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CLASSIFICATION OF FLAT PSEUDO-KÄHLER SUBMANIFOLDS IN COMPLEX PSEUDO-EUCLIDEAN SPACES

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Abstract. Calabi [1] gave a classification of Kähler imbeddings of complete, simply-connected *definite* complex space forms into complete, simply-connected *definite* complex space forms. The local version of Calabi's result was obtained by Nakagawa and Ogiue in [5]. In contrast, no classification results were known for pseudo-Kähler immersions between *indefinite* complex space forms.

In this article, we initiate the study of the classification problem on pseudo-Kähler immersions between indefinite complex space forms. As a consequence, three classification theorems for pseudo-Kähler immersions between flat indefinite complex space forms are obtained.

1. Introduction

A pseudo-Riemannian metric g on a complex manifold M is called *pseudo-Hermitian* if the metric g and the almost complex structure J are compatible, that is,

$$g(JX, JY) = g(X, Y), \quad X, Y \in T_pM,\ p \in M. \tag{1.1}$$

It follows from (1.1) that the index of g is an even integer $2t$ with $0 \leq t \leq m$, $m = \dim_{\mathbf{C}} M$. The integer t is called the *complex index*.

The *fundamental* 2*-form* Ω of a pseudo-Hermitian manifold (M, g) is defined by

$$\Omega(X, Y) = g(X, JY), \quad X, Y \in TM. \tag{1.2}$$

A pseudo-Hermitian manifold is called *pseudo-Kähler* if its fundamental 2-form Ω is closed, that is, $d\Omega = 0$. The corresponding metric is called *pseudo-Kähler*.

A plane section on a pseudo-Kähler manifold is called *holomorphic* if it is spanned by $\{v, Jv\}$ for some non-null vector $v \in TM$. The sectional curvature $K(v \wedge Jv)$ of a holomorphic section is called the *holomorphic sectional curvature* at v, which is denoted by $H(v)$.

A pseudo-Kähler manifold with positive complex index is called an *indefinite complex space form* if it has constant holomorphic sectional curvature. We denote

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by $M_t^m(4c)$ a complex m-dimensional indefinite complex space form of constant holomorphic sectional curvature $4c$ and with complex index t.

The simplest example of indefinite complex space form is the flat complex pseudo-Euclidean m-space $\mathbf{C}_t^m$ with complex index t which is the complex m-space $\mathbf{C}^m$ endowed with the flat metric

$$g_0 = -\sum_{j=1}^{t} dz_j d\bar{z}_j + \sum_{j=t+1}^{m} dz_j d\bar{z}_j. \tag{1.3}$$

A pseudo-Riemannian submanifold N_s^n of a pseudo-Kähler manifold M_t^m is called a *complex submanifold* if each of its tangent spaces is invariant under the action of the almost complex structure J of M_t^m. By a *pseudo-Kähler submanifold* we mean a complex submanifold of a pseudo-Kähler manifold with its induced pseudo-Kählerian structure.

E. Calabi gave in [1] a classification of Kähler imbeddings of complete and simply connected *definite* complex space forms into complete and simply-connected *definite* complex space forms. The local version of Calabi's result was obtained by Nakagawa and Ogiue in [5]. In contrast, no classification results were known for pseudo-Kähler immersions between *indefinite* complex space forms.

In this article we initiate the study of the classification problem on pseudo-Kähler immersions between indefinite complex space forms. As a consequence, three classification theorems for pseudo-Kähler immersions between flat indefinite complex space forms are obtained.

2. Preliminaries

2.1. **Basic formulas.** Let $M_t^m(4c)$ denote a complete simply-connected pseudo-Kähler m-manifold with complex index t and with constant holomorphic sectional curvature $4c$. Then the curvature tensor $\tilde{R}$ of the indefinite complex space form $M_t^m(4c)$ satisfies

$$\begin{aligned}\tilde{R}(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &- g(JX,Z)JY + 2g(X,JY)JZ\}.\end{aligned} \tag{2.1}$$

Assume that N_s^n is a pseudo-Kähler submanifold of $M_t^m(4c)$. We denote the Levi-Civita connections of N_s^n and $M_t^m(4c)$ by ∇ and $\tilde{\nabla}$, respectively. The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.3}$$

for vector fields X and Y tangent to N_s^n and vector field ξ normal to N_s^n, where D is the normal connection.

The second fundamental form h is related to A_ξ by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle\,. \tag{2.4}$$

If we denote the curvature tensors of ∇ and D by R and R^D, respectively, then the equations of Gauss and Codazzi are given by

$$\langle R(X,Y)Z,W\rangle = \langle A_{h(Y,Z)}X,W\rangle - \langle A_{h(X,Z)}Y,W\rangle + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \tag{2.5}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.6}$$

$$R^D(X,Y;\xi,\eta) = \tilde{R}(X,Y;\xi,\eta) + \langle [A_\xi, A_\eta](X), Y\rangle, \tag{2.7}$$

where X,Y,Z,W (respectively, η and ξ) are vector fields tangent (respectively, normal) to N^n_s; and ∇h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{2.8}$$

For pseudo-Kähler submanifolds the following results are well-known (see, for instance, [3]).

Lemma 2.1. *The second fundamental form h, the shape operator A and the normal connection D of a pseudo-Kähler submanifold N^n_s of a pseudo-Kähler manifold M^m_t satisfy*

$$h(JX,Y) = h(X,JY) = Jh(X,Y), \tag{2.9}$$

$$A_{J\xi} = JA_\xi, \quad JA_\xi = -A_\xi J, \tag{2.10}$$

$$D_X J\xi = JD_X\xi, \tag{2.11}$$

for X,Y tangent to N^n_s and ξ normal to N^n_s.

Proof. Equations (2.9) and (2.10) can be found in [3, page 187]. Equation (2.11) follows immediately from $\tilde{\nabla}_X J\xi = J\tilde{\nabla}_X\xi$ and formula (2.3) of Weingarten. □

2.2. **Reduction theorem.** Let $\mathbf{R}^n_{i,j}$ denote the affine n-space equipped with the metric whose canonical form is

$$\begin{pmatrix} O_j & & \\ & -I_i & \\ & & I_{n-i-j} \end{pmatrix},$$

where I_k is the $k\times k$ identity matrix and O_j is the $j\times j$ zero matrix.

The metric is non-degenerate if and only if $j=0$. The j in $\mathbf{R}^n_{i,j}$ measures the degenerate part. The metric of $\mathbf{R}^n_{i,j} = \mathbf{R}^j_{0,j}\times\mathbb{E}^{n-j}_i$ is degenerate on the first factor $\mathbf{R}^j_{0,j}$ and it is the standard pseudo-Euclidean metric with index i on the second factor $\mathbb{E}^{n-j}_i$.

Similar notation holds for the complex space $\mathbf{C}^n_{i,j} = \mathbf{C}^j_{0,j}\times\mathbf{C}^{n-j}_i$.

Denote the natural embedding $\iota : \mathbf{C}^n_{i,j}\to\mathbf{C}^{n+j}_{i+j}$ given by

$$\iota((z_1,z_2,\ldots,z_n)) = (z_1,\ldots,z_j,z_{j+1},\ldots,z_n,z_j,\ldots,z_1)\in\mathbf{C}^{n+j}_{i+j} \tag{2.12}$$

for $(z_1,\ldots,z_n)\in\mathbf{C}^n_{i,j}$.

Let $\phi : N\to M$ be an isometric immersion of a pseudo-Riemannian manifold into another pseudo-Riemannian manifold. At each point $p\in N$, the *first normal space* at p, denoted by $\operatorname{Im} h_p$, is defined by

$$\operatorname{Im} h_p = \{h(u,v) : u,v\in T_p(N)\}.$$

The following result is known as the reduction theorem of Erbacher-Magid (cf. [4] or [3, page 40]).

Theorem 2.1. *Let $\phi : N_i^n \to \mathbb{E}_t^m$ be an isometric immersion of a pseudo-Riemannian n-manifold N_i^n with index i into the pseudo-Euclidean m-space $\mathbb{E}_t^m$. If the first normal spaces are parallel, then there exists a complete $(n+k)$-dimensional totally geodesic submanifold E^* such that $\psi(N_i^n) \subset E^*$, where k is the dimension of the first normal spaces.*

In Erbacher-Magid's reduction theorem, $E^* = \mathbf{R}_{s,t}^{n+k}$ for some s, t and t need not be zero.

3. Flat pseudo-Kähler submanifolds $N_s^n(0)$ in $\mathbf{C}_t^{n+2}$

A pseudo-Riemannian submanifold N_s^n of a pseudo-Riemannian manifold is called *isotropic* if $\langle h(v,v), h(v,v)\rangle$ is independent of the choice of the unit vector $v \in T_p(N_s^n)$ at each point $p \in N_s^n$, where h denotes the second fundamental form of N_s^n. Moreover, the pseudo-Riemannian submanifold N_s^n is called *null-isotropic* if its second fundamental form h satisfies

$$\langle h(u,u), h(u,u)\rangle = 0 \tag{3.1}$$

for any $u \in T(N_s^n)$.

The following result can be found in [3, page 189].

Proposition 3.1. *A pseudo-Kähler submanifold N_s^n of an indefinite complex space form $M_t^m(4c)$ has constant holomorphic sectional curvature $4c$ if and only if N_s^n is null-isotropic.*

To prove the main results, we need the following proposition.

Proposition 3.2. *Let N_s^n be a pseudo-Kähler submanifold of an indefinite complex space form $M_t^m(4c)$. Then the following three statements are equivalent:*

(i) *N_s^n is flat;*

(ii) *the second fundamental form h of N_s^n satisfies*

$$\langle h(u,v), h(w,\theta)\rangle = 0, \tag{3.2}$$

for any $u, v, w, \theta \in T(N_s^n)$;

(iii) *the shape operator A of N_s^n satisfies $A_\xi = 0$ for each $\xi \in \operatorname{Im} h$.*

Proof. Assume that N_s^n is a pseudo-Kähler submanifold of an indefinite complex space form $M_t^m(4c)$.

If N_s^n is a flat pseudo-Kähler submanifold, then N_s^n is null-isotropic according to Proposition 3.1. Thus condition (3.1) holds. After replacing the u in (3.1) by $u+v$, we find

$$\begin{aligned} &2\langle h(u,u), h(u,v)\rangle + 2\langle h(v,v), h(u,v)\rangle \\ &\quad + 2\langle h(u,v), h(u,v)\rangle + \langle h(u,u), h(v,v)\rangle = 0. \end{aligned} \tag{3.3}$$

Similarly, after replacing the u by $u-v$, we have

$$\begin{aligned} &2\langle h(u,u), h(u,v)\rangle + 2\langle h(v,v), h(u,v)\rangle \\ &\quad - 2\langle h(u,v), h(u,v)\rangle - \langle h(u,u), h(v,v)\rangle = 0. \end{aligned} \tag{3.4}$$

Thus, after combining (3.3) and (3.4), we get

$$\langle h(u,u), h(u,v)\rangle + \langle h(v,v), h(u,v)\rangle = 0, \tag{3.5}$$

$$2\langle h(u,v), h(u,v)\rangle + \langle h(u,u), h(v,v)\rangle = 0. \tag{3.6}$$

On the other hand, since N_s^n is a pseudo-Kähler submanifold of constant holomorphic sectional curvature $4c$ in an indefinite complex space form $M_t^m(4c)$ of the same constant holomorphic sectional curvature, it follows from (2.1) and Gauss' equation that

$$\langle h(u,v),h(u,v)\rangle - \langle h(u,u),h(v,v)\rangle = 0. \tag{3.7}$$

Thus, by combining (3.6) and (3.7), we obtain

$$\langle h(u,u),h(v,v)\rangle = 0 \tag{3.8}$$

for any u,v tangent to N_s^n. Now, after applying polarization we obtain (3.2) from (3.8). This shows that (i) implies (ii).

(ii) $\Longrightarrow$ (i) is obvious.

Now, let us assume that (ii) holds. Then we find from (2.4) and (3.2) that

$$\left\langle A_{h(u,v)}w,\theta\right\rangle = 0 \quad \forall u,v,w,\theta \in T(N_s^n). \tag{3.9}$$

Since N_s^n is pseudo-Riemannian, (3.9) implies that $A_\xi = 0$ for any $\xi \in \operatorname{Im} h$. This proves (iii).

Conversely, it follows from (2.4) that (iii) implies (ii). □

Theorem 3.1. *Let $\psi : N_s^n(0) \to \mathbf{C}_t^{n+2}$ be a flat pseudo-Kähler submanifold of the complex pseudo-Euclidean $(n+2)$-space $\mathbf{C}_t^{n+2}$. Then either*

(i) $t \in \{s, s+1, s+2\}$ *and ψ is a totally geodesic pseudo-Kähler immersion, or*

(ii) $t = s+1$ *and ψ is locally congruent to the following pseudo-Kähler immersion:*

$$\psi(z_1,\ldots,z_n) = \big(f(z), z_1,\ldots,z_n, f(z)\big),$$

where $f(z) := f(z_1,\ldots,z_n)$ is a non-trivial holomorphic function in $z_j = x_j + \mathrm{i}y_j$, $i = 1,\ldots,n$.

Proof. Let $\psi : N_s^n(0) \to \mathbf{C}_t^{n+2}$ be a pseudo-Kähler immersion of flat pseudo-Kähler manifold $N_s^n(0)$ into the complex pseudo-Euclidean $(n+2)$-space $\mathbf{C}_t^{n+2}$. Then ψ is null-isotropic according to Proposition 3.1.

Since $N_s^n(0)$ is a flat indefinite complex space form, $N_s^n(0)$ is locally holomorphically isometric to $\mathbf{C}_s^n$. Hence there exists local complex coordinates $z_1,\ldots,z_n$ such that the metric of $N_s^n(0)$ is given by

$$g_0 = -\sum_{k=1}^{s} dz_k d\bar z_k + \sum_{k=s+1}^{n} dz_k d\bar z_k. \tag{3.10}$$

Put

$$\partial_{x_i} = \frac{\partial}{\partial x_i},\ \ \partial_{y_i} = \frac{\partial}{\partial y_i},\ \ i = 1,\ldots,n.$$

Then we have

$$\partial_{y_i} = J\partial_{x_i},\ \ i = 1,\ldots,n, \tag{3.11}$$

Moreover, if we put

$$h\left(\partial_{x_i},\partial_{x_j}\right) = \eta_{ij},\ \ i,j = 1,\ldots,n. \tag{3.12}$$

we derive from (2.9) and (3.12) that

$$h\left(\partial_{x_i},\partial_{y_j}\right) = J\eta_{ij},\ \ h\left(\partial_{y_i},\partial_{y_j}\right) = -\eta_{ij},\ \ i,j = 1,\ldots,n. \tag{3.13}$$

It follows from Proposition 3.2 and (3.12) that, at a given point in $N_s^n(0)$, each η_{ij} is either zero or light-like.

Since $z_1, \dots, z_n$ are the natural complex coordinates of $\mathbf{C}^n_s$, we have

$$\nabla_{\partial_{x_i}}\partial_{x_j} = \nabla_{\partial_{x_i}}\partial_{y_j} = \nabla_{\partial_{y_i}}\partial_{y_j} = 0, \quad i, j = 1, \dots, n. \tag{3.14}$$

If ψ is totally geodesic, we get case (i).

Next, let us assume that ψ is non-totally geodesic. Then there exists at least one pair $(i, j), i, j \in \{1, \dots, n\}$ with $i \le j$ such that $\eta_{ij} \ne 0$. For simplicity, let us denote this η_{ij} by ξ_0.

It is clear that $\xi_0, J\xi_0$ span a light-like complex line subbundle of the normal bundle $T^\perp(N^n_s(0))$, that is,

$$\langle \xi_0, \xi_0 \rangle = \langle \xi_0, J\xi_0 \rangle = \langle J\xi_0, J\xi_0 \rangle = 0, \quad \xi_0 \ne 0. \tag{3.15}$$

Thus, the complex index of the normal bundle must be one. Therefore, we obtain $t = s + 1$.

If there exists another pair $(k, \ell) \ne (i, j)$ with $k \le \ell$ and $\eta_{k\ell} \ne 0$, then we also have

$$\langle \eta_{k\ell}, \eta_{k\ell} \rangle = \langle \eta_{k\ell}, J\eta_{k\ell} \rangle = \langle J\eta_{k\ell}, J\eta_{k\ell} \rangle = 0. \tag{3.16}$$

It follows from Proposition 3.2 that

$$\langle \xi_0, \eta_{k\ell} \rangle = \langle J\xi_0, J\eta_{k\ell} \rangle = 0. \tag{3.17}$$

Moreover, from Lemma 2.1 and Proposition 3.2, we also have

$$\langle \xi_0, J\eta_{k\ell} \rangle = 0. \tag{3.18}$$

From these we conclude that the first normal bundle, $\{\operatorname{Im} h_p, p \in N^n_s(0)\}$, is spanned by $\{\xi_0, J\xi_0\}$. Moreover, $\operatorname{Im} h$ is a complex line subbundle equipped with degenerate induced metric.

Now, by differentiating (3.15), we obtain

$$\langle D_X\xi_0, \xi_0 \rangle = \langle D_X J\xi_0, J\xi_0 \rangle = 0 \tag{3.19}$$

for any $X \in T(N^n_s(0))$.

On the other hand, after applying (3.12), (3.13), (3.14) and the equation of Codazzi we find

$$D_{\partial_{x_i}} J\xi_0 = D_{\partial_{y_i}}\xi_0, \quad D_{\partial_{y_i}} J\xi_0 = -D_{\partial_{x_i}}\xi_0. \tag{3.20}$$

By combining (3.19) and (3.20) we also have

$$\langle D_X\xi_0, J\xi_0 \rangle = 0, \quad X \in T(N^n_s(0)). \tag{3.21}$$

It follows from (3.19) and (3.21) that $D_X\xi_0, D_X J\xi_0 \in \operatorname{Im} h$. Therefore the first normal space is complex one-dimensional and it is light-like. Moreover, it is parallel in the normal bundle. Consequently, $\psi(N^n_s(0))$ is contained in a degenerate complex hyperplane $\mathbf{C}^{n+1}_{s,1}$ of $\mathbf{C}^{n+1}_{s+1}$ according to the Reduction Theorem of Erbacher-Magid (see Theorem 2.1 or [3, page 40]).

It is well-known that $\mathbf{C}^{n+1}_{s,1}$ is holomorphically isometric to the complex affine $(n+1)$-space $\mathbf{C}^1_{0,1} \times \mathbf{C}^n_s$, where $\mathbf{C}^1_{0,1}$ is equipped with a degenerate metric.

On the other hand, it follows from Proposition 3.2 that the shape operator satisfies

$$A_{\xi_0} = 0, \quad \forall \xi \in \operatorname{Im} h \tag{3.22}$$

Thus, we derive from (2.3) and (3.13) that

$$\tilde{\nabla}_X \xi_0, \tilde{\nabla}_X J\xi_0 \in \operatorname{Im} h \tag{3.23}$$

for every vector $X \in T(N_s^n(0))$. Hence the first normal bundle $\operatorname{Im} h$ is a constant complex line $\mathbf{C}^1_{0,1}$ with a degenerate metric in $\mathbf{C}^{n+1}_{s,1}$. Therefore, there exists a non-trivial complex-valued function $f(z)$ such that $\psi(N_s^n(0))$ is realized as the set of points

$$(f(z), z_1, \ldots, z_n) \tag{3.24}$$

in $\mathbf{C}^{n+1}_{s,1} = \mathbf{C}^1_{0,1} \times \mathbf{C}^n_s$. Because $\mathbf{C}^{n+1}_{s,1}$ can be holomorphically isometric embedded into $\mathbf{C}^{n+2}_{s+1}$ via the map

$$(w_0, w_1, \ldots, w_n) \mapsto (w_0, w_1, \ldots, w_n, w_0), \tag{3.25}$$

we conclude that the immersion $\psi : N_s^n(0) \to \mathbf{C}^{n+2}_{s+1}$ is congruent to

$$\psi(z_1, \ldots, z_n) = \big(f(z), z_1, \ldots, z_n, f(z)\big). \tag{3.26}$$

Now, if we put $f(z) = u(z) + \mathrm{i}v(z)$, then after applying the assumption that $N_s^n(0)$ is a complex submanifold of $\mathbf{C}^{n+2}_{s+1}$, we find

$$\mathrm{i}\,\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j} = \frac{\partial u}{\partial y_j} + \mathrm{i}\,\frac{\partial v}{\partial y_j}.$$

Hence u and v satisfy the Cauchy-Riemann equations, that is,

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j},\ \ \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j},\ \ j = 1, \ldots, n.$$

Therefore, $f(z_1, \ldots, z_n)$ is a holomorphic function. Consequently, we obtain case (ii) of the theorem. □

4. Flat pseudo-Kähler submanifolds fully in $\mathbf{C}^m_t$

A pseudo-Kähler immersion $\psi : N_s^n \to \mathbf{C}^m_t$ is said to be *full* if $\psi(N_s^n)$ does not lie in any totally geodesic complex pseudo-Euclidean subspace $\mathbf{C}^{m'}_{t'}$ of $\mathbf{C}^m_t$ with $m' < m$ and $t' \le t$.

Because the following theorem can be proved in the same spirit as Theorem 3.1, we only provide the key steps of the proof.

Theorem 4.1. *Let $\psi : N_s^2(0) \to \mathbf{C}^m_t$ be a flat pseudo-Kähler surface of the complex pseudo-Euclidean m-space $\mathbf{C}^m_t$. If the immersion is full, then ψ is locally congruent to the immersion:*

$$\psi(z_1, z_2) = \big(f_1(z), \ldots, f_r(z), z_1, z_2, f_r(z), \ldots, f_1(z)\big) \tag{4.1}$$

for some positive integer r, where $f_1, \ldots, f_r$ are non-trivial holomorphic functions.

Proof. Let $\psi : N_s^2(0) \to \mathbf{C}^m_t$ be a pseudo-Kähler immersion of a flat pseudo-Kähler surface $N_s^2(0)$ into the complex pseudo-Euclidean m-space $\mathbf{C}^m_t$. Then it follows from Proposition 3.2 that (3.2) holds. Hence, at each point $p \in N_s^2(0)$, the first normal space, $\operatorname{Im} h_p$, is a complex subspace of the normal space $T^\perp(N_s^2(0))$ with a degenerate induced metric.

Now, let us assume that the pseudo-Kähler immersion ψ is full. The we have $\mathbf{C}^m_t = \mathbf{C}^{2+2r}_{s+r}$, where r is the complex rank of the first normal bundle $\operatorname{Im} h$.

Since $N_s^2(0)$ is a flat indefinite complex space form, $N_s^2(0)$ is locally holomorphically isometric to $\mathbf{C}_s^2$. Hence we may assume that the metric is given by (3.10) with $n = 2$. As in the proof of Theorem 3.1, we have

$$\nabla_{\partial_{x_i}}\partial_{x_j} = \nabla_{\partial_{x_i}}\partial_{y_j} = \nabla_{\partial_{y_i}}\partial_{y_j} = 0, \tag{4.2}$$

$$\partial_{y_i} = J\partial_{x_i}, \quad i,j = 1,2. \tag{4.3}$$

As before, if we put $\eta_{ij} = h\left(\partial_{x_i}, \partial_{x_j}\right)$, then we have

$$h\left(\partial_{x_i}, \partial_{y_j}\right) = J\eta_{ij}, \quad h\left(\partial_{y_i}, \partial_{y_j}\right) = -\eta_{ij}. \tag{4.4}$$

From Proposition 3.2 we find

$$\langle D_{\partial x_k}\eta_{ij}, \eta_{ij}\rangle = \langle D_{\partial y_k}\eta_{ij}, \eta_{ij}\rangle = 0, \quad i,j,k = 1,2. \tag{4.5}$$

On the other hand, by (2.8) and (4.2), we have

$$\begin{aligned}
(\nabla_{\partial_{x_i}} h)(\partial_{x_j}, \partial_{x_k}) &= D_{\partial_{x_j}} h(\partial_{x_i}, \partial_{x_k}),\\
(\nabla_{\partial_{y_i}} h)(\partial_{x_j}, \partial_{x_k}) &= D_{\partial_{x_j}} h(\partial_{y_i}, \partial_{x_k}),\\
(\nabla_{\partial_{y_i}} h)(\partial_{y_j}, \partial_{x_k}) &= D_{\partial_{y_j}} h(\partial_{y_i}, \partial_{x_k}),\\
(\nabla_{\partial_{y_i}} h)(\partial_{y_j}, \partial_{y_k}) &= D_{\partial_{y_j}} h(\partial_{y_i}, \partial_{y_k}).
\end{aligned} \tag{4.6}$$

Thus, by Codazzi's equation and (4.6) we get

$$\begin{aligned}
D_{\partial_{x_i}}h(\partial_{x_j}, \partial_{x_k}) &= D_{\partial_{x_j}}h(\partial_{x_i}, \partial_{x_k}) = D_{\partial_{x_k}}h(\partial_{x_i}, \partial_{x_j}),\\
D_{\partial_{y_i}}h(\partial_{x_j}, \partial_{x_k}) &= D_{\partial_{x_j}}h(\partial_{y_i}, \partial_{x_k}) = D_{\partial_{x_k}}h(\partial_{y_i}, \partial_{x_j}),\\
&\cdots \textit{etc.},
\end{aligned} \tag{4.7}$$

Now, by applying (4.3), (4.5), (4.6) and Lemma 2.1, we find

$$\begin{aligned}
&\langle D_{\partial_{x_k}}\eta_{ij}, J\eta_{ij}\rangle = \langle D_{\partial_{y_k}}\eta_{ij}, \eta_{ij}\rangle = 0,\\
&\langle D_{\partial_{x_i}}\eta_{jj}, \eta_{ik}\rangle = \langle D_{\partial_{x_k}}\eta_{ij}, \eta_{ij}\rangle = 0, \quad i,j,k = 1,2.
\end{aligned} \tag{4.8}$$

Since the first normal bundle $\operatorname{Im} h$ of $N_s^2(0)$ in $\mathbf{C}_{s+r}^{2+r}$ is of complex rank r endowed with a degenerate induced metric, it follows from (4.5) and (4.8) that

$$D_X(\operatorname{Im} h) \subset \operatorname{Im} h, \quad \forall X \in T(N_s^2(0)). \tag{4.9}$$

Therefore, the first normal bundle is parallel in the normal bundle. Consequently, it follows from the Reduction Theorem of Erbacher-Magid that $N_s^2(0)$ is immersed in a totally geodesic complex $(2+r)$-subspace $\mathbf{C}_{s,r}^{2+r}$ of $\mathbf{C}_{s+r}^{2+2r} = \mathbf{C}_{0,r}^r \times \mathbf{C}_s^2$. Now, by applying (4.9), Proposition 3.2 and Weingarten's formula, we obtain

$$\tilde{\nabla}_X\xi_0, \tilde{\nabla}_X J\xi_0 \in \operatorname{Im} h \tag{4.10}$$

for every vector $X \in T(N_s^n(0))$. Hence the first normal bundle is a constant complex r-subspace $\mathbf{C}_{0,r}^r \subset \mathbf{C}_{s,r}^{2+r}$ with a degenerate metric. Thus, $\psi(N_s^2(0))$ can be realized as the set of points

$$(f_1(z), \ldots, f_r(z), z_1, z_2) \tag{4.11}$$

in $\mathbf{C}_{s,r}^{2+r}$ for some non-trivial complex-valued functions $f_1, \ldots, f_r$. Now, because $\mathbf{C}_{s,r}^{2+r}$ can be holomorphically isometric embedded into $\mathbf{C}_{s+r}^{2+2r}$ via

$$(\zeta_1, \ldots, \zeta_r, w_1, w_2) \mapsto (\zeta_1, \ldots, \zeta_r, w_1, w_2, \zeta_r, \ldots, \zeta_1), \tag{4.12}$$

we conclude that $\psi : N_s^2(0) \to \mathbf{C}_{s+r}^{2+r}$ is locally congruent to (4.1). As in the proof of Theorem 3.1, we may conclude that $f_1, \ldots, f_r$ are holomorphic functions by using the assumption that $N_s^2(0)$ is a complex surface in $\mathbf{C}_{s+r}^{2+r}$. □

A pseudo-Riemannian submanifold of a pseudo-Riemannian manifold is called *parallel* if it has parallel second fundamental form, that is, $\bar{\nabla} h = 0$ identically.

Theorem 4.2. *Let $\psi : N_s^n(0) \to \mathbf{C}_t^m$ be a flat pseudo-Kähler submanifold of the complex pseudo-Euclidean m-space $\mathbf{C}_t^m$. If the immersion is full and parallel, then ψ is locally congruent to the immersion:*

$$\psi(z_1, \ldots, z_n) = \big(f_1(z), \ldots, f_r(z), z_1, \ldots, z_n, f_r(z), \ldots, f_1(z)\big) \tag{4.13}$$

for some positive integer r, where $f_1, \ldots, f_r$ are non-trivial polynomial functions of degree ≤ 2 in $z_1, \ldots, z_n$.

Proof. Under the hypothesis of the theorem and using the same notations as before, we have $D\eta_{ij} = 0$ for $i, j = 1, \ldots, n$. Therefore, after applying the same argument given in the proofs of Theorem 3.1 and Theorem 4.1 we know that the first normal bundle $\operatorname{Im} h$ is a parallel degenerate subbundle of the normal bundle. Consequently, according to the Reduction Theorem of Erbacher-Magid, $N_s^n(0)$ is immersed in a totally geodesic complex $(n + r)$-subspace $\mathbf{C}_{s,r}^{n+r}$ of $\mathbf{C}_{s+r}^{n+2r} = \mathbf{C}_{0,r}^r \times \mathbf{C}_s^n$, where $r = \operatorname{rank}_{\mathbf{C}}(\operatorname{Im} h)$. As before, we may also prove that the first normal bundle is a constant complex r-subspace of $\mathbf{C}_{s+r}^{n+2r}$. Thus, ψ is locally congruent to (4.13) for some non-trivial holomorphic functions $f_1, \ldots, f_r$.

Moreover, since ψ is a parallel pseudo-Kähler immersion, it follows from (4.2), (4.6), (4.13) and Proposition 3.2(iii) that the third derivatives of $f_1, \ldots, f_r$ vanishes identically. Consequently, $f_1, \ldots, f_r$ are polynomial functions of degree ≤ 2 in $z_1, \ldots, z_n$. □

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