \xi] \in \mathcal{D}$ implies $0 = [X, \xi](\lambda) = X(\xi(\lambda)) = -\lambda X(\mu)$.

Instead, to prove $X(\mu) = 0$ in the case of generalized (k, μ)-nullity condition, we use the second Bianchi identity. We have, for $X, Y \in \mathcal{D}$

$$(\nabla_X R)(Y, \xi, \xi) - (\nabla_Y R)(X, \xi, \xi) + (\nabla_\xi R)(X, Y, \xi) = 0. \tag{3.10}$$

The nullity condition implies the vanishing of the third term. Using (3.4) and taking into account that $\nabla_X \xi \in \mathcal{D}$, we find

$$\begin{aligned}(\nabla_X R)(Y, \xi, \xi) &= \nabla_X (R(Y, \xi)\xi) - R(\nabla_X Y, \xi)\xi - R(Y, \xi)\nabla_X \xi \\ &= X(k)Y + X(\mu)hY + \mu(\nabla_X h)Y + \mu g(hY, X + h\varphi X)\xi.\end{aligned}$$

Thus, since $X(k) = Y(k) = 0$, (3.10) becomes

$$0 = X(\mu)hY - Y(\mu)hX + \mu((\nabla_X h)Y - (\nabla_Y h)X - 2g(hX, h\varphi Y)\xi). \tag{3.11}$$

Now, from (3.9) and (3.8), one easily deduces

$$(\nabla_X h)Y - (\nabla_Y h)X \in [\xi], \tag{3.12}$$

for any $X, Y \in \mathcal{D}$ and, by a simple computation, we obtain

$$(\nabla_X h)Y - (\nabla_Y h)X = 2g(hX, h\varphi Y)\xi.$$

Therefore, (3.11) reduces to $X(\mu)hY - Y(\mu)hX = 0$.
Finally, choosing $X \in [\lambda]$ and $Y \in [-\lambda]$, we get $\lambda\big(X(\mu)Y + Y(\mu)X\big) = 0$. Then $\lambda \neq 0$ and X, Y linearly independent imply $X(\mu) = Y(\mu) = 0$. This means that $Z(\mu) = 0$ for any $Z \in \mathcal{D}$, since $\mathcal{D} = [\lambda] \oplus [-\lambda]$. □

Examples with non constant k, μ are constructed in last section.

Proposition 3.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold, with $h \neq 0$ and $n > 1$, satisfying either the generalized (k, μ)-nullity condition or the generalized $(k, \mu)'$-nullity condition. Then, h and h' are η-parallel, that is for any $X, Y, Z \in \mathcal{D}$, one has*

$$g((\nabla_X h)Y, Z) = 0 \qquad g((\nabla_X h')Y, Z) = 0$$

Proof. We notice that the two nullity conditions imply (3.9) and (3.12). We discuss the η-parallelism of h. If $X, Y, Z \in \mathcal{D}$, (3.12) implies

$$g((\nabla_X h)Y, Z) = g((\nabla_Y h)X, Z). \tag{3.13}$$

Then, if $X \in \mathcal{D}$ and Y, Z are in the same eigenspace of h, since $X(\lambda) = 0$, a direct computation gives $g\big((\nabla_X h)Y, Z\big) = 0$. By the symmetry of $\nabla_X h$ with respect to g, using (3.13), for any $X, Y \in [\lambda]$ and $Z \in \mathcal{D}$, we also have $g((\nabla_X h)Y, Z) = g(Y, (\nabla_X h)Z) = g(Y, (\nabla_Z h)X) = 0$.
Therefore, $(\nabla_X h)Y$, with $X, Y \in [\lambda]$, has only the component along ξ. Since $g((\nabla_X h)Y, \xi) = -g(Y, h(X - \varphi hX)) = -\lambda g(X, Y)$, being $\varphi X \in [-\lambda]$, we obtain $(\nabla_X h)Y = -\lambda g(X, Y)\xi$, for $X, Y \in [\lambda]$.
Analogously, if $X, Y \in [-\lambda]$, we get $(\nabla_X h)Y = \lambda g(X, Y)\xi$, and, finally, if $X \in [\lambda]$ and $Y \in [-\lambda]$, we have $(\nabla_X h)Y = \lambda^2 g(X, \varphi Y)\xi$.
Now writing $Z \in \mathcal{D}$ as $Z_\lambda + Z_{-\lambda}$, where Z_λ and $Z_{-\lambda}$ denote the component of Z in

$[\lambda]$ and $[-\lambda]$, respectively, it follows that, for any $X,Y,Z \in \mathcal{D}$, $g((\nabla_X h)Y,Z)=0$ and h is η-parallel. Moreover, we have

$$g((\nabla_X h)Y,\xi)=g(Y,(\nabla_X h)\xi)=-g(Y,h(\nabla_X\xi))=-g(hX+h^2\varphi X,Y),$$

hence $(\nabla_X h)Y=-g(hX+h^2\varphi X,Y)\xi$.
Since $h'=h\circ\varphi$, being φ η-parallel, the η-parallelism of h' follows. We get

$$g((\nabla_X h')Y,\xi)=g(Y,(\nabla_X h')\xi)=-g(Y,h'(\nabla_X\xi))=-g(h'X+h'^2X,Y)$$

and $(\nabla_X h')Y=-g(h'X+h'^2X,Y)\xi$.

Finally, differentiating (3.1), for $X,Y\in\mathcal{D}$, we get $(\nabla_X h^2)Y=-(k+1)\eta(\nabla_X Y)$. Being $\mathcal{D}$ integrable and $h^2=h'^2$ we obtain

$$(\nabla_X h^2)Y-(\nabla_Y h^2)X=0\,,\quad (\nabla_X h'^2)Y-(\nabla_Y h'^2)X=0. \tag{3.14}$$

for all $X,Y\in\mathcal{D}$. □

4. The generalized (k,μ)-nullity condition

In this section we deeply analyze almost Kenmotsu manifolds M^{2n+1}, of dimension $2n+1\geq 5$, whose characteristic vector field ξ belongs to the generalized (k,μ)-nullity distribution.

Proposition 4.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost Kenmotsu manifold satisfying the generalized (k,μ)-nullity condition with $h\neq 0$ and $n>1$. Then, for any X,Y, one has*

$$(\nabla_X h)Y=-g(hX+h^2\varphi X,Y)\xi-\eta(Y)(hX+h^2\varphi X)-2\eta(X)hY-\mu\eta(X)\varphi hY\,.$$

Proof. From Proposition 3.3 we know that $(\nabla_X h)Y=-g(hX+h^2\varphi X,Y)\xi$, for $X,Y\in\mathcal{D}$. Then, we write any vector field X on M^{2n+1} as $X_{\mathcal{D}}+\eta(X)\xi$, $X_{\mathcal{D}}$ denoting the component of X in $\mathcal{D}$ and, using (3.3), we have

$$\begin{aligned}(\nabla_X h)Y &= (\nabla_{X_{\mathcal{D}}}h)Y_{\mathcal{D}}+\eta(Y)(\nabla_{X_{\mathcal{D}}}h)\xi+\eta(X)(-2hY-\mu\varphi hY)\\ &= -g(hX+h^2\varphi X,Y)\xi-\eta(Y)(hX+h^2\varphi X)+\eta(X)(-2hY-\mu\varphi hY),\end{aligned}$$

which concludes the proof. □

We now state some curvature properties.

Lemma 4.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$, $n>1$, be an almost Kenmotsu manifold satisfying the generalized (k,μ)-nullity condition and $h\neq 0$. Then, for any $X,Y,Z\in\mathcal{D}$,*

$$\begin{aligned}R_{XY}hZ-hR_{XY}Z &= -g(hY+h^2\varphi Y,Z)(X+h\varphi X)\\ &\quad +g(hX+h^2\varphi X,Z)(Y+h\varphi Y)\\ &\quad -g(X+h\varphi X,Z)(hY+h^2\varphi Y)\\ &\quad +g(Y+h\varphi Y,Z)(hX+h^2\varphi X).\end{aligned}$$

Proof. We consider the Ricci identity for h and applying Proposition 4.1, by direct computation, we obtain

$$\begin{aligned}R_{XY}hZ-hR_{XY}Z &= -g((\nabla_X h)Y-(\nabla_Y h)X,Z)\xi\\ &\quad +g((\nabla_X h^2)Y-(\nabla_Y h^2)X,\varphi Z)\xi\\ &\quad -g(\nabla_Y Z,\xi)(hX+h^2\varphi X)+g(\nabla_X Z,\xi)(hY+h^2\varphi Y)\\ &\quad -g(hY+h^2\varphi Y,Z)\nabla_X\xi+g(hX+h^2\varphi X,Z)\nabla_Y\xi\\ &\quad -g(h^2X,(\nabla_Y\varphi)Z)\xi+g(h^2Y,(\nabla_X\varphi)Z)\xi.\end{aligned}$$

Therefore, taking into account (3.14), the η-parallelism of φ and h, and completing the above computation, we obtain the desired formula. □

Lemma 4.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$, be an almost Kenmotsu manifold satisfying the generalized (k, μ)-nullity condition with $h \neq 0$. Then, for any $X, Y, Z \in \mathcal{D}$, one has*

$$\begin{aligned} R_{XY}\varphi Z - \varphi R_{XY}Z \quad &= g(\varphi Y + hY, Z)(X + h\varphi X) - g(\varphi X + hX, Z)(Y + h\varphi Y) \\ &\quad - g(X + h\varphi X, Z)(\varphi Y + hY) + g(Y + h\varphi Y, Z)(\varphi X + hX). \end{aligned}$$

Proof. Owing to the CR-integrability, each integral submanifold M' of $\mathcal{D}$ is a Kähler manifold, so that $R'_{XY}\varphi Z - \varphi R'_{XY}Z = 0$, for any $X, Y, Z \in \Gamma(TM')$, where R' denotes the curvature tensor field of M'. On the other side, as hypersurface of M^{2n+1}, M' has the Weingarten operator A given by $AX = -\nabla_X \xi = -(X + h\varphi X)$. Hence, using the Gauss equation, we get

$$R_{XY}Z = R'_{XY}Z + g(X + h\varphi X, Z)(Y + h\varphi Y) - g(Y + h\varphi Y, Z)(X + h\varphi X).$$

and the required formula follows. □

We shall prove that the generalized (k, μ)-nullity condition completely determines the curvature tensor field. To this aim, taking into account the nullity condition and (3.4), it is enough to determine R on $\mathcal{D}$.

Theorem 4.1. *The curvature tensor of an almost Kenmotsu manifold M^{2n+1}, $n > 1$, satisfying the generalized (k, μ)-nullity condition with $h \neq 0$, satisfies, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]$:*

$$\begin{aligned} R_{X_\lambda Y_\lambda}Z_{-\lambda} &= \lambda[g(\varphi Y_\lambda, Z_{-\lambda})X_\lambda - g(\varphi X_\lambda, Z_{-\lambda})Y_\lambda] \\ &\quad - \lambda^2[g(\varphi Y_\lambda, Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda, Z_{-\lambda})\varphi Y_\lambda], \end{aligned}$$

$$\begin{aligned} R_{X_{-\lambda} Y_{-\lambda}}Z_\lambda &= -\lambda^2[g(\varphi Y_{-\lambda}, Z_\lambda)\varphi X_{-\lambda} - g(\varphi X_{-\lambda}, Z_\lambda)\varphi Y_{-\lambda}] \\ &\quad + \lambda[g(\varphi X_{-\lambda}, Z_\lambda)Y_{-\lambda} - g(\varphi Y_{-\lambda}, Z_\lambda)X_{-\lambda}], \end{aligned}$$

$$\begin{aligned} R_{X_\lambda Y_{-\lambda}}Z_{-\lambda} &= -g(Y_{-\lambda}, Z_{-\lambda})X_\lambda - \lambda^2 g(\varphi X_\lambda, Z_{-\lambda})\varphi Y_{-\lambda} \\ &\quad + \lambda[g(Y_{-\lambda}, Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda, Z_{-\lambda})Y_{-\lambda}], \end{aligned}$$

$$\begin{aligned} R_{X_\lambda Y_{-\lambda}}Z_\lambda &= g(X_\lambda, Z_\lambda)Y_{-\lambda} + \lambda^2 g(\varphi Y_{-\lambda}, Z_\lambda)\varphi X_\lambda \\ &\quad + \lambda[g(X_\lambda, Z_\lambda)\varphi Y_{-\lambda} - g(\varphi Y_{-\lambda}, Z_\lambda)X_\lambda], \end{aligned}$$

$$\begin{aligned} R_{X_\lambda Y_\lambda}Z_\lambda &= -g(Y_\lambda, Z_\lambda)X_\lambda + g(X_\lambda, Z_\lambda)Y_\lambda \\ &\quad + \lambda[g(Y_\lambda, Z_\lambda)\varphi X_\lambda - g(X_\lambda, Z_\lambda)\varphi Y_\lambda], \end{aligned}$$

$$\begin{aligned} R_{X_{-\lambda} Y_{-\lambda}}Z_{-\lambda} &= -g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} + g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda} \\ &\quad - \lambda[g(Y_{-\lambda}, Z_{-\lambda})\varphi X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})\varphi Y_{-\lambda}]. \end{aligned}$$

Proof. First of all, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]$, applying Lemma 4.1, we get

$$\lambda R_{X_\lambda Y_\lambda}Z_\lambda - hR_{X_\lambda Y_\lambda}Z_\lambda = 2\lambda^2[g(Y_\lambda, Z_\lambda)\varphi X_\lambda - g(X_\lambda, Z_\lambda)\varphi Y_\lambda]$$

and, by scalar multiplication with $W_{-\lambda} \in [-\lambda]$, one has

$$2\lambda g(R_{X_\lambda Y_\lambda}Z_\lambda, W_{-\lambda}) = 2\lambda^2[g(Y_\lambda, Z_\lambda)g(\varphi X_\lambda, W_{-\lambda}) - g(X_\lambda, Z_\lambda)g(\varphi Y_\lambda, W_{-\lambda})]$$

from which, being $\lambda \neq 0$,

$$g(R_{X_\lambda Y_\lambda} Z_\lambda, W_{-\lambda}) = \lambda[g(Y_\lambda, Z_\lambda) g(\varphi X_\lambda, W_{-\lambda}) - g(X_\lambda, Z_\lambda) g(\varphi Y_\lambda, W_{-\lambda})]. \tag{4.1}$$

With a similar argument, for any $X_\lambda, W_\lambda \in [\lambda]$ and $Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]$, we also obtain

$$g(R_{X_\lambda Y_{-\lambda}} Z_{-\lambda}, W_\lambda) = -g(Y_{-\lambda}, Z_{-\lambda}) g(X_\lambda, W_\lambda) - \lambda^2 g(\varphi X_\lambda, Z_{-\lambda}) g(\varphi Y_{-\lambda}, W_\lambda) \tag{4.2}$$

and, from (4.1), by symmetries of the tensor field R, for any $X_\lambda, Y_\lambda, W_\lambda \in [\lambda]$ and $Z_{-\lambda} \in [-\lambda]$

$$g(R_{X_\lambda Y_\lambda} Z_{-\lambda}, W_\lambda) = \lambda[g(\varphi Y_\lambda, Z_{-\lambda}) g(X_\lambda, W_\lambda) - g(\varphi X_\lambda, Z_{-\lambda}) g(Y_\lambda, W_\lambda)]. \tag{4.3}$$

Next, fixed a local φ-basis $(\xi, e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n)$, with $e_i \in [\lambda]$ we compute $R_{X_\lambda Y_\lambda} Z_{-\lambda}$. The nullity condition implies $g(R_{X_\lambda Y_\lambda} Z_{-\lambda}, \xi) = 0$, while, using the first Bianchi identity, (4.2) and (4.3), we get

$$g(R_{X_\lambda Y_\lambda} Z_{-\lambda}, e_i) = \lambda[g(\varphi Y_\lambda, Z_{-\lambda}) g(X_\lambda, e_i) - g(\varphi X_\lambda, Z_{-\lambda}) g(Y_\lambda, e_i)]$$

$$g(R_{X_\lambda Y_\lambda} Z_{-\lambda}, \varphi e_i) = -\lambda^2 [g(X_\lambda, \varphi Z_{-\lambda}) g(Y_\lambda, e_i) - g(\varphi Z_{-\lambda}, Y_\lambda) g(X_\lambda, e_i)],$$

so that, summing on i, the expression for $R_{X_\lambda Y_\lambda} Z_{-\lambda}$ follows.
The terms $R_{X_{-\lambda} Y_{-\lambda}} Z_\lambda$ and $R_{X_\lambda Y_{-\lambda}} Z_{-\lambda}$ are computed in a similar manner.
Now, acting by φ on the formula just proved and using Lemma 4.2, we get

$$R_{X_\lambda Y_\lambda} \varphi Z_{-\lambda} = g(\varphi Y_\lambda, Z_{-\lambda}) X_\lambda - g(\varphi X_\lambda, Z_{-\lambda}) Y_\lambda - \lambda[g(\varphi Y_\lambda, Z_{-\lambda}) \varphi X_\lambda - g(\varphi X_\lambda, Z_{-\lambda}) \varphi Y_\lambda].$$

Writing this formula for φZ_λ, by the compatibility condition, we have the result for $R_{X_\lambda Y_\lambda} Z_\lambda$. Similar computation yields $R_{X_{-\lambda} Y_{-\lambda}} Z_{-\lambda}$. Analogously, using the third formula and Lemma 4.2 we obtain $R_{X_\lambda Y_{-\lambda}} Z_\lambda$. □

Now, we are able to compute some sectional curvatures.

Proposition 4.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$, $n > 1$, be an almost Kenmotsu manifold satisfying the generalized (k, μ)-nullity condition with $h \neq 0$. Then, for the sectional curvatures we have:*

a) $K(X, \xi) = \begin{cases} k + \lambda\mu & \text{if } X \in [\lambda] \\ k - \lambda\mu & \text{if } X \in [-\lambda]. \end{cases}$

b) $K(X, Y) = \begin{cases} -1 & \text{if } X, Y \in [\lambda] \text{ or } X, Y \in [-\lambda] \\ -1 + \lambda^2 \big(g(X, \varphi Y)\big)^2 & \text{if } X \in [\lambda], Y \in [-\lambda]. \end{cases}$

c) *The scalar curvature is given by $Sc = 2nk - 4n^2$.*

Proof. From the nullity condition, one immediately obtains the sectional curvature of a plane section containing ξ. The expressions in b) easily follow from the previous proposition. Finally, choosing a local orthonormal frame $(\xi, e_i, \varphi e_i)_{1 \leq i \leq n}$, we get the Ricci tensor expressions

$$Ric(\xi, \xi) = 2nk, \; Ric(e_i, e_i) = \lambda\mu - 2n, \; Ric(\varphi e_i, \varphi e_i) = -\lambda\mu - 2n,$$

and the scalar curvature is given by $Sc = 2nk - 4n^2$. □

Corollary 4.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold. If $n > 1$ and ξ belongs to the generalized (k, μ)-nullity distribution with $h \neq 0$, then the leaves of $\mathcal{D}$ are flat Kählerian manifolds.*

Proof. By Remark 3.1, we already know that each of the integral submanifolds of $\mathcal{D}$ is a Kählerian manifold. Now, we prove the flatness. Let $\bar{M}$ be a leaf of $\mathcal{D}$. As hypersurface of M^{2n+1}, $\bar{M}$ has the Weingarten operator A given by $AX = -\nabla_X \xi = -(X - \varphi h X)$. Hence, applying the Gauss equation, we get

$$\bar{K}(X,Y) = K(X,Y) + \big(1 + g(hY, \varphi Y)\big)\big(1 + g(hX, \varphi X)\big) - g(hX, \varphi Y)^2 ,$$

for any orthonormal $X, Y \in \Gamma(T\bar{M})$. Therefore, the result follows from the previous proposition. □

5. The generalized $(k,\mu)'$-nullity condition

This section is devoted to study almost Kenmotsu manifolds, of dimension at least 5, satisfying the generalized $(k,\mu)'$-nullity condition with $h' \neq 0$.

Being h' η-parallel, from Proposition 1 in [9] we soon obtain the explicit expression of $\nabla h'$.

Proposition 5.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$, $n > 1$, be an almost Kenmotsu manifold satisfying the generalized $(k,\mu)'$-nullity condition with $h' \neq 0$. Then, we have:*

$$(\nabla_X h')Y = -g(h'X + h'^2 X, Y)\xi - \eta(Y)(h'X + h'^2 X) - (\mu + 2)\eta(X)h'Y,$$

for any $X, Y \in \Gamma(TM^{2n+1})$.

Proof. Proposition 1 in [9] states that

$$(\nabla_X h')Y = -g(h'X + h'^2 X, Y)\xi - \eta(Y)(h'X + h'^2 X) + \eta(X)(\nabla_\xi h')Y$$

and using (3.5) we get the required formula. □

Lemma 5.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$, $n > 1$, be an almost Kenmotsu manifold satisfying the generalized $(k,\mu)'$-nullity condition with $h' \neq 0$. Then, for any $X, Y, Z \in \mathcal{D}$, one has*

$$\begin{aligned} R_{XY} h'Z - h'R_{XY} Z &= (k+2)[g(Y,Z)h'X - g(X,Z)h'Y \\ &\quad + g(h'X, Z)Y - g(h'Y, Z)X]. \end{aligned}$$

Proof. We can consider the Ricci identity for h' and obtain the formula by direct computation using Proposition 5.1 or, since $h' = h \circ \varphi$ implies

$$(R_{XY} h')Z = (R_{XY} h)(\varphi Z) + h((R_{XY}\varphi)Z),$$

then, using Proposition 3.3, Proposition 4.1 and Lemma 4.2, we obtain

$$\begin{aligned} (R_{XY} h')Z \;=\; & -g(Z, X + h'X)(h'Y + h'^2 Y) + g(Z, Y + h'Y)(h'X + h'^2 X) \\ & -g(h'Y + h'^2 Y, Z)(X + h'X) + g(h'X + h'^2 X, Z)(Y + h'Y). \end{aligned}$$

Finally, using $h'^2 = (k+1)\varphi^2$, we conclude the proof. □

Since the link between R_{XY} and φ does not involve the nullity conditions and, being $h' \neq 0$, the structure is CR-integrable, then Lemma 4.2 still holds, writing h' instead of $h \circ \varphi$.

Lemma 5.2. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$, be an almost Kenmotsu manifold satisfying the generalized $(k,\mu)'$-nullity condition with $h'\neq 0$. Then, for any $X,Y,Z\in\mathcal{D}$, one has*

$$\begin{aligned} R_{XY}\varphi Z-\varphi R_{XY}Z &= g(\varphi Y+\varphi h'Y,Z)(X+h'X)\\ &\quad - g(\varphi X+\varphi h'X,Z)(Y+h'Y)\\ &\quad - g(X+h'X,Z)(\varphi Y+\varphi h'Y)\\ &\quad + g(Y+h'Y,Z)(\varphi X+\varphi h'X). \end{aligned}$$

Theorem 5.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost Kenmotsu manifold satisfying the generalized $(k,\mu)'$-nullity condition with $h'\neq 0$ and $n>1$. Then, for any $X_\lambda,Y_\lambda,Z_\lambda\in[\lambda]'$ and $X_{-\lambda},Y_{-\lambda},Z_{-\lambda}\in[-\lambda]'$, the curvature tensor satisfies:*

$$\begin{aligned} &R_{X_\lambda Y_\lambda}Z_{-\lambda}=0,\quad R_{X_{-\lambda}Y_{-\lambda}}Z_\lambda=0,\\ &R_{X_\lambda Y_{-\lambda}}Z_\lambda=(2+k)g(X_\lambda,Z_\lambda)Y_{-\lambda},\\ &R_{X_\lambda Y_{-\lambda}}Z_{-\lambda}=-(2+k)g(Y_{-\lambda},Z_{-\lambda})X_\lambda,\\ &R_{X_\lambda Y_\lambda}Z_\lambda=(k-2\lambda)[g(Y_\lambda,Z_\lambda)X_\lambda-g(X_\lambda,Z_\lambda)Y_\lambda],\\ &R_{X_{-\lambda}Y_{-\lambda}}Z_{-\lambda}=(k+2\lambda)[g(Y_{-\lambda},Z_{-\lambda})X_{-\lambda}-g(X_{-\lambda},Z_{-\lambda})Y_{-\lambda}]. \end{aligned}$$

Proof. Applying Lemma 5.1, for any $X_\lambda\in[\lambda]'$ and $Y_{-\lambda},Z_{-\lambda}\in[-\lambda]'$, we have

$$-\lambda R_{X_\lambda Y_{-\lambda}}Z_{-\lambda}-h'R_{X_\lambda Y_{-\lambda}}Z_{-\lambda}=2\lambda(k+2)\big(g(Y_{-\lambda},Z_{-\lambda})X_\lambda+g(X_\lambda,Z_{-\lambda})Y_{-\lambda}\big).$$

At the same time, by scalar multiplication with $W_\lambda\in[\lambda]'$, being $\lambda\neq 0$, we get

$$g(R_{X_\lambda Y_{-\lambda}}Z_{-\lambda},W_\lambda)=-(k+2)g(Y_{-\lambda},Z_{-\lambda})g(X_\lambda,W_\lambda). \tag{5.1}$$

Lemma 5.1 implies that $R_{X_\lambda Y_\lambda}Z_\lambda\in[\lambda]'$ and $R_{X_{-\lambda}Y_{-\lambda}}Z_{-\lambda}\in[-\lambda]'$. Now, in order to compute $R_{X_\lambda Y_\lambda}Z_{-\lambda}$, we consider a local orthonormal frame $\{\xi,e_i,\varphi e_i\}$, with $e_i\in[\lambda]'$. The nullity condition gives $g(R_{X_\lambda Y_\lambda}Z_{-\lambda},\xi)=0$, while $R_{X_\lambda Y_\lambda}e_i\in[\lambda]'$ implies $g(R_{X_\lambda Y_\lambda}Z_{-\lambda},e_i)=0$. Using the first Bianchi identity and (5.1), we have

$$\begin{aligned} g(R_{X_\lambda Y_\lambda}Z_{-\lambda},\varphi e_i)&=g(R_{Y_\lambda Z_{-\lambda}}\varphi e_i,X_\lambda)-g(R_{X_\lambda Z_{-\lambda}}\varphi e_i,Y_\lambda)\\ &=-(k+2)\big(g(Z_{-\lambda},\varphi e_i)g(Y_\lambda,X_\lambda)\\ &\quad -g(Z_{-\lambda},\varphi e_i)g(X_\lambda,Y_\lambda)\big)=0, \end{aligned}$$

so that $R_{X_\lambda Y_\lambda}Z_{-\lambda}=0$. The terms $R_{X_{-\lambda}Y_{-\lambda}}Z_\lambda, R_{X_\lambda Y_{-\lambda}}Z_\lambda$ and $R_{X_\lambda Y_{-\lambda}}Z_{-\lambda}$ are computed in a similar way.

Now, by Lemma 5.2, using $R_{X_\lambda Y_\lambda}Z_{-\lambda}=0$, we get

$$R_{X_\lambda Y_\lambda}\varphi Z_{-\lambda}=(1+\lambda)^2\big(g(\varphi Y_\lambda,Z_{-\lambda})X_\lambda-g(\varphi X_\lambda,Z_{-\lambda})Y_\lambda\big).$$

Writing this formula for $\varphi Z_\lambda\in[-\lambda]'$, being $(1+\lambda)^2=-k+2\lambda$, we have

$$R_{X_\lambda Y_\lambda}Z_\lambda=-R_{X_\lambda Y_\lambda}\varphi(\varphi Z_\lambda)=(k-2\lambda)\big(g(Y_\lambda,Z_\lambda)X_\lambda-g(X_\lambda,Z_\lambda)Y_\lambda\big).$$

In the same manner, we obtain the result for $R_{X_{-\lambda}Y_{-\lambda}}Z_{-\lambda}$. □

Proposition 5.2. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$, $n>1$, be an almost Kenmotsu manifold satisfying the generalized $(k,\mu)'$-nullity condition with $h'\neq 0$. Then, for the sectional curvatures we have:*

a) $K(X,\xi)=\begin{cases} k+\lambda\mu & \text{if } X\in[\lambda]' \\ k-\lambda\mu & \text{if } X\in[-\lambda]'; \end{cases}$

b) $K(X,Y) = \begin{cases} k-2\lambda & \text{if } X,Y \in [\lambda]' \\ k+2\lambda & \text{if } X,Y \in [-\lambda]' \\ -(k+2) & \text{if } X \in [\lambda]', Y \in [-\lambda]'; \end{cases}$

c) *The scalar curvature Sc is given by $Sc = 2n(k-2n)$.*

Proof. The expressions in a) follow immediately from the nullity condition. In fact, for any unit $X \in [\lambda]'$, $R_{X\xi}\xi = (k+\lambda\mu)X$ and, taking the scalar product with X, one has $K(X,\xi) = k+\lambda\mu$. Now, Proposition 5.1 implies that, for any orthonormal $X,Y \in [\lambda]'$, $R_{XY}Y = (k-2\lambda)X$ which implies $K(X,Y) = k-2\lambda$. The remaining cases in b) are similar. Finally, the expression of the Ricci tensor in a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$ with $e_i \in [\lambda]'$ is given by

$$\begin{aligned} Ric(\xi,\xi) &= 2nk\,, \\ Ric(e_i,e_i) &= -2n(1+\lambda)+\lambda(\mu+2)\,, \\ Ric(\varphi e_i,\varphi e_i) &= -2n(1-\lambda)-\lambda(\mu+2). \end{aligned}$$

So, the scalar curvature is $Sc = 2n(k-2n)$. □

Corollary 5.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost Kenmotsu manifold. If $n>1$ and ξ belongs to the generalized $(k,\mu)'$-nullity distribution with $h' \neq 0$, then the leaves of $\mathcal{D}$ are flat bi-Lagrangian Kählerian manifolds.*

Proof. Let $\bar{M}$ be a leaf of $\mathcal{D}$. By Remark 3.1, we already know that $\bar{M}$ is a Kählerian manifold. Moreover, from Theorem 2.2, we get that $\bar{M}$ is locally the Riemannian product $M_\lambda \times M_{-\lambda}$ and $M_\lambda \times M_{-\lambda}$ is a bi-Lagrangian Kähler manifold. Now, to prove the flatness, we use the Gauss equation for a hypersurface and we get

$$\bar{K}(X,Y) = K(X,Y) + \big(1+g(h'Y,Y)\big)\big(1+g(h'X,X)\big) - g(X,h'Y)^2\,,$$

for any orthonormal $X,Y \in \Gamma(T\bar{M})$. Therefore, the result follows from the previous proposition. □

6. Examples

6.1. **Examples in any dimension.**

Example 6.1. Let $n \in \mathbb{N}, n \geq 1$ and consider on $\mathbb{R}^{2n+1}$ the following vector fields:

$$\xi = e^{-z}\frac{\partial}{\partial z}, \quad X_i = e^{-(z+e^z)}\frac{\partial}{\partial x_i}, \quad Y_i = e^{z-e^z}\frac{\partial}{\partial y_i}, \quad i=1,\dots,n,$$

where $(x_1,\dots x_n, y_1,\dots y_n, z)$ are the standard coordinates of $\mathbb{R}^{2n+1}$. They make up a global frame which satisfies

$$\begin{gathered} [\xi,X_i] = -(1+e^{-z})X_i, \qquad [\xi,Y_i] = -(1-e^{-z})Y_i, \\ [X_i,X_j] = [X_i,Y_j] = [Y_i,X_j] = [Y_i,Y_j] = 0, \end{gathered} \tag{6.1}$$

for any $i,j \in \{1,\dots,n\}$. We consider the 1-form η dual to ξ and the $(1,1)$-tensor field φ defined by putting $\varphi(\xi)=0$, $\varphi(X_i)=Y_i$, $\varphi(Y_i)=-X_i$. Let g be the Riemannian metric on $\mathbb{R}^{2n+1}$ such that the basis $\{\xi,X_i,Y_i\}_{1\leq i\leq n}$ is orthonormal. It can be easily verified that $(\mathbb{R}^{2n+1},\varphi,\xi,\eta,g)$ is an almost Kenmotsu manifold. Notice that X_i and Y_i are eigenvectors of h' with eigenvalues $\lambda = e^{-z}$ and $-\lambda$, respectively.

Now, we check that ξ belongs to the generalized $(k,\mu)'$-nullity distribution with

$k = -(1+e^{-2z})$ and $\mu = -2+e^{-z}$. If ∇ is the Levi-Civita connection of g, using the Koszul formula and (6.1), we obtain

$$\nabla_{X_i}X_j = -(1+e^{-z})\delta_{ij}\xi, \qquad \nabla_{X_i}Y_j = 0, \qquad \nabla_{X_i}\xi = (1+e^{-z})X_i,$$
$$\nabla_{Y_i}Y_j = -(1-e^{-z})\delta_{ij}\xi, \qquad \nabla_{Y_i}X_j = 0, \qquad \nabla_{Y_i}\xi = (1-e^{-z})Y_i,$$
$$\nabla_\xi X_i = 0, \qquad \nabla_\xi Y_i = 0, \qquad \nabla_\xi \xi = 0.$$

It follows that

$$\begin{aligned} R(X_i,X_j)\xi &= \nabla_{X_i}(\nabla_{X_j}\xi) - \nabla_{X_j}(\nabla_{X_i}\xi) \\ &= (1+e^{-z})\nabla_{X_i}X_j - (1+e^{-z})\nabla_{X_j}X_i \\ &= (1+e^{-z})[X_i,X_j] = 0. \end{aligned}$$

Analogously, we get $R(Y_i,Y_j)\xi = 0$ and $R(X_i,Y_j)\xi = 0$. Finally, we compute

$$\begin{aligned} R(X_i,\xi)\xi &= -\nabla_\xi(\nabla_{X_i}\xi) - \nabla_{[X_i,\xi]}\xi \\ &= -\xi(e^{-z})X_i - (1+e^{-z})\nabla_\xi X_i - (1+e^{-z})\nabla_{X_i}\xi \\ &= e^{-2z}X_i - (1+e^{-z})^2 X_i \\ &= -(1+e^{-2z})X_i + (-2+e^{-z})h'X_i. \end{aligned}$$

A similar computation gives $R(Y_i,\xi)\xi = -(1+e^{-2z})Y_i + (-2+e^{-z})h'Y_i$. This shows the announced assertion.

Example 6.2. Consider the open submanifold of $\mathbb{R}^{2n+1}$

$$M = \{(x_1,\ldots,x_n,y_1,\ldots,y_n,t) \in \mathbb{R}^{2n+1} \mid t > 0\}$$

with standard coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n,t)$. Let us consider

$$\xi = -t\frac{\partial}{\partial t}, \quad X_i = t(1+t^2)\frac{\partial}{\partial x_i} + t^3\frac{\partial}{\partial y_i}, \quad Y_i = -t^3\frac{\partial}{\partial x_i} + t(1-t^2)\frac{\partial}{\partial y_i}$$

as a global basis of M. The brackets of these vector fields are all zero except for

$$(6.2) \qquad [\xi,X_i] = -(1+2t^2)X_i - 2t^2Y_i, \quad [\xi,Y_i] = 2t^2X_i - (1-2t^2)Y_i,$$

for any $i = 1,\ldots,n$. We consider the 1-form $\eta = -(1/t)dt$, dual to ξ, and define a $(1,1)$-tensor field φ by putting $\varphi\xi = 0$, $\varphi X_i = Y_i$, $\varphi Y_i = -X_i$. Finally, let g be the Riemannian metric that makes the basis $\{\xi,X_i,Y_i\}_{1\le i\le n}$ orthonormal. It can be easily checked that (φ,ξ,η,g) is an almost Kenmotsu structure on M. Computing the operators h and h' by means of (6.2), we get

$$h\xi = 0, \qquad hX_i = 2t^2Y_i, \qquad hY_i = 2t^2X_i,$$
$$h'\xi = 0, \qquad h'X_i = 2t^2X_i, \qquad h'Y_i = -2t^2Y_i.$$

Since h and h' have the same eigenvalues, it follows that $\lambda = 2t^2$ and $-\lambda$ are eigenfunctions for h and the corresponding eigendistributions have constant dimension n. Now, we compute the Levi-Civita connection of g, by using the above orthonormal frame and the Koszul formula. Applying (2.2) to X_i and Y_i, we soon obtain

$$\nabla_{X_i}\xi = (1+\lambda)X_i, \qquad \nabla_{Y_i}\xi = (1-\lambda)Y_i.$$

Furthermore

$$\begin{aligned} \nabla_{X_i}X_j &= g(\nabla_{X_i}X_j,\xi)\xi = -g(X_j,\nabla_{X_i}\xi)\xi = -(1+\lambda)\delta_{ij}\xi, \\ \nabla_{Y_i}Y_j &= -g(Y_j,\nabla_{Y_i}\xi)\xi = -(1-\lambda)\delta_{ij}\xi, \\ \nabla_{X_i}Y_j &= -g(X_i,\nabla_{Y_j}\xi)\xi = 0. \end{aligned}$$

As the connection is torsion-free, we also get

$$\nabla_\xi X_i = \nabla_{X_i}\xi + [\xi, X_i] = -\lambda Y_i, \qquad \nabla_\xi Y_i = \nabla_{Y_i}\xi + [\xi, Y_i] = \lambda X_i.$$

Then we compute the curvature tensor R of g, obtaining

$$\begin{aligned} R(X_i, X_j)\xi &= \nabla_{X_i}(\nabla_{X_j}\xi) - \nabla_{X_j}(\nabla_{X_i}\xi) \\ &= (1+\lambda)\nabla_{X_i}X_j - (1+\lambda)\nabla_{X_j}X_i \\ &= (1+\lambda)[X_i, X_j] = 0. \end{aligned}$$

Similarly, we get $R(Y_i, Y_j)\xi = 0$ and $R(X_i, Y_j)\xi = 0$. Finally, we have

$$\begin{aligned} R(X_i, \xi)\xi &= -\nabla_\xi \nabla_{X_i}\xi - \nabla_{[X_i,\xi]}\xi \\ &= -\xi(\lambda)X_i - (1+\lambda)\nabla_\xi X_i + [\xi, X_i] + h'[\xi, X_i] \\ &= 4t^2 X_i + \lambda(1+\lambda)Y_i - (1+\lambda)X_i - \lambda Y_i - \lambda(1+\lambda)X_i + \lambda^2 Y_i \\ &= 2\lambda X_i - (1+\lambda)^2 X_i + 2\lambda^2 Y_i \\ &= -(1+\lambda^2)X_i + 2\lambda h X_i. \end{aligned}$$

In a similar manner one finds $R(Y_i, \xi)\xi = -(1+\lambda^2)Y_i + 2\lambda h Y_i$.
Hence ξ belongs to the generalized (k, μ)-nullity distribution with $k = -(1 + 4t^4)$ and $\mu = 4t^2$.

6.2. **Three dimensional examples.** We describe two examples of 3-dimensional almost Kenmotsu manifold satisfying the generalized $(k, \mu)'$-nullity condition and the generalized k-nullity condition, respectively.

Example 6.3. Let us denote the canonical coordinates on $\mathbb{R}^3$ by (x, y, z), and take the three dimensional manifold $M^3 \subset \mathbb{R}^3$ defined by

$$M^3 := \{(x, y, z) \in \mathbb{R}^3 | z > 0\}.$$

One may easily verify that putting

$$\xi := \frac{\partial}{\partial z}, \quad \eta := dz, \quad g := ze^{2z}dx^2 + \frac{e^{2z}}{z}dy^2 + dz^2,$$

$$\varphi(\xi) = 0\,, \quad \varphi\left(\frac{\partial}{\partial x}\right) = z\frac{\partial}{\partial y}\,, \quad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{1}{z}\frac{\partial}{\partial x},$$

(φ, ξ, η, g) is an almost contact metric structure on M^3. We shall check that $(M^3, \varphi, \xi, \eta, g)$ is an almost Kenmotsu manifold satisfying the generalized $(k, \mu)'$-nullity condition, with k, μ non-constant smooth functions.
Obviously, $d\eta = 0$ and to verify the condition $d\Phi = 2\eta \wedge \Phi$, we have only to evaluate it on the canonical basis of $\mathbb{R}^3$. Considering that all the Φ_{ij}'s are zero except for $\Phi_{12} := g\left(\frac{\partial}{\partial x}, \varphi\frac{\partial}{\partial y}\right) = -\frac{1}{z}g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = -e^{2z}$, a direct computation yields $d\Phi\left(\xi, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -\frac{2}{3}e^{2z}$ and $\eta \wedge \Phi\left(\xi, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -\frac{1}{3}e^{2z}$, which show that the condition is satisfied. Now, computing h', we obtain

$$h'(\xi) = 0\,, \quad h'\left(\frac{\partial}{\partial x}\right) = \frac{1}{2z}\frac{\partial}{\partial x}\,, \quad h'\left(\frac{\partial}{\partial y}\right) = -\frac{1}{2z}\frac{\partial}{\partial y}\,. \tag{6.3}$$

We point out that h' does not vanish, so that M^3 is not Kenmotsu, and $\lambda = 1/2z$ and $-\lambda$ are the non-zero eigenvalues of h'. According to the general theory of almost

Kenmotsu manifolds, for the Levi-Civita connection, we soon obtain $\nabla_\xi \xi = 0$ and, since $[\xi, \frac{\partial}{\partial x}] = 0 = [\xi, \frac{\partial}{\partial y}]$, by using (2.2), we get

$$\nabla_\xi \frac{\partial}{\partial x} = \nabla_{\frac{\partial}{\partial x}} \xi = \left(1 + \frac{1}{2z}\right)\frac{\partial}{\partial x}, \quad \nabla_\xi \frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}} \xi = \left(1 - \frac{1}{2z}\right)\frac{\partial}{\partial y}.$$

By means of the Koszul formula, since the g_{ij}'s are zero or depend only on z, we have $g\left(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}, \xi\right) = 0$, $g\left(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) = 0$, $g\left(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = 0$ from which it follows that $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = 0$.
For the curvature tensor we obtain $R_{\frac{\partial}{\partial x}\frac{\partial}{\partial y}}\xi = 0$, and
$R_{\frac{\partial}{\partial x}\xi}\xi = -\nabla_\xi\left((1 + \frac{1}{2z})\frac{\partial}{\partial x}\right) = \frac{1}{2z^2}\frac{\partial}{\partial x} - \left(1 + \frac{1}{2z}\right)^2 \frac{\partial}{\partial x} = \left(\frac{1}{2z^2} - 1 - \frac{1}{z} - \frac{1}{4z^2}\right)\frac{\partial}{\partial x}$. Analogously, one has $R_{\frac{\partial}{\partial y}\xi}\xi = \left(-\frac{1}{2z^2} - 1 + \frac{1}{z} - \frac{1}{4z^2}\right)\frac{\partial}{\partial y}$. Taking (6.3) into account, these two relations may be rewritten as

$$R_{\frac{\partial}{\partial x}\xi}\xi = \left(-1 - \frac{1}{4z^2}\right)\frac{\partial}{\partial x} + \left(\frac{1}{z} - 2\right)h'\left(\frac{\partial}{\partial x}\right)$$
$$R_{\frac{\partial}{\partial y}\xi}\xi = \left(-1 - \frac{1}{4z^2}\right)\frac{\partial}{\partial y} + \left(\frac{1}{z} - 2\right)h'\left(\frac{\partial}{\partial y}\right).$$

Thus, we see that ξ belongs to the generalized $(k, \mu)'$-nullity distribution, where k, μ are non-constant functions given by $k = -1 - \frac{1}{4z^2}$ and $\mu = -2 + \frac{1}{z}$, as provided by Proposition 3.2.

Example 6.4. Let us consider $\mathbb{R}^3$ with the natural coordinates x, y, z, and let $f(z) = -e^{-2z}$. We define on $\mathbb{R}^3$ an almost contact metric structure (φ, ξ, η, g) as follows:

$$\xi := \frac{\partial}{\partial z}, \qquad \eta := dz, \qquad g := e^{f(z)+2z}dx^2 + e^{2z-f(z)}dy^2 + dz^2,$$

$$\varphi(\xi) = 0\,, \quad \varphi\left(\frac{\partial}{\partial x}\right) = e^{f(z)}\frac{\partial}{\partial y}\,, \quad \varphi\left(\frac{\partial}{\partial y}\right) = -e^{-f(z)}\frac{\partial}{\partial x}.$$

Working as in the previous example, one sees that $\mathbb{R}^3$ with the structure (φ, ξ, η, g) is an almost Kenmotsu structure, which is not Kenmotsu, since the operator h does not vanish. In fact, it is given by

$$h(\xi) = 0, \qquad h\left(\frac{\partial}{\partial x}\right) = -f(z)e^{f(z)}\frac{\partial}{\partial y}, \qquad h\left(\frac{\partial}{\partial y}\right) = -\frac{f(z)}{e^{f(z)}}\frac{\partial}{\partial x}.$$

The positive eigenvalue λ of h is given by $\lambda = \frac{1}{2}f'(z) = e^{-2z}$, and one easily verifies that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are eigenvectors of h' with eigenvalues λ and $-\lambda$, respectively. Thus, denoted by ∇ the Levi-Civita connection, we get $\nabla_{\frac{\partial}{\partial x}}\xi = (1 + e^{-2z})\frac{\partial}{\partial x}$, $\nabla_{\frac{\partial}{\partial y}}\xi = (1 - e^{-2z})\frac{\partial}{\partial y}$, and using the Koszul formula, $\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = 0$. Finally, for the curvature tensor R, we have

$$R_{\frac{\partial}{\partial x}\frac{\partial}{\partial y}}\xi = 0\,, \qquad R_{\frac{\partial}{\partial x}\xi}\xi = \left(-1 - \frac{1}{e^{4z}}\right)\frac{\partial}{\partial x}\,, \qquad R_{\frac{\partial}{\partial y}\xi}\xi = \left(-1 - \frac{1}{e^{4z}}\right)\frac{\partial}{\partial y}.$$

Therefore, $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ is an almost Kenmotsu manifold with ξ belonging to the generalized k-nullity distribution, $k = -1 - e^{-4z}$ non-constant function, and $h \neq 0$.

References

[1] Blair, D. E., Riemannian geometry of contact and symplectic manifolds, Progr. Math., 203, Birkhäuser Boston, Boston, MA, 2002.

[2] Blair, D. E., Koufogiorgos, T. and Papantoniou, B. J., Contact metric manifolds satisfying a nullity condition, Israel J. Math. **91** (1995), 189–214.

[3] Boeckx, E., A full classification of contact (k, μ)–spaces, Illinois J. Math. **44** (2000), 212–219.

[4] Dacko, P., On almost cosymplectic manifolds with the structure vector field ξ belonging to the k-nullity distribution, BJGA, **5** No.2 (2000) 47–60.

[5] Dacko, P. and Olszak, Z., On almost cosymplectic (k, μ, ν)-spaces, in: PDEs, Submanifolds and Affine Differential Geometry, Banach Center Publications, Vol **69**, pp. 211–220, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 2005.

[6] Dacko, P. and Olszak, Z., On almost cosymplectic $(-1, \mu, 0)$-space, Centr. Eur. J. Math. **3** No. 2 (2005), 318–330.

[7] Dileo, G. and Pastore, A. M., Almost Kenmotsu manifolds and local symmetry, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 343–354.

[8] Dileo, G. and Pastore, A. M., Almost Kenmotsu manifolds and nullity distributions, J. Geom. **93** (2009), 46–61.

[9] Dileo, G. and Pastore, A. M., Almost Kenmotsu manifolds with a condition of η-parallelism, Differential Geom. Appl. **27** (2009), 671–679.

[10] Falcitelli, M. and Pastore, A. M., Almost Kenmotsu f-manifolds, Balkan J. Geom. Appl. **12** (2007), no. 1, 32–43.

[11] Gouli-Andreou, F. and Xenos, P. J., A class of contact metric 3–manifolds with $\xi \in N(k, \mu)$ and k, μ functions, Algebras Groups Geom. **17** (2000), 401–407.

[12] Gray, A., Spaces of constancy of curvature operators, Proc. Amer. Math. Soc. **17** (1966), 897–902.

[13] Janssens, D. and Vanhecke, L., Almost contact structures and curvatures tensors, Kodai Math. J. **4** (1981), no. 1, 1–27.

[14] Kenmotsu, K., A class of almost contact Riemannian manifolds, Tôhoku Math. J. **24** (1972), 93–103.

[15] Kim, T. W. and Pak, H. K., Canonical foliations of certain classes of almost contact metric structures, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 4, 841–846.

[16] Kobayashi, S. and Nomizu, K., Foundations of Differential Geometry, Vol. I, II, Interscience Publishers, New York, 1963, 1969.

[17] Koufogiorgos, T. and Tsichlias, C., On the existence of a new class of contact metric manifolds, Canad. Math. Bull. Vol. **43** (2000), no. 4, 440–447.

[18] Olszak, Z., Locally conformal almost cosymplectic manifolds, Colloq. Math. **57** (1989), 73–87.

[19] Pastore, A. M. and Saltarelli, V., Almost Kenmotsu manifolds with conformal Reeb foliation, Bull. Belg. Math. Soc. Simon Stevin 18 (2011) (to appear).

[20] Tanno, S., Some differential equations on Riemannian manifolds, J. Math. Soc. Japan, **30** (1978), 509–531.

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