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ON THE GEOMETRY OF THE PRODUCT RIEMANNIAN MANIFOLD WITH THE POISSON STRUCTURE

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Abstract. Let (M, π_M, g_M) and (N, π_N, g_N) be two Poisson manifolds, with g_M and g_N are The Riemanniann metric of M and N respectively. Let $(M \times N, g_M \oplus g_N)$ be their product manifold equipped with the Poisson tensor $\pi_M \oplus \pi_N$ and the product Riemannian metric $g_{M\times N} = g_M \oplus g_N$. In this paper, we discuss the Poisson structure of the product Riemannian manifold and we describe all its geometric properties of the product in terms of each Poisson Riemannian manifold of the basis. Some interesting consequences are also given.

1. Introduction

Let M be a Poisson manifold with Poisson tensor π. A pseudo-metric of signature (p, q) on the cotangent bundle T^*M is a smooth symmetric contravariant 2-form g on M such that, at each point $x \in M$, g_x is non degenerate on T^*M with signature (p, q). For vector bundles, the corresponding operational of a contravariant derivative had been introduced by I.Vaisman ([7]).

The contravariant connection D^π, in the sense of Fernandes [3] and Vaisman, is defined by formula (4) (see below) for any pseudo-metric g on T^*M. The notion of Poisson manifolds with compatible pseudo-metric, introduced by M. Boucetta in [2], is called a pseudo-Riemannian Poisson manifold (for more details the reader is referred to [1]).

The purpose of this work is to study the geometry of product Poisson manifold $(M \times N, \pi_{M\times N} = \pi_M \oplus \pi_N)$ with a Riemannian metric. We calculate their Lie bracket, Schouten-Nijenhuis bracket, contravariant connection and several curvatures on product manifold endowed with the natural class of Riemannian metric $g_{M\times N} = g_M \oplus g_N$. This leads to some interesting relation between the geometry of the Poisson manifolds (M, π_M, g_M) and (N, π_N, g_N) where g_M and g_N are the Riemannian metrics of T^*M and T^*N respectively, and its product Poisson $(M \times N, \pi_{M\times N}, g_{M\times N})$ endowed with a Riemannian metric on $T^*(M \times N)$.

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2. PRELIMINARIES

We start by giving the basic tools needed to prove the main results in this paper. We also assume the reader is familiar with definitions and results about the Riemannian geometry of Poisson. In the second subsection we present also some recent general results concerning product Poisson manifolds. These results are important for us in what is left in our work. We will also express the product of Poisson manifolds in terms of bivector fields $\pi_{M\times N}$ which satisfy some special conditions. Some general references are [3], [4] and [7].

2.1. **Riemannian Poisson manifold.** Let M be a manifold equipped with the Riemannian metric g_M and a bivector field π_M. We associate with the bivector field π_M (in a natural way) a sharp map $\sharp_{\pi_M} : T^*M \to TM$ defined by $\beta(\sharp_{\pi_M}(\alpha)) = \pi(\alpha, \beta)$ and a Poisson bracket on the 1-forms defined by

$$[\alpha,\beta]_{\pi_M} = L_{\sharp_{\pi_M}(\alpha)}\beta - L_{\sharp_{\pi_M}(\beta)}\alpha - d(\pi_M(\alpha,\beta)) \tag{2.1}$$

The Schouten-Nijenhuis bracket $[\pi_M, \pi_M]_S$ is the obstruction so that $\sharp_{\pi_M}$ be a homomorphism between $\Omega^1(M)$ endowed with the bracket $[\cdot,\cdot]_{\pi_M}$ and $\mathcal{X}^1(M)$ endowed with the Lie bracket. We have (see [1])

$$[\pi_M, \pi_M]_S(\alpha,\beta,\gamma) = \gamma(\sharp_{\pi_M}([\alpha,\beta]_{\pi_M}) - [\sharp_{\pi_M}(\alpha), \sharp_{\pi_M}(\beta)]) \tag{2.2}$$

The bivector field π_M defines a Poisson structure on M if and only if

$$[\pi_M, \pi_M]_S = 0. \tag{2.3}$$

Let ${\sharp_g}^{-1} : TM \to T^*M$ be the bundle isomorphism defined by $({\sharp_g}^{-1}(X))(Y)$ and $\sharp_g : T^*M \to TM$ its inverse. We can then define the metric $\tilde{g}$ on the bundle cotangent by $\tilde{g}(\alpha,\beta) = g(\sharp_g(\alpha), \sharp_g(\beta))$. This latter is a non-degenerate definie positive symmetric bilinear form.

For each Riemannian metric on T^*M, one considers the contravariante connection introduced in the sense of Fernandes [3] by

$$\begin{aligned} 2\tilde{g}(D_\alpha\beta,\gamma) &= \sharp_{\pi_M}(\alpha).\tilde{g}(\beta,\gamma) + \sharp_{\pi_M}(\beta).\tilde{g}(\alpha,\gamma) - \sharp_{\pi_M}(\gamma).\tilde{g}(\alpha,\beta) \\ &\quad + \tilde{g}([\alpha,\beta]_{\pi_M},\gamma) + \tilde{g}([\gamma,\alpha]_{\pi_M},\beta) - \tilde{g}([\gamma,\beta]_{\pi_M},\alpha) \end{aligned} \tag{2.4}$$

where $\alpha, \beta, \gamma \in \Omega^1(M)$ and the Lie bracket $[\cdot,\cdot]_{\pi_M}$ is given by (2.1).

D will be called the Levi-Civita Contravariante connection associated with the couple $(\pi_M, \tilde{g})$. Further, D satisfies:

(1) $D_\alpha\beta - D_\beta\alpha = [\alpha,\beta]_{\pi_M}$; (Torsion-free).
(2) $\sharp_{\pi_M}(\alpha).\tilde{g}(\beta,\gamma) = \tilde{g}(D_\alpha\beta,\gamma) + \tilde{g}(\beta, D_\alpha\gamma)$; ($D$ compatible with $\tilde{g}$).

With the notation above, the triplet $(M, \pi_M, \tilde{g})$ is called a Riemannian Poisson manifold if, for any $\alpha, \beta, \gamma \in \Omega^1(M)$

$$D\pi_M(\alpha,\beta,\gamma = \sharp_{\pi_M}(\alpha).\pi_M(\beta,\gamma) - (\pi_M(D_\alpha\beta,\gamma) - \pi_M(\beta, D_\alpha\gamma) = 0 \tag{2.5}$$

(π is compatible with $\tilde{g}$).

By $R^{\pi_M}, r^{\pi_M}, \sigma^{\pi_M}(x)$ and $K^{\pi_M}(V_x)$ we denote respectively the Riemannian curvature tensor, the Ricci tensor, the scalar curvature and the sectional curvature of m

dimensional Riemannian manifold (M,g). Then $R^{\pi_M}, r^{\pi_M}, \sigma^{\pi_M}(x)$ and $K^{\pi_M}(V_x)$ are defined by

$$R^{\pi_M}(\alpha,\beta)\gamma = D_\alpha D_\beta\gamma - D_\beta D_\alpha\gamma - D_{[\alpha,\beta]}\gamma \tag{2.6}$$

$$r^{\pi_M}(\alpha,\beta) = Tr(\gamma\mapsto R^{\pi_M}(\alpha,\beta)\pi) = \sum_{i=1}^{m}\tilde{g}(R^{\pi_M}(E_i,\alpha)\beta,E_i) \tag{2.7}$$

$$\sigma^{\pi_M}(x) = \sum_{i=1}^{m} r^{\pi_M}(E_i,E_i)(x) \text{ with } E_i(x)=e_i \tag{2.8}$$

and

$$K^{\pi_M}(V_x) = \frac{\tilde{g}(R^{\pi_M}(\alpha_1,\beta_1)\alpha_1,\beta_1)}{\tilde{g}(\alpha_1,\alpha_1).g(\beta_1,\beta_1)-g(\alpha_1,\beta_1)^2}$$

where $\{E_1,E_2,...,E_m\}$ is an orthogonal basis in T^*M, $\alpha,\beta,\gamma\in T^*M$ and α_1,β_1 are two linearly independent forms at a point $x\in\omega^1(M)$. V_x is the plane section spanned by α_1 and β_1.

2.2. **Product of Poisson manifolds.** Let (M,g_M) and (N,g_N) be two Riemannian manifolds with the dimensions m and n respectively. $M\times N$ is the product manifold of M and N.
Let π and σ be the projections mappings C^∞ of $M\times N$ to M and N respectively

Lemma 2.1 ([6]). *The tangent space of $M\times N$ at a point (x,y) is the direct sum of subspaces $T_{(x,y)}M\times\{y\}$ and $T_{(x,y)}\{x\}\times N$, i.e.*

$$T_{(x,y)}M\times N = T_{(x,y)}M\times\{y\}\oplus T_{(x,y)}\{x\}\times N$$

Proposition 2.1. *The cotangent bundle of $M\times N$ at a point (x,y) is the direct sum of subspaces $T^*_{(x,y)}M\times\{y\}$ and $T^*_{(x,y)}\{x\}\times N$, i.e.*

$$T^*_{(x,y)}M\times N = T^*_{(x,y)}M\times\{y\}\oplus T^*_{(x,y)}\{x\}\times N \tag{2.9}$$

Proof.

(1) Let $\pi_y=\pi^y\circ\pi$ and $\sigma_x=\sigma^x\circ\pi$ be the mappings from $M\times N$ into $M\times\{y\}$ and $\{x\}\times N$ respectively. Set $P=d_{(x,y)}\pi_y$ and $Q=d_{(x,y)}\sigma_x$. According to the previous lemma, P and Q are projections of $T_{(x,y)}M\times N$ into $T_{(x,y)}M\times\{y\}$ and $T_{(x,y)}\{x\}\times N$ respectively. We then have

$$P+Q=I, P\circ P=P, Q\circ Q=Q, P_{/T_{(x,y)}\{x\}\times N}=0, Q_{/T_{(x,y)}M\times N}=0$$

For $\alpha\in T^*_{(x,y)}M\times N, W\in T_{(x,y)}M\times N$ with $W=W_1+W_2=P(W)+Q(W)$, we have

$$\alpha(W)=\alpha(W_1)+\alpha(W_2)=\alpha\circ P(W)+\alpha\circ Q(W)=(\alpha_1+\alpha_2)(W)$$

where $\alpha_1\in T^*_{(x,y)}M\times N$ and $\alpha_2\in T^*_{(x,y)}M\times N$ such that $(\alpha_1)_{/T_{(x,y)}\{x\}\times N}=0$ and $(\alpha_2)_{/T_{(x,y)}M\times\{y\}}=0$.

Setting $H=\{\omega\in T^*_{(x,y)}M\times N/(\omega)_{/T_{(x,y)}\{x\}\times N}=0\}$ and $G=\{\omega\in T^*_{(x,y)}M\times N/(\omega)_{/T_{(x,y)}M\times\{y\}}=0\}$, we can easily see $H\simeq T^*_{(x,y)}M\times\{y\}$ and $G\simeq T^*_{(x,y)}\{x\}\times N$.

(2) The dimension $dim(T^*_{(x,y)}M\times N)=dim(T^*_{(x,y)}M\times\{y\})+dim(T^*_{(x,y)}\{x\}\times N)$, establishing the desired formula.

□

Remark 2.1. Let $\alpha \in \Omega^1(M \times N)$, $\alpha(x,y) \in T^*_{(x,y)}M \times N$ for any $(x,y) \in M \times N$. From Proposition (1), we have
$\alpha(x,y) = \alpha(x,y) \circ d_{(x,y)}\pi_y + \alpha(x,y) \circ d_{(x,y)}\sigma_x \in H \oplus G \simeq T^*_x M \oplus T^*_y N$
If we put $(P^* \circ \alpha)(x,y) = \alpha(x,y) \circ d_{(x,y)}\pi_y \in T^*_x M$ and $(Q^* \circ \alpha)(x,y) = \alpha(x,y) \circ d_{(x,y)}\sigma_x \in T^*_y N$, then we can easily see that P^* and Q^* are the projection mapping of $T^*(M \times N)$ into T^*M and T^*N respectively.

Remark 2.2. P and Q satisfy

$$P^* + Q^* = I, P^* \circ P^* = P^*, Q^* \circ Q^* = Q^*, P^* \circ Q^* = Q^* \circ P^* = 0 \tag{2.10}$$

We Put $J^* = P^* - Q^*$. It is easy to see that

$$J^* \circ J^* = I, P^* \circ J^* = J^* \circ P^* = P^*, Q^* \circ J^* = J^* \circ Q^* = -Q^* \tag{2.11}$$

Now, let (M, g_M, π_M) and (N, g_N, π_N) be two m and n dimensional Poisson manifolds respectively. We define a Riemannian metric and Poisson tensor of $M \times N$ respectively by

$$\tilde{g}_{M\times N}(\alpha, \beta) = \tilde{g}_M(P^*\alpha, P^*\beta) + \tilde{g}_N(Q^*\alpha, Q^*\beta) \tag{2.12}$$

such that

- $(\tilde{g}_{M\times N}(P^*\alpha, P^*\beta)) = \tilde{g}_M(P^*\alpha, P^*\beta)$.
- $\tilde{g}_{M\times N}(Q^*\alpha, Q^*\beta)) = \tilde{g}_N(Q^*\alpha, Q^*\beta)$.
- $\tilde{g}_{M\times N}(P^*\alpha, Q^*\beta)) = \tilde{g}_{M\times N}(Q^*\alpha, P^*\beta)) = 0$

and

$$\pi_{M\times N}(\alpha, \beta) = \pi_M(P^*\alpha, P^*\beta) + \pi_N(Q^*\alpha, Q^*\beta) \tag{2.13}$$

such that

- $\pi_{M\times N}(P^*\alpha, P^*\beta) = \pi_M(P^*\alpha, P^*\beta)$.
- $\pi_{M\times N}(Q^*\alpha, Q^*\beta) = \pi_N(Q^*\alpha, Q^*\beta)$.
- $\pi_{M\times N}(P^*\alpha, Q^*\beta) = \pi_{M\times N}(Q^*\alpha, P^*\beta) = 0$.

We will also denote the Poisson bracket by $\pi_{M\times N}$: $\{f, g\}_{M\times N} = \pi_{M\times N}(df, dg)$. By direct calculations we obtain

$$\{f, g\}_{M\times N}(x, y) = \{f_y, g_y\}_M(x) + \{f_x, g_x\}_N(y) \tag{2.14}$$

where $f, g \in C^\infty(M \times N), \tilde{g}_M, \tilde{g}_N$, are Riemannian metrics on the cotangent bundle T^*M and T^*N respectively, and π_M, π_N are Poisson tensors of M and N respectively.

Remark 2.3. (see [5])

- Let X, Y be two vectors in $T(M\times N)$. If for any $\alpha \in T^*(M\times N)$, $P^*\alpha(X) = Y$ and $Q^*(X) = 0$, then $X = Y$.
- We can easily verify that this tensor is indeed a Poisson tensor on $M \times N$.
- With respect to this product Poisson structure, the projection maps $\pi : M \times N \to M$ and $\sigma : M \times N \to N$ are Poisson maps.
- For any $f \in C^\infty(M \times N)$,

$$d_{(x,y)}f = df(x,y) = df_y(x) + df_x(y) = d_x f_y + d_y f_x.$$

- The formula (13) is equivalent to

$$\sharp_{\pi M\times N}(\alpha) = \sharp_{\pi M\times N}(P^*\alpha) + \sharp_{\pi M\times N}(Q^*\alpha).$$

Proposition 2.2. *Let $(M, \pi_M), (N, \pi_N)$ be Poisson manifolds and let $\phi : M \to N$ be a Poisson map which is a diffeomorphism. Then for any $f, g \in C^\infty(N)$ and $\alpha, \beta \in \Omega^1(N)$, we have*

(1) $\phi_*^{-1}(\sharp_{\pi_N}(\alpha)) = \phi^*(\sharp_{\pi_N}(\alpha)) = \sharp_{\pi_N}(\phi^*\alpha)$
(2) $L_{\sharp_{\pi_N}(\phi^*\beta)}\sharp_{\pi_N}(\phi^*\alpha) = \phi^* L_{\sharp_{\pi_N}(\beta)}\sharp_{\pi_N}(\alpha)$
(3) $\phi^*[\alpha, \beta]_{\pi_N} = [\phi^*\alpha, \phi^*\beta]_{\pi_M}$

Proof. Using the fact that φ is a Poisson map and seeing that $\varphi_*(X) = d\varphi \circ X \circ \varphi^{-1}, \varphi_*^{-1}(Y) = d\varphi^{-1} \circ Y \circ \varphi$, $\varphi_*(X)(f) = X(f \circ \varphi) \circ \varphi^{-1}, \varphi_*^{-1}(Y)(h) = X(h \circ \varphi^{-1}) \circ \varphi, \varphi^*(dg) = d(\varphi^* g)$ and $(\varphi^*(\alpha))(Y) = \alpha(\varphi_* Y) \circ \varphi$, 1), 2) and 3) follow immediately. □

3. On the connection of product Poisson manifolds

We will now prove the first results of this paper.

Proposition 3.1. *Let $(M \times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ be a product Riemannian manifold of the Poisson manifolds $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$. Then for any $\alpha, \beta, \gamma \in T^*(M \times N)$, we have*

(1) $\sharp_{\pi_{M\times N}}(P^*\alpha) = \sharp_{\pi_M}(P^*\alpha)$
(2) $\sharp_{\pi_{M\times N}}(Q^*\alpha) = \sharp_{\pi_M}(Q^*\alpha)$
(3) $\sharp_{\pi_{M\times N}}(P^*\alpha)(\pi_{M\times N}(Q^*\beta, Q^*\gamma)) = 0$
(4) $L_{\sharp_{\pi_{M\times N}}(P^*\alpha)}(P^*\beta) = L_{\sharp_{\pi_{M\times N}}(P^*\beta)}(P^*\alpha) = 0$
(5) $P^*[\alpha, \beta]_{\pi_{M\times N}} = [P^*\alpha, P^*\beta]_{\pi_{M\times N}} = [P^*\alpha, P^*\beta]_{\pi_M}$
(6) $Q^*[\alpha, \beta]_{\pi_{M\times N}} = [Q^*\alpha, Q^*\beta]_{\pi_{M\times N}} = [Q^*\alpha, Q^*\beta]_{\pi_N}$
(7) $[P^*\alpha, Q^*\beta]_{\pi_{M\times N}} = 0$
(8) $[J^*\alpha, \beta]_{\pi_{M\times N}} = J^*[\alpha, \beta]_{\pi_{M\times N}}$
(9) $[J^*\alpha, J^*\beta]_{\pi_{M\times N}} = [\alpha, \beta]_{\pi_{M\times N}}$

Proof. For any $\beta \in T^*(M \times N)$, we have $P^*\beta(\sharp_{\pi_{M\times N}}(P^*\alpha)) = \pi_{M\times N}(P^*\alpha, P^*\beta) = \pi_M(P^*\alpha, P^*\beta) = P^*\beta(\sharp_{\pi_M}(P^*\alpha))$ and $Q^*\beta(\sharp_{\pi_{M\times N}}(P^*\alpha)) = 0$. This combined with Remark 1 now yield the first three assertions.

The fourth statement follows directly from the definition of the Lie derivative and from (1) and (2).

The last five points follow directly from the definition of the Lie bracket in $T^*(M \times N)$ and also from (1) and (2).

□

Proposition 3.2. *Let $(M\times N, \pi_{M\times N}=\pi_M+\pi_N, \tilde{g}_{M\times N}=\tilde{g}_M+\tilde{g}_N)$ be a product Poisson manifold of the Poisson manifolds $(M,\pi_M,\tilde{g}_M)$ and $(N,\pi_N,\tilde{g}_N)$ with $\tilde{g}_{M\times N}$ the product metric of $M\times N$ and $\tilde{g}_M,\tilde{g}_N$ are the Riemannian metric of M,N respectively. By $D^{\pi_{M\times N}}, D^{\pi_M}$ and D^{π_N} we denote the Levi-Civita connection of the metric $\tilde{g}_{M\times N}, \tilde{g}_M$ and $\tilde{g}_N$ respectively. Then for any $\alpha,\beta,\gamma\in\Omega^1(M\times N)$, we have*

(1) $P^*D^{\pi_{M\times N}}_{\alpha}\beta = D^{\pi_{M\times N}}_{P^*\alpha}P^*\beta = D^{\pi_M}_{P^*\alpha}P^*\beta$

(2) $Q^*D^{\pi_{M\times N}}_{\alpha}\beta = D^{\pi_{M\times N}}_{Q^*\alpha}Q^*\beta = D^{\pi_N}_{Q^*\alpha}Q^*\beta$

(3) $D^{\pi_{M\times N}}_{P^*\alpha}Q^*\beta = D^{\pi_{M\times N}}_{Q^*\alpha}P^*\beta = 0$

(4) $D^{\pi_{M\times N}}_{J^*\alpha}J^*\beta = D^{\pi_{M\times N}}_{\alpha}\beta$

(5) $D^{\pi_{M\times N}}_{\alpha}J^*\beta = J^*D^{\pi_{M\times N}}_{\alpha}\beta$

Proof. Using the equation (4), we obtain

$$\begin{aligned}
2\tilde{g}_{\pi M\times N}(D^{\pi_{M\times N}}_{P^*\alpha}P^*\beta, P^*\gamma) =&\\
&\sharp_{\pi_{M\times N}}(P^*\alpha).\tilde{g}_{\pi M\times N}(P^*\beta,P^*\gamma) + \sharp_{\pi_{M\times N}}(P^*\beta).\tilde{g}_{\pi M\times N}(P^*\alpha,P^*\gamma)\\
&-\sharp_{\pi_{M\times N}}(P^*\gamma).\tilde{g}_{\pi M\times N}(P^*\alpha,P^*\beta) + \tilde{g}_{\pi M\times N}([P^*\alpha,P^*\beta]_{\pi_{M\times N}},P^*\gamma)\\
&+\tilde{g}_{\pi M\times N}([P^*\gamma,P^*\alpha]_{\pi_{M\times N}},P^*\beta) - \tilde{g}_{\pi M\times N}([P^*\gamma,P^*\beta]_{\pi_{M\times N}},P^*\alpha)\\
&=\sharp_{\pi_M}(P^*\alpha).\tilde{g}_{\pi M}(P^*\beta,P^*\gamma) + \sharp_{\pi_M}(P^*\beta).\tilde{g}_{\pi M}(P^*\alpha,P^*\gamma)\\
&-\sharp_{\pi_M}(P^*\gamma).\tilde{g}_{\pi M}(P^*\alpha,P^*\beta) + \tilde{g}_{\pi M}([P^*\alpha,P^*\beta]_{\pi_M},P^*\gamma)\\
&+\tilde{g}_{\pi M}([P^*\gamma,P^*\alpha]_{\pi_M},P^*\beta) - \tilde{g}_{\pi M}([P^*\gamma,P^*\beta]_{\pi_M},P^*\alpha)\\
&=2\tilde{g}_{\pi M}(D^{\pi_M}_{P^*\alpha}P^*\beta,P^*\gamma)
\end{aligned}$$

and

$$2\tilde{g}_{\pi M\times N}(D^{\pi_{M\times N}}_{P^*\alpha}P^*\beta, P^*\gamma) = 0,$$

establishing the first statement.

Now we prove (2). We have

$$\begin{aligned}
2\tilde{g}_{\pi M\times N}(D^{\pi_{M\times N}}_{Q^*\alpha}Q^*\beta, Q^*\gamma) =&\\
&\sharp_{\pi_{M\times N}}(Q^*\alpha).\tilde{g}_{\pi M\times N}(Q^*\beta,Q^*\gamma) + \sharp_{\pi_{M\times N}}(Q^*\beta).\tilde{g}_{\pi M\times N}(Q^*\alpha,Q^*\gamma)\\
&-\sharp_{\pi_{M\times N}}(Q^*\gamma).\tilde{g}_{\pi M\times N}(Q^*\alpha,Q^*\beta) + \tilde{g}_{\pi M\times N}([Q^*\alpha,Q^*\beta]_{\pi_{M\times N}},Q^*\gamma)\\
&+\tilde{g}_{\pi M\times N}([Q^*\gamma,Q^*\alpha]_{\pi_{M\times N}},Q^*\beta) - \tilde{g}_{\pi M\times N}([Q^*\gamma,Q^*\beta]_{\pi_{M\times N}},Q^*\alpha)\\
&=2\tilde{g}_{\pi N}(D^{\pi_N}_{Q^*\alpha}Q^*\beta,Q^*\gamma)
\end{aligned}$$

and

$$2\tilde{g}_{\pi M\times N}D^{\pi_{M\times N}}_{Q^*\alpha}Q^*\beta, Q^*\gamma) = 0,$$

proving (2).

In the same way we can prove $D^{\pi_{M\times N}}_{P^*\alpha}Q^*\beta = D^{\pi_{M\times N}}_{Q^*\alpha}P^*\beta = 0$.

We can say that $D^{\pi_{M\times N}}_{P^*\alpha}P^*\beta$ and $D^{\pi_{M\times N}}_{Q^*\alpha}Q^*\beta$ are also the Levi-Civita contravariant connection introduced in the sense of Fernandes of the Poisson manifolds $(M,\pi_M,\tilde{g}_M)$ and $(N,\pi_N,\tilde{g}_N)$.

Using Equation (4), we find

$$\begin{aligned}
\tilde{g}_{M\times N}(D^{\pi_{M\times N}}_{Q^*\alpha}P^*\beta, Q^*\gamma) &= \sharp_{\pi_{M\times N}}(Q^*\alpha).\tilde{g}_{M\times N}(P^*\beta,Q^*\gamma) - \tilde{g}_{M\times N}(P^*\beta, D^{\pi_{M\times N}}_{Q^*\alpha}Q^*\gamma)\\
&= 0.
\end{aligned}$$

Then we have
$$Q^*(D^{\pi_{M\times N}}_{Q^*\alpha}P^*\beta)=0$$
and
$$P^*(D^{\pi_{M\times N}}_{Q^*\alpha}P^*\beta)=D^{\pi_{M\times N}}_{Q^*\alpha}P^*\beta.$$
Similarly
$$P^*(D^{\pi_{M\times N}}_{P^*\alpha}Q^*\beta)=0$$
and
$$Q^*(D^{\pi_{M\times N}}_{P^*\alpha}Q^*\beta)=D^{\pi_{M\times N}}_{P^*\alpha}Q^*\beta.$$
Since
$$\begin{aligned}P^*D^{\pi_{M\times N}}_{\alpha}\beta &= D^{\pi_{M\times N}}_{P^*\alpha}P^*\beta+D^{\pi_{M\times N}}_{Q^*\alpha}P^*\beta+D^{\pi_{M\times N}}_{P^*\alpha}Q^*\beta+D^{\pi_{M\times N}}_{Q^*\alpha}Q^*\beta\\ &=(D^{\pi_{M\times N}}_{P^*\alpha}P^*\beta)+(D^{\pi_{M\times N}}_{Q^*\alpha}P^*\beta),\end{aligned}$$
we have
$$D^{\pi_{M\times N}}_{\alpha}P^*\beta=P^*D^{\pi_{M\times N}}_{\alpha}\beta.$$

For the same reason $D^{\pi_{M\times N}}_{\alpha}Q^*\beta=Q^*D^{\pi_{M\times N}}_{\alpha}\beta$.
As $J^*=P^*-Q^*$, then we have
$$D^{\pi_{M\times N}}_{\alpha}J^*\beta=J^*D^{\pi_{M\times N}}_{\alpha}\beta.$$
Then from Theorem (1) and the fact that $D^{\pi_{M\times N}}$ is torsion free, we have
$$\begin{aligned}D^{\pi_{M\times N}}_{J^*\alpha}J^*\beta &= J^*D^{\pi_{M\times N}}_{J^*\alpha}\beta\\ &= J^*(D^{\pi_{M\times N}}_{\beta}J^*\alpha+[J^*\alpha,\beta]_{\pi_{M\times N}})\\ &= J^*(J^*D^{\pi_{M\times N}}_{\beta}\alpha+J^*[\alpha,\beta]_{\pi_{M\times N}})\\ &= D^{\pi_{M\times N}}_{\beta}\alpha+[\alpha,\beta]_{\pi_{M\times N}}\\ &= D^{\pi_{M\times N}}_{\alpha}\beta.\end{aligned}$$

□

4. On the curvature tensor of product Poisson manifolds

We denote the Poisson curvature tensor of product Riemannian Poisson manifold $(M\times N,\pi_{M\times N}=\pi_M+\pi_N,\tilde g_{M\times N}=\tilde g_M+\tilde g_N)$ by $R^{\pi_{M\times N}}$. Then we have the following

Theorem 4.1. *Let $(M\times N,\pi_{M\times N}=\pi_M+\pi_N,\tilde g_{M\times N}=\tilde g_M+\tilde g_N)$ be a product Poisson manifold of the Poisson manifolds $(M,\pi_M,\tilde g_M)$ and $(N,\pi_N,\tilde g_N)$ with $\tilde g_{M\times N}$ the product metric of $M\times N$ and $\tilde g_M,\tilde g_N$ are the Riemannian metric of M,N respectively. Then for any $\alpha,\beta,\gamma\in\Omega^1(M\times N)$, $R^{\pi_{M\times N}}$ satisfies the following properties:*

1. $R^{\pi_{M\times N}}(\alpha,\beta)P^*\gamma=P^*R^{\pi_{M\times N}}(\alpha,\beta)\gamma\in\Omega^1(M)$
2. $R^{\pi_{M\times N}}(\alpha,\beta)Q^*\gamma=Q^*R^{\pi_{M\times N}}(\alpha,\beta)\gamma\in\Omega^1(N)$
3. $R^{\pi_{M\times N}}(\alpha,\beta)J^*\gamma=J^*R^{\pi_{M\times N}}(\alpha,\beta)\gamma\in\Omega^1(M\times N)$
4. $R^{\pi_{M\times N}}(J^*\alpha,J^*\beta)\gamma=R^{\pi_{M\times N}}(\alpha,\beta)\gamma$
5. $R^{\pi_{M\times N}}(J^*\alpha,\beta)\gamma=R^{\pi_{M\times N}}(\alpha,J^*\beta)\gamma$
6. $R^{\pi_{M\times N}}(\alpha,\beta)\gamma=R_1(P^*\alpha,P^*\beta)P^*\gamma+R_2(Q^*\alpha,Q^*\beta)Q^*\gamma$

where

$$R^{\pi_{M\times N}}(P^*\alpha,P^*\beta)P^*\gamma, R^{\pi_{M\times N}}(Q^*\alpha,Q^*\beta)Q^*\gamma = R_1(P^*\alpha,P^*\beta)P^*\gamma, R_2(Q^*\alpha,Q^*\beta)Q^*\gamma.$$

Proof. It follows directly from Proposition 3 and the definition of curvature tensor in $T^*(M\times N)$ that

$$\begin{aligned}
R^{\pi_{M\times N}}(\alpha,\beta)P^*\gamma &= D^{\pi_{M\times N}}_\alpha D^{\pi_{M\times N}}_\beta P^*\gamma - D^{\pi_{M\times N}}_\beta D^{\pi_{M\times N}}_\alpha P^*\gamma - D^{\pi_{M\times N}}_{[\alpha,\beta]_{\pi_{M\times N}}} P^*\gamma\\
&= D^{\pi_{M\times N}}_\alpha P^* D^{\pi_{M\times N}}_\beta \gamma - D^{\pi_{M\times N}}_\beta P^* D^{\pi_{M\times N}}_\alpha \gamma - P^* D^{\pi_{M\times N}}_{[\alpha,\beta]_{\pi_{M\times N}}} \gamma\\
&= P^* D^{\pi_{M\times N}}_\alpha D^{\pi_{M\times N}}_\beta \gamma - P^* D^{\pi_{M\times N}}_\beta D^{\pi_{M\times N}}_\alpha \gamma - P^* D^{\pi_{M\times N}}_{[\alpha,\beta]_{\pi_{M\times N}}} \gamma\\
&= P^*(D^{\pi_{M\times N}}_\alpha D^{\pi_{M\times N}}_\beta \gamma - D^{\pi_{M\times N}}_\beta D^{\pi_{M\times N}}_\alpha \gamma - D^{\pi_{M\times N}}_{[\alpha,\beta]_{\pi_{M\times N}}} \gamma)\\
&= P^* R^{\pi_{M\times N}}(\alpha,\beta)\gamma.
\end{aligned}$$

The proof of the second statement is similar to the one of (1).

To prove (3), we note that from $J^* = P^* - Q^*$, Propositions 2 and 3, and direct calculations, we have that

$$\begin{aligned}
R^{\pi_{M\times N}}(J^*\alpha,J^*\beta)\gamma &= D^{\pi_{M\times N}}_{J^*\alpha} D^{\pi_{M\times N}}_{J^*\beta} \gamma - D^{\pi_{M\times N}}_{J^*\beta} D^{\pi_{M\times N}}_{J^*\alpha} \gamma - D^{\pi_{M\times N}}_{[J^*\alpha,J^*\beta]_{\pi_{M\times N}}} \gamma\\
&= D^{\pi_{M\times N}}_{J^*\alpha} J^* D^{\pi_{M\times N}}_\beta \gamma - D^{\pi_{M\times N}}_{J^*\beta} J^* D^{\pi_{M\times N}}_\alpha \gamma - D^{\pi_{M\times N}}_{[\alpha,\beta]_{\pi_{M\times N}}} \gamma\\
&= D^{\pi_{M\times N}}_\alpha D^{\pi_{M\times N}}_\beta \gamma - D^{\pi_{M\times N}}_\beta D^{\pi_{M\times N}}_\alpha \gamma - D^{\pi_{M\times N}}_{[\alpha,\beta]_{\pi_{M\times N}}} \gamma\\
&= R^{\pi_{M\times N}}(\alpha,\beta)\gamma.
\end{aligned}$$

The other points are treated with similar calculations as above. □

5. On the Ricci tensor of product Poisson manifolds

By $r^{\pi_{M\times N}}$ we denote the poisson curvature tensor of product Riemannian Poisson manifold $(M\times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$. Then we have the following

Theorem 5.1. *Let $(M\times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ be a product Poisson manifold of the Poisson manifolds $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ with $\tilde{g}_{M\times N}$ the product metric of $M\times N$ and $\tilde{g}_M, \tilde{g}_N$ are the Riemannian metric of M, N respectively. Then for any $\alpha,\beta,\gamma \in \Omega^1(M\times N)$, the following hold*

(1) $r^{\pi_{M\times N}}(P^*\alpha,P^*\beta) = r^{\pi_M}(P^*\alpha,P^*\beta)$
(2) $r^{\pi_{M\times N}}(Q^*\alpha,Q^*\beta) = r^{\pi_N}(Q^*\alpha,Q^*\beta)$
(3) $r^{\pi_{M\times N}}(J^*\alpha,J^*\beta) = r^{\pi_{M\times N}}(\alpha,\beta)$
(4) $r^{\pi_{M\times N}}(P^*\alpha,Q^*\beta) = r^{\pi_{M\times N}}(Q^*\alpha,P^*\beta) = 0$
(5) $r^{\pi_{M\times N}}(\alpha,\beta) = r^{\pi_{M\times N}}(P^*\alpha,P^*\beta) + r^{\pi_{M\times N}}\ (Q^*\alpha,Q^*\beta)$

Proof. We choose a local form of adapted orthonormal frames $\{dx_1, dx_2, ...dx_m\}$ and $\{dy_1, dy_2, ...dy_m\}$ in T^*M and T^*N respectively. Then $\{e_1 = \pi^*dx_1, e_2 = \pi^*dx_2, ...e_m = \pi^*dx_m, e_{m+1} = \sigma^*dy_1, \sigma^*dy_2....., e_{m+n} = \sigma^*dy_n\}$ is an orthonormal frame in $T^*M\times N$.

We have

$$\begin{aligned}
r^{\pi_{M\times N}}(P^*\alpha, P^*\beta) &= \sum_{i=1}^{m+n} \tilde{g}(R^{\pi_{M\times N}}(e_i, P^*\alpha)P^*\beta, e_i) \\
&= \sum_{i=1}^{m+n} \tilde{g}(P^* R^{\pi_{M\times N}}(e_i, P^*\alpha)\beta, e_i) \\
&= \sum_{i=1}^{m+n} \tilde{g}(R^{\pi_{M\times N}}(P^*e_i, P^*\alpha)P^*\beta, e_i) \\
&= \sum_{i=1}^{m+n} \tilde{g}(R^{\pi_{M\times N}}(P^*e_i, P^*\alpha)P^*\beta, P^*e_i) \\
&= \sum_{i=1}^{m} g_M(R^{\pi_{M\times N}}(dx_i, P^*\alpha)P^*\beta, dx_i) \\
&= r^{\pi_m}(P^*\alpha, P^*\beta).
\end{aligned}$$

The proof of formulae (2), (3), (4) and (5) is akin to the foregoing one. □

6. On the contravariant derivative of the curvature tensor of product Poisson manifolds

Let $(M\times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ be a product Poisson manifold of the Poisson manifolds $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ with $\tilde{g}_{M\times N}$ the product metric of $M \times N$ and $\tilde{g}_M, \tilde{g}_N$ are the Riemannian metric of M, N respectively.

For $R^{\pi_{M\times N}}$, which is a tensor of type (3.1), we define its contravariant derivative, denoted by $D^{\pi_{M\times N}}R^{\pi_{M\times N}}$, as follows

$$\begin{aligned}
D_\alpha^{\pi_{M\times N}}R^{\pi_{M\times N}}(\beta,\gamma,\delta) =& D_\alpha^{\pi_{M\times N}}(R^{\pi_{M\times N}}(\beta,\gamma)\delta) - R^{\pi_{M\times N}}(D_\alpha^{\pi_{M\times N}}\beta,\gamma)\delta \\
&- R^{\pi_{M\times N}}(\beta, D_\alpha^{\pi_{M\times N}}\gamma)\delta - R^{\pi_{M\times N}}(\beta,\gamma)D_\alpha^{\pi_{M\times N}}\delta
\end{aligned}$$

Then we have the following

Theorem 6.1. *under the same assumptions as those of Theorem 2 and for any $\alpha, \beta, \gamma, \delta \in \Omega^1(M \times N)$, the following hold*

(1) $P^*(D_\alpha^{\pi_{M\times N}}R^{\pi_{M\times N}}(\beta,\gamma,\delta)) = D_{P^*\alpha}^{\pi_{M\times N}}R^{\pi_{M\times N}}(P^*\beta, P^*\gamma, P^*\delta)$
(2) $Q^*[D_\alpha^{\pi_{M\times N}}R^{\pi_{M\times N}}(\beta,\gamma,\delta)] = D_{Q^*\alpha}^{\pi_{M\times N}}R^{\pi_{M\times N}}(Q^*\beta, Q^*\gamma, Q^*\delta)$
(3) $D_{P^*\alpha}^{\pi_{M\times N}}R^{\pi_{M\times N}}(Q^*\beta,\gamma,\delta) = D_{P^*\alpha}^{\pi_{M\times N}}R^{\pi_{M\times N}}(\beta, Q^*\gamma,\delta) = 0$
(4) $D_{P^*\alpha}^{\pi_{M\times N}}R^{\pi_{M\times N}}(\beta,\gamma,Q^*\delta) = D_{Q^*\alpha}^{\pi_{M\times N}}R^{\pi_{M\times N}}(P^*\beta,\gamma,\delta) = 0$
(5) $D_{Q^*\alpha}^{\pi_{M\times N}}R^{\pi_{M\times N}}(\beta,P^*\gamma,\delta) = D_{Q^*\alpha}^{\pi_{M\times N}}R^{\pi_{M\times N}}(\beta,\gamma,P^*\delta) = 0$
(6) $D_\alpha^{\pi_{M\times N}}R^{\pi_{M\times N}} = D_{P^*\alpha}^{\pi_M}R_1(P^*\beta, P^*\gamma, P^*\delta) + D_{Q^*\alpha}^{\pi_N}R_2(Q^*\beta, Q^*\gamma, Q^*\delta)$

Proof. From Formula (12), Propositions 3&4 and the previous theorem, we obtain

$$\begin{aligned}
(D^{\pi_{M\times N}}_{P^*\alpha}R^{\pi_{M\times N}})(P^*\beta,P^*\gamma,P^*\delta) =&\\
& D^{\pi_{M\times N}}_{P^*\alpha}(R^{\pi_{M\times N}}(P^*\beta,P^*\gamma)P^*\delta) - R^{\pi_{M\times N}}(D^{\pi_{M\times N}}_{P^*\alpha}P^*\beta,P^*\gamma)P^*\delta\\
& - R^{\pi_{M\times N}}(P^*\beta,D^{\pi_{M\times N}}_{P^*\alpha},P^*\gamma)P^*\delta - R^{\pi_{M\times N}}(P^*\beta,P^*\gamma)D^{\pi_{M\times N}}_{P^*\alpha}P^*\delta\\
=& D^{\pi_{M\times N}}_{P^*\alpha}(P^*R^{\pi_{M\times N}}(\beta,\gamma)\delta) - R^{\pi_{M\times N}}(P^*D^{\pi_{M\times N}}_{\alpha}\beta,P^*\gamma)P^*\delta\\
& - R^{\pi_{M\times N}}(P^*\beta,P^*D^{\pi_{M\times N}}_{\alpha}\gamma)P^*\delta - R^{\pi_{M\times N}}(P^*\beta,P^*\gamma)P^*D^{\pi_{M\times N}}_{\alpha}\delta\\
=& P^*D^{\pi_{M\times N}}_{\alpha}(R^{\pi_{M\times N}}(\beta,\gamma)\delta) - P^*R^{\pi_{M\times N}}(D^{\pi_{M\times N}}_{\alpha}\beta,\gamma)\delta\\
& - P^*R^{\pi_{M\times N}}(\beta,D^{\pi_{M\times N}}_{\alpha}\gamma)\delta - P^*R^{\pi_{M\times N}}(\beta,\gamma)D^{\pi_{M\times N}}_{\alpha}\delta\\
=& P^*[D^{\pi_{M\times N}}_{\alpha}(R^{\pi_{M\times N}}(\beta,\gamma)\delta) - R^{\pi_{M\times N}}(D^{\pi_{M\times N}}_{\alpha}\beta,\gamma)\delta\\
& - R^{\pi_{M\times N}}(\beta,D^{\pi_{M\times N}}_{\alpha}\gamma)\delta - R^{\pi_{M\times N}}(\beta,\gamma)D^{\pi_{M\times N}}_{\alpha}\delta]\\
=& P^*[(D^{\pi_{M\times N}}_{\alpha}R^{\pi_{M\times N}})(\beta,\gamma,\delta)].
\end{aligned}$$

The proof of (2) is similar to the previous one while the proofs of (3), (4), (5) and (6) are based on Propositions 4&3 and on Theorem 1. □

7. On the contravariant derivative of the Poisson tensor $\pi_{M\times N}$ of product Poisson manifolds

The contravariant derivative of the Poisson tensor $\pi_{M\times N}$ is denoted by $D^{\pi_{M\times N}}\pi_{M\times N}$, which is defined for any $\alpha,\beta,\gamma\in\Omega^1(M\times N)$ by

$$\begin{aligned}
D^{\pi_{M\times N}}\pi_{M\times N}(\alpha,\beta,\gamma) &= \sharp_{\pi_{M\times N}}(\alpha).\pi_{M\times N}(\beta,\gamma) - \pi_{M\times N}(D^{\pi_{M\times N}}_{\alpha}\beta,\gamma)\\
&\quad -\pi_{M\times N}(\beta,D^{\pi_{M\times N}}_{\alpha}\gamma).
\end{aligned}$$

Theorem 7.1. *Under the same assumptions of Theorem 3, $D^{\pi_{M\times N}}\pi_{M\times N}$ verifies the following*

(1) $D^{\pi_{M\times N}}\pi_{M\times N}(P^*\alpha,P^*\beta,P^*\gamma) = D^{\pi_M}\pi_M(P^*\alpha,P^*\beta,P^*\gamma)$
(2) $D^{\pi_{M\times N}}\pi_{M\times N}(Q^*\alpha,Q^*\beta,Q^*\gamma) = D^{\pi_N}\pi_N(Q^*\alpha,Q^*\beta,Q^*\gamma)$
(3) $D^{\pi_{M\times N}}\pi_{M\times N}(P^*\alpha,Q^*\beta,\gamma) = D^{\pi_{M\times N}}\pi_{M\times N}(P^*\alpha,\beta,Q^*\gamma) = 0$
(4) $D^{\pi_{M\times N}}\pi_{M\times N}(Q^*\alpha,P^*\beta,\gamma) = D^{\pi_{M\times N}}\pi_{M\times N}(Q^*\alpha,\beta,Q^*\gamma) = 0$
(5) $D^{\pi_{M\times N}}\pi_{M\times N}(\alpha,\beta,\gamma) = D^{\pi_{M\times N}}\pi_{M\times N}(P^*\alpha,P^*\beta,P^*\gamma)+D^{\pi_{M\times N}}\pi_{M\times N}(Q^*\alpha,Q^*\beta,Q^*\gamma)$

Proof. From Propositions 2&3&4, we have

$$\begin{aligned}
D^{\pi_{M\times N}}\pi_{M\times N}(P^*\alpha,P^*\beta,P^*\gamma) &= \sharp_{\pi_{M\times N}}(P^*\alpha).\pi_{M\times N}(P^*\beta,P^*\gamma) - \pi_{M\times N}(D^{\pi_{M\times N}}_{P^*\alpha}P^*\beta,P^*\gamma)\\
&\quad - \pi_{M\times N}(P^*\beta,D^{\pi_{M\times N}}_{P^*\alpha}P^*\gamma)\\
&= \sharp_{\pi_M}(P^*\alpha).\pi_M(P^*\beta,P^*\gamma) - \pi_M(D^{\pi_M}_{P^*\alpha}\beta,P^*\gamma)\\
&\quad - \pi_M(P^*\beta,D^{\pi_M}_{P^*\alpha}P^*\gamma)\\
&= D^{\pi_M}\pi_M(P^*\alpha,P^*\beta,P^*\gamma),
\end{aligned}$$

establishing (1). We also have

$$\begin{aligned} D^{\pi_{M\times N}}\pi_{M\times N}(Q^*\alpha, Q^*\beta, Q^*\gamma) =& \sharp_{\pi_{M\times N}}(Q^*\alpha).\pi_{M\times N}(Q^*\beta, Q^*\gamma) - \pi_{M\times N}(D^{\pi_{M\times N}}_{Q^*\alpha}\beta, Q^*\gamma) \\ &- \pi_{M\times N}(Q^*\beta, D^{\pi_{M\times N}}_{Q^*\alpha}Q^*\gamma) \\ &= \sharp_{\pi_N}(Q^*\alpha).\pi_N(Q^*\beta, Q^*\gamma) - \pi_N(D^{\pi_N}_{Q^*\alpha}\beta, Q^*\gamma) \\ &- \pi_N(Q^*\beta, D^{\pi_N}_{Q^*\alpha}Q^*\gamma) \\ &= D^{\pi_N}\pi_N(Q^*\alpha, Q^*\beta, Q^*\gamma), \end{aligned}$$

proving (2). Lastly,

$$\begin{aligned} D^{\pi_{M\times N}}\pi_{M\times N}(P^*\alpha, Q^*\beta, \gamma) =& \sharp_{\pi_{M\times N}}(P^*\alpha).\pi_{M\times N}(Q^*\beta, \gamma) - \pi_{M\times N}(D^{\pi_{M\times N}}_{P^*\alpha}Q^*\beta, \gamma) \\ &- \pi_{M\times N}(D^{\pi_{M\times N}}_{P^*\alpha}Q^*\beta, \gamma) \\ &= \sharp_{\pi_{M\times N}}(P^*\alpha).\pi_{M\times N}(Q^*\beta, Q^*\gamma) = 0, \end{aligned}$$

showing (3).

As for the proof of (4), we imitate the proof of (3). Finally, (5) is an immediate consequence of (1), (2), (3) and (4). The proof of the theorem is over. □

8. On the Riemannian-Christoffel curvature tensor of product Poisson manifolds

Theorem 8.1. *Under the same assumptions of Theorem 4,* $\tilde{g}(R^{\pi_{M\times N}}(\alpha, \beta, \gamma), \delta)$ *verifies the following*

(1) $\tilde{g}_{M\times N}(R^{\pi_{M\times N}}(P^*\alpha, P^*\beta)P^*\gamma, P^*\delta) = \tilde{g}_M(R^{\pi_M}(P^*\alpha, P^*\beta)P^*\gamma, P^*\delta)$
(2) $\tilde{g}_{M\times N}(R^{\pi_{M\times N}}(Q^*\alpha, Q^*\beta)Q^*\gamma, Q^*\delta) = \tilde{g}_M(R^{\pi_M}(Q^*\alpha, Q^*\beta)Q^*\gamma, Q^*\delta)$
(3) $\tilde{g}_{M\times N}(R^{\pi_{M\times N}}(Q^*\alpha, \beta)\gamma, P^*\delta) = \tilde{g}_N(R^{\pi_N}(Q^*\alpha, P^*\beta)\gamma, \delta) = 0$
(4) $\tilde{g}_{M\times N}(R^{\pi_{M\times N}}(Q^*\alpha, \beta)P^*\gamma, Q^*\delta) = \tilde{g}_N(R^{\pi_N}(P^*\alpha, \beta)\gamma, Q^*\delta)=0$
(5) $\tilde{g}_{M\times N}(R^{\pi_{M\times N}}(P^*\alpha, Q^*\beta)\gamma, \delta) = \tilde{g}_N(R^{\pi_N}(P^*\alpha, \beta)Q^*\gamma, \delta) = 0$
(6) $\tilde{g}_{M\times N}(R^{\pi_{M\times N}}(\alpha, Q^*\beta)\gamma, P^*\delta) = \tilde{g}_N(R^{\pi_N}(\alpha, \beta)Q^*\gamma, P^*\delta) = 0$
(7) $\tilde{g}_{M\times N}(R^{\pi_{M\times N}}(^*\alpha, ^*\beta)^*\gamma, ^*\delta) = \tilde{g}_M(R^{\pi_M}(P^*\alpha, P^*\beta)P^*\gamma, P^*\delta)$
$+ \tilde{g}_M(R^{\pi_M}(Q^*\alpha, Q^*\beta)Q^*\gamma, Q^*\delta)$.

Proof. Formula (13), Theorem 1, Proposition 4 and direct calculations yield the desired formulae. □

9. On the scalar curvature of Product Poisson

Theorem 9.1. *Under the same assumptions of Theorem 5,* $\sigma^{\pi_{M\times N}}(x, y)$ *verifies the following equation ,*

$$\sigma^{\pi_{M\times N}}(x, y) = \sigma^{\pi_M}(x) + \sigma^{\pi_N}(y)$$

Proof. We choose a local form of adapted orthonormal frames $\{dx_1, dx_2, ...dx_m\}$ and $\{dy_1, dy_2, ...dy_m\}$ in T^*M and T^*N respectively.
Then $\{E_1 = \pi^*dx_1, E_2, ...e_m = \pi^*dx_m, E_{m+1} = \sigma^*dy_1, E_{m+2}....., E_{m+n} = \sigma^*dy_n\}$ is an orthonormal frames in $T^*M \times N$.

From theorems (5),(2) and (1), we have

$$\begin{aligned}\sigma^{\pi_{M\times N}}(x,y) &= \sum_{i=1}^{m+n} r^{\pi_{M\times N}}(E_i,E_i)(x,y)\\ &= \sum_{i=1}^{m} r^{\pi_{M\times N}}(E_i,E_i)(x,y) + \sum_{i=1}^{n} r^{\pi_{M\times N}}(E_{i+m},E_{i+m})(x,y)\\ &= \sum_{i=1}^{m} r^{\pi_{M}}(E_i,E_i)(x) + \sum_{i=1}^{n} r^{\pi_{N}}(E_{i+m},E_{i+m})(y)\\ &= \sigma^{\pi_M}(x) + \sigma^{\pi_N}(y)\end{aligned}$$

□

10. Geometric Consequences

Proposition 10.1. *Under the same hypotheses as those of Theorem 2, the product Poisson manifolds $(M\times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ is a Riemannian Poisson manifold if and only if $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ are Riemannian Poisson manifolds.*

Proof. If $(M\times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ is a Riemannian Poisson manifolds, then for any α, β and $\gamma \in T^*(M\times N)$, $D^{\pi_{M\times N}}\pi_{M\times N}(\beta,\gamma,\delta) = 0$. From Theorem 4, we have

$$0 = D^{\pi_{M\times N}}\pi_{M\times N}(P^*\beta, P^*\gamma, P^*\delta) = D^{\pi_M}\pi_M(P^*\beta, P^*\gamma, P^*\delta)$$

and

$$0 = D^{\pi_{M\times N}}\pi_{M\times N}(Q^*\beta, Q^*\gamma, Q^*\delta) = D^{\pi_N}\pi_N(Q^*\beta, Q^*\gamma, Q^*\delta).$$

The last two displayed equations imply that $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ are both Riemannian Poisson manifolds.

Conversely, if $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ are locally Riemannian Poisson manifolds, then

$$0 = D^{\pi_M}\pi_M(P^*\beta, P^*\gamma, P^*\delta) = D^{\pi_{M\times N}}\pi_{M\times N}(P^*\beta, P^*\gamma, P^*\delta)$$

and

$$0 = D^{\pi_N}\pi_N(Q^*\beta, Q^*\gamma, Q^*\delta) = D^{\pi_{M\times N}}\pi_{M\times N}(Q^*\beta, Q^*\gamma, Q^*\delta).$$

Using Theorem 4 again, we see that

$$D^{\pi_{M\times N}}\pi_{M\times N}(P^*\beta, P^*\gamma, P^*\delta) + D^{\pi_{M\times N}}\pi_{M\times N}(Q^*\beta, Q^*\gamma, Q^*\delta) = D^{\pi_{M\times N}}\pi_{M\times N}(\beta,\gamma,\delta)$$

and so $(M\times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ is a Riemannian Poisson manifold. □

Proposition 10.2. *Under the same hypotheses as those of Theorem 2, the product $(M\times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ is a locally symmetric Poisson manifold if and only if $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ are locally symmetric Poisson manifolds.*

Proof. If $(M\times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ is a locally symmetric manifold, then for any α, β and $\gamma \in T^*(M\times N)$, $D^{\pi_{M\times N}}_{\alpha} R^{\pi_{M\times N}}(\beta,\gamma,\delta) = 0$. Now, Theorem 3 gives us

$$0 = D^{\pi_{M\times N}}_{P^*\alpha} R^{\pi_{M\times N}}(P^*\beta, P^*\gamma, P^*\delta) = D^{\pi_M}_{P^*\alpha} R^{\pi_M}(P^*\beta, P^*\gamma, P^*\delta)$$

and

$$0 = D^{\pi_{M\times N}}_{Q^*\alpha} R^{\pi_{M\times N}}(Q^*\beta, Q^*\gamma, Q^*\delta) = D^{\pi_N}_{Q^*\alpha} R^{\pi_N}(Q^*\beta, Q^*\gamma, Q^*\delta).$$

These last two equations yield $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ are locally symmetric manifolds (i.e $D^{\pi_M} R^{\pi_M} = D^{\pi_N} R^{\pi_N} = 0$).

Conversely, if $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ are locally symmetric manifolds, then

$$0 = D^{\pi_M}_{P^*\alpha} R^{\pi_M}(P^*\beta, P^*\gamma, P^*\delta) = D^{\pi_{M\times N}}_{P^*\alpha} R^{\pi_{M\times N}}(P^*\beta, P^*\gamma, P^*\delta)$$

and

$$0 = D^{\pi_N}_{Q^*\alpha} R^{\pi_N}(Q^*\beta, Q^*\gamma, Q^*\delta) = D^{\pi_{M\times N}}_{Q^*\alpha} R^{\pi_{M\times N}}(Q^*\beta, Q^*\gamma, Q^*\delta).$$

From Theorem 3, we have

$$D^{\pi_{M\times N}}_{P^*\alpha} R^{\pi_{M\times N}}(P^*\beta, P^*\gamma, P^*\delta) + D^{\pi_{M\times N}}_{Q^*\alpha} R^{\pi_{M\times N}}(Q^*\beta, Q^*\gamma, Q^*\delta) = D^{\pi_{M\times N}}_{\alpha} R^{\pi_{M\times N}}(\beta, \gamma, \delta).$$

Thus $(M \times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ is a locally symmetric Poisson manifold. □

Proposition 10.3. *Let $(M \times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ be a product Poisson manifold of the Poisson manifolds $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ with $\tilde{g}_{M\times N}$ the product metric of $M \times N$ and $\tilde{g}_M, \tilde{g}_N$ are the Riemannian metric of M, N respectively. Then $(M \times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ is flat if and only if $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ are flats.*

Proof. The proof is based upon Theorem 1. □

Proposition 10.4. *Let $(M \times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ be a product Poisson manifold of the Poisson manifolds $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ with $\tilde{g}_{M\times N}$ the product metric of $M \times N$ and $\tilde{g}_M, \tilde{g}_N$ are the Riemannian metric of M, N respectively. Then $(M \times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ is a Ricci flat Poisson manifold if and only if $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ are Ricci flats Poisson manifolds.*

Proof. The proof is very similar (with obvious changes) to that of Proposition 6 and it uses Theorems 2&4. □

Theorem 10.1. *Let $(M \times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ be a product Poisson manifold of the Poisson manifolds $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ with $\tilde{g}_{M\times N}$ the product metric of $M \times N$ and $\tilde{g}_M, \tilde{g}_N$ are the Riemannian metric of M, N respectively. If $(M \times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ has constant sectional curvature, then $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ have the same constant sectional curvature.*

Proof. Theorem 5 and Formula (13) are used to prove this theorem. Let $\{P^*\alpha, P^*\beta\}$ and $\{Q^*\alpha, Q^*\beta\}$ be two linearly independent forms in $M \times N$. We have

$$\begin{aligned} k &= \frac{\tilde{g}_{\pi_{M\times N}}(R^{\pi_{M\times N}}(P^*\alpha, P^*\beta)P^*\beta, P^*\alpha)}{\tilde{g}_{\pi_{M\times N}}(P^*\alpha, P^*\alpha).\tilde{g}_{\pi_{M\times N}}(P^*\beta, P^*\beta) - (\tilde{g}_{\pi_{M\times N}}(P^*\alpha, P^*\beta))^2} \\ &= \frac{\tilde{g}_{\pi_M}(R^{\pi_M}(P^*\alpha, P^*\beta)P^*\beta, P^*\alpha)}{\tilde{g}_{\pi_M}(P^*\alpha, P^*\alpha).\tilde{g}_{\pi_M}(P^*\beta, P^*\beta) - (\tilde{g}_{\pi_M}(P^*\alpha, P^*\beta))^2} \end{aligned}$$

and

$$\begin{aligned} k &= \frac{\tilde{g}_{\pi_{M\times N}}(R^{\pi_{M\times N}}(Q^*\alpha, Q^*\beta)P^*\beta, Q^*\alpha)}{\tilde{g}_{\pi_{M\times N}}(Q^*\alpha, Q^*\alpha).\tilde{g}_{\pi_{M\times N}}(Q^*\beta, Q^*\beta) - (\tilde{g}_{\pi_{M\times N}}(Q^*\alpha, Q^*\beta))^2} \\ &= \frac{\tilde{g}_{\pi_N}(R^{\pi_N}(Q^*\alpha, Q^*\beta)Q^*\beta, Q^*\alpha)}{\tilde{g}_{\pi_N}(Q^*\alpha, Q^*\alpha).\tilde{g}_{\pi_N}(Q^*\beta, Q^*\beta) - (\tilde{g}_{\pi_N}(Q^*\alpha, Q^*\beta))^2} \end{aligned}$$

This means that $(M, \pi_M, \tilde{g}_M)$ and $(N, \pi_N, \tilde{g}_N)$ have the same constant sectional curvature k.

□

Corollary 10.1. *With the same assumptions as in the previous theorem and assuming that* $(M, \pi_M, \tilde{g}_M)$ *and* $(N, \pi_N, \tilde{g}_N)$ *have as constant sectional curvatures* k *and* k' $(k \neq k')$ *respectively, then* $(M \times N, \pi_{M\times N} = \pi_M + \pi_N, \tilde{g}_{M\times N} = \tilde{g}_M + \tilde{g}_N)$ *does not have constant sectional curvature.*

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