International Electronic Journal of Geometry
Volume 3 No. 2 pp. 15-25 (2010) ©IEJG

# ON A SEMI-SYMMETRIC NON-METRIC CONNECTION IN AN LP-SASAKIAN MANIFOLD

SELCEN YÜKSEL PERKTAŞ, EROL KILIÇ, SADIK KELEŞ

(Communicated by Cihan ÖZGÜR)

Abstract. In this paper we study some properties of curvature tensor, projective curvature tensor, $v$-Weyl projective tensor, concircular curvature tensor, conformal curvature tensor, quasi-conformal curvature tensor with respect to semi-symmetric non-metric connection in a Lorentzian para-Sasakian (briefly LP-Sasakian) manifold. It is shown that an LP-Sasakian manifold $(M^n, g)(n > 3)$ with the semi-symmetric non-metric connection is an $\eta$-Einstein manifold.

## 1. Introduction

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [5]. A linear connection $\widetilde{\nabla}$ in an $n$-dimensional differentiable manifold $M$ is said to be semi-symmetric connection if its torsion $\widetilde{T}$ is of the form

$$\widetilde{T}(X,Y) = u(X)Y - u(Y)X, \tag{1.1}$$

where $u$ is a 1-form. The connection $\widetilde{\nabla}$ is a metric connection if there is a Riemannian metric $g$ in $M$ such that $\widetilde{\nabla}g = 0$, otherwise it is non-metric. In 1932, H. A. Hayden [6] defined a semi-symmetric metric connection on a Riemannian manifold and this was further developed by K. Yano [21]. In [1], Agashe and Chafle introduced a semi-symmetric non-metric connection on a Riemannian manifold and this was further studied by U. C. De and D. Kamilya [3], J. Sengupta, U. C. De and T. Q. Binh [12], S. C. Biswas and U. C. De[2] and others. In [12], the authors defined a semi-symmetric non-metric connection on a Riemannian manifold which generalizes the notion of semi-symmetric non-metric connection introduced by Agashe and Chafle. M. M. Tripathi [15] studied the semi-symmetric metric connection in a Kenmotsu manifolds. In [16], the semi-symmetric non-metric connection in a Kenmotsu manifold was studied by M. M. Tripathi and N. Nakkar. Also in [18], M. M. Tripathi proved the existence of a new connection and he showed that in particular

2000 *Mathematics Subject Classification.* 53B15, 53C50, 53D15.

*Key words and phrases.* Lorentzian para-Sasakian manifold, $\eta$-Einstein manifold, semi-symmetric non-metric connection, projective curvature tensor,$v$-Weyl projective tensor, concircular curvature tensor, conformal curvature tensor, quasi-conformal curvature tensor, deformation algebra.

cases, this connection reduces to semi-symmetric connections; even some of them are not introduced so far. On the other hand, there is a class of almost paracontact metric manifolds, namely Lorentzian para-Sasakian manifolds. In 1989, K. Matsumoto [8], introduced the notion of Lorentzian para-Sasakian manifold. Then I. Mihai and R. Rosca [9] introduced the same notion independently and they obtained several results on this manifold. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [7], U. C. De and et al.,[10], A. A. Shaikh and S. Biswas [13], M. M. Tripathi and U. C. De [17].

In this paper, we study the semi-symmetric non-metric connection in a Lorentzian para-Sasakian manifold. Section 2 is devoted to preliminaries. In section 3, we find the expression for curvature tensor (resp. Ricci tensor) with respect to the semi-symmetric non-metric connection and investigate relations between curvature tensor (resp. Ricci tensor) with respect to the semi-symmetric non-metric connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. In this section we also show that a LP-Sasakian manifold with the semi-symmetric non-metric connection is an $\eta$-Einstein manifold. In section 4 projective curvature tensor of the semi-symmetric non-metric connection is studied and it is shown that in a LP-Sasakian manifold the projective curvature tensor of the manifold with respect to the semi-symmetric non-metric connection is equal to the projective curvature tensor of the manifold. The concircular curvature tensor, conformal curvature tensor, quasi conformal curvature tensor of a LP-Sasakian manifold with respect to the semi-symmetric non-metric connection are investigated in section 5, section 6 and section 7, respectively. In the last section deformation algebra of the Levi-Civita connection and the semi-symmetric non-metric connection of a Lorentzian para-Sasakian manifold is established.

## 2. Preliminaries

A differentiable manifold of dimension $n$ is called Lorentzian para-Sasakian (briefly, LP-Sasakian)[8, 9], if it admits a $(1,1)$-tensor field $\phi$, a contravariant vector field $\xi$, a 1-form $\eta$ and a Lorentzian metric $g$ which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$
$$\phi^2(X) = X + \eta(X)\xi, \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$
$$g(X,\xi) = \eta(X), \tag{2.4}$$
$$\nabla_X \xi = \phi X, \tag{2.5}$$
$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

where $\nabla$ denotes the covariant differentiation with respect to the Lorentzian metric $g$.

It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \tag{2.7}$$
$$rank\ \phi = n-1. \tag{2.8}$$

If we put

$$\Phi(X,Y) = g(X, \phi Y) \tag{2.9}$$

for any vector fields $X$ and $Y$, then the tensor field $\Phi(X,Y)$ is a symmetric $(0,2)$ tensor field [8]. Also since the 1-form $\eta$ is closed in an LP-Sasakian manifold, we have [8, 10]

$$(\nabla_X\eta)(Y) = \Phi(X,Y), \quad \Phi(X,\xi) = 0 \tag{2.10}$$

for all $X$ , $Y \in TM$.

Also in an LP-Sasakian manifold, the following relations hold [7, 10]:

$$\begin{aligned}
g(R(X,Y)Z,\xi) &= \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), && (2.11)\\
R(\xi,X)Y &= g(X,Y)\xi - \eta(Y)X, && (2.12)\\
R(X,Y)\xi &= \eta(Y)X - \eta(X)Y, && (2.13)\\
R(\xi,X)\xi &= X + \eta(X)\xi, && (2.14)\\
S(X,\xi) &= (n-1)\eta(X), && (2.15)\\
S(\phi X,\phi Y) &= S(X,Y) + (n-1)\eta(X)\eta(Y) && (2.16)
\end{aligned}$$

for any vector fields $X, Y, Z$ where $R$ and $S$ are the Riemannian curvature and the Ricci tensors of the manifold, respectively.

Let $M$ be an $n$-dimensional LP-Sasakian manifold and $\nabla$ be the Levi-Civita connection on $M$. A linear connection $\widetilde{\nabla}$ on $M$ is said to be semi-symmetric if the torsion tensor

$$\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]$$

satisfies

$$\widetilde{T}(X,Y) = \eta(Y)X - \eta(X)Y \tag{2.17}$$

for all $X$ , $Y \in TM$. A semi-symmetric non-metric connection $\widetilde{\nabla}$ in an LP-Sasakian manifold can be defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X, \tag{2.18}$$

where $\nabla$ is Levi-Civita connection on $M$ (see [1]).

## 3. Curvature Tensor

Let $M$ be an $n$-dimensional LP-Sasakian manifold. The curvature tensor $\widetilde{R}$ of $M$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ is defined by

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z. \tag{3.1}$$

From (2.18) and (3.1) we have,

$$\widetilde{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y \tag{3.2}$$

where $\alpha$ is a tensor field of $(0,2)$ type defined by

$$\alpha(X,Y) = (\widetilde{\nabla}_X\eta)Y = (\nabla_X\eta)Y - \eta(X)\eta(Y). \tag{3.3}$$

**Lemma 3.1.** *Let $M$ be an $n$-dimensional LP-Sasakian manifold with the semi-symmetric non-metric connection $\widetilde{\nabla}$. Then*

$$\begin{aligned}
(\widetilde{\nabla}_X\phi)Y &= (\nabla_X\phi)Y - \eta(Y)\phi X, && (3.4)\\
\widetilde{\nabla}_X\xi &= -X + \phi X, && (3.5)\\
(\widetilde{\nabla}_X\eta)Y &= \Phi(X,Y) - \eta(X)\eta(Y) = \alpha(X,Y) && (3.6)
\end{aligned}$$

*Proof.* By using (2.18) and (2.7), we obtain (3.4). From (2.18) and (2.5) we get (3.5). Finally by using (2.18) and (2.6) we have (3.6). □

From (3.6) we have

**Corollary 3.1.** *In an LP-Sasakian manifold, the tensor field $\alpha$ satisfies*

$$\alpha(X,\xi)=\eta(X)=g(X,\xi).$$

Let $K$ and $\widetilde{K}$ be the curvature tensors of $(0,4)$ type given by

$$K(X,Y,Z,U)=g(R(X,Y)Z,U) \tag{3.7}$$

and

$$\widetilde{K}(X,Y,Z,U)=g(\widetilde{R}(X,Y)Z,U). \tag{3.8}$$

**Theorem 3.1.** *In an LP-Sasakian manifold with semi-symmetric non-metric connection $\widetilde{\nabla}$ we have*

$$\widetilde{R}(X,Y)Z+\widetilde{R}(Y,Z)X+\widetilde{R}(Z,X)Y = 0, \tag{3.9}$$

$$\widetilde{K}(X,Y,Z,U)+\widetilde{K}(Y,X,Z,U) = 0, \tag{3.10}$$

$$\begin{aligned}\widetilde{K}(X,Y,Z,U)-\widetilde{K}(Z,U,X,Y) &= \alpha(X,U)g(Y,Z)\\ &\quad-\alpha(Y,Z)g(X,U).\end{aligned} \tag{3.11}$$

*Proof.* By using (3.2) and the first Bianchi identity

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0$$

with respect to Levi-Civita connection $\nabla$, we obtain

$$\begin{aligned}\widetilde{R}(X,Y)Z+\widetilde{R}(Y,Z)X+\widetilde{R}(Z,X)Y &= (\alpha(Z,Y)-\alpha(Y,Z))X\\ &\quad+(\alpha(X,Z)-\alpha(Z,X))Y\\ &\quad+(\alpha(Y,X)-\alpha(X,Y))Z.\end{aligned}$$

Since $\alpha$ is symmetric then we obtain (3.9).

From (3.2), (3.7) and (3.8) we have

$$\begin{aligned}\widetilde{K}(X,Y,Z,U) &= g(\widetilde{R}(X,Y)Z,U)\\ &= K(X,Y,Z,U)-\alpha(Y,Z)g(X,U)+\alpha(X,Z)g(Y,U).\end{aligned} \tag{3.12}$$

If we change the role of $X$ and $Y$ in (3.12) we write

$$\widetilde{K}(Y,X,Z,U)=K(Y,X,Z,U)-\alpha(X,Z)g(Y,U)+\alpha(Y,Z)g(X,U).$$

Since $K(X,Y,Z,U)+K(Y,X,Z,U)=0$ and $\alpha$ is symmetric then we get (3.10).

Finally, if we use (3.12) and $K(X,Y,Z,U)=K(Z,U,X,Y)$ we can easily obtain (3.11). □

**Lemma 3.2.** *Let $M$ be a n-dimensional LP-Sasakian manifold with the semi-symmetric non-metric connection $\widetilde{\nabla}$. Then*

$$\widetilde{R}(\xi,X)Y = (g(X,Y)-\alpha(X,Y))\xi, \tag{3.13}$$

$$\widetilde{R}(\xi,X)\xi = 0, \tag{3.14}$$

$$\widetilde{R}(X,Y)\xi = 0. \tag{3.15}$$

*Proof.* (3.13) follows from (2.12) and (3.2). By using (2.14) and (3.2) we have (3.14). From (2.13) and (3.2) we obtain (3.15). □

The Ricci tensor $\widetilde{S}$ of the manifold $M$ with respect to the the semi-symmetric non-metric connection $\widetilde{\nabla}$ is defined by

$$\widetilde{S}(X,Y) = Trace\ of\ the\ map : X \to \widetilde{R}(X,Y)Z,$$

that is,

$$\widetilde{S}(X,Y) = \sum_{i=1}^{n} \varepsilon_i g(\widetilde{R}(e_i,X)Y,e_i) \tag{3.16}$$

and the scalar curvature of $M$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ is given by

$$\widetilde{r} = \sum_{i=1}^{n} \varepsilon_i \widetilde{S}(e_i,e_i) \tag{3.17}$$

where $X$ , $Y \in TM$, $\{e_1,e_2,...,e_n\}$ is an orthonormal frame and $\varepsilon_i = g(e_i,e_i)$.

**Theorem 3.2.** *Let $M$ be an $n$-dimensional LP-Sasakian manifold with semi-symmetric non-metric connection $\widetilde{\nabla}$. Then the Ricci tensor $\widetilde{S}$ and the scalar curvature of the manifold $M$ with respect to $\widetilde{\nabla}$ are given by*

$$\widetilde{S}(X,Y) = S(X,Y) - (n-1)\alpha(X,Y) \tag{3.18}$$

*and*

$$\widetilde{r} = r - (n-1)trace(\alpha), \tag{3.19}$$

*respectively. Here $S$ and $r$ are the Ricci tensor and the scalar curvature of the manifold with respect to the Levi-Civita connection $\nabla$, respectively. Consequently, $\widetilde{S}$ is symmetric.*

*Proof.* From (3.2) and (3.16) we have (3.18). (3.19) follows from (3.18). Since $\alpha$ and $S$ are symmetric, from (3.18) $\widetilde{S}$ is symmetric, too. □

**Lemma 3.3.** *In an LP-Sasakian manifold with semi-symmetric non-metric connection $\widetilde{\nabla}$ we have*

$$\widetilde{S}(X,\xi) = 0, \tag{3.20}$$

$$\widetilde{S}(\phi X,\phi Y) = \widetilde{S}(X,Y). \tag{3.21}$$

*Proof.* By using (2.15) and (3.18) we have (3.20). Since $\Phi(\phi X,\phi Y) = \Phi(X,Y)$, from (2.16) and (3.18) we get (3.21). □

In [13], the authors stated the following theorems:

**Theorem 3.3.** [13] *An LP-Sasakian manifold $(M^n,g)(n>3)$ satisfying the condition $S(X,\xi).R = 0$ is an $\eta$-Einstein manifold.*

**Theorem 3.4.** [13] *Let $(M^n,g)(n>3)$ be an LP-Sasakian manifold satisfying the condition $S(X,\xi).R = 0$. Then the scalar curvature of the manifold is constant if and only if the vector field $\xi$ is harmonic.*

From (3.20) and Theorem (3.3) we have the following corollary

**Corollary 3.2.** *An LP-Sasakian manifold $(M^n,g)(n>3)$ with the semi-symmetric non-metric connection is an $\eta$-Einstein manifold.*

By using (3.20) again and from Theorem (3.4) we get

**Corollary 3.3.** *Let $(M^n, g)(n > 3)$ be an LP-Sasakian manifold with the semi-symmetric non-metric connection. Then the scalar curvature of the manifold with respect to the semi-symmetric non-metric connection is constant if and only if the vector field $\xi$ is harmonic.*

## 4. Projective Curvature Tensor

Let $M$ be an $n$-dimensional LP-Sasakian manifold. The projective curvature tensor $\widetilde{P}$ of type (1,3) of $M$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ is defined by [11]

$$\begin{aligned} \widetilde{P}(X,Y)Z &= \widetilde{R}(X,Y)Z + \frac{1}{n+1}\{\widetilde{S}(X,Y) - \widetilde{S}(Y,X)\}Z \\ &\quad - \frac{n}{n^2-1}\{\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y\} \\ &\quad - \frac{1}{n^2-1}\{\widetilde{S}(Z,Y)X - \widetilde{S}(Z,X)Y\}. \end{aligned} \tag{4.1}$$

**Proposition 4.1.** *In a LP-Sasakian manifold $M$ the projective curvature tensor of $M$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ is equal to the projective curvature tensor of the manifold with respect to the Levi-Civita connection.*

*Proof.* Let $P$ and $\widetilde{P}$ denote the projective curvature tensors with respect to $\widetilde{\nabla}$ and $\nabla$, respectively. Then by using (3.2) and (3.18) in (4.1) we have

$$\begin{aligned} \widetilde{P}(X,Y)Z &= R(X,Y)Z - \alpha(Y,Z)X - \alpha(X,Z)Y \\ &\quad + \frac{1}{n+1}\{S(X,Y) - (n-1)\alpha(X,Y) \\ &\quad - S(Y,X) + (n-1)\alpha(Y,X)\}Z \\ &\quad - \frac{n}{n^2-1}\{S(Y,Z)X - (n-1)\alpha(Y,Z)X \\ &\quad - S(X,Z)Y + (n-1)\alpha(X,Z)Y\} \\ &\quad - \frac{1}{n^2-1}\{S(Z,Y)X - (n-1)\alpha(Z,Y)X \\ &\quad - S(Z,X)Y + (n-1)\alpha(Z,X)Y\} \\ &= R(X,Y)Z + \frac{1}{n+1}\{S(X,Y) - S(Y,X)\}Z \\ &\quad - \frac{n}{n^2-1}\{S(Y,Z)X - S(X,Z)Y\} \\ &\quad - \frac{1}{n^2-1}\{S(Z,Y)X - S(Z,X)Y\} \\ &\quad + (-1 + \frac{n}{n+1} + \frac{1}{n+1})\{\alpha(Y,Z)X - \alpha(X,Z)Y\} \\ &= P(X,Y)Z. \end{aligned}$$

This completes the proof. □

The projective curvature tensor of $M$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ satisfies the following algebraic properties

$$\widetilde{P}(X,Y)Z + \widetilde{P}(Y,X)Z = 0 \tag{4.2}$$

and

$$\widetilde{P}(X,Y)Z + \widetilde{W}(Y,Z)X + \widetilde{P}(Z,X)Y = 0. \tag{4.3}$$

**Theorem 4.1.** *Let $M$ be a LP-Sasakian manifold. If the Ricci tensor of $M$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ vanishes, then the curvature tensor of $M$ with respect to $\widetilde{\nabla}$ is equal to the projective curvature tensor of the manifold with respect to the Levi-Civita connection.*

*Proof.* If $\widetilde{S} = 0$, from (4.1) we have $\widetilde{P}(X,Y)Z = \widetilde{R}(X,Y)Z$. Since

$$\widetilde{P}(X,Y)Z = P(X,Y)Z$$

we get

$$\widetilde{R}(X,Y)Z = P(X,Y)Z. \tag{4.4}$$

This gives the assertion of the theorem. □

**Theorem 4.2.** *If the curvature tensor of a LP-Sasakian manifold with respect to the semi-symmetric non-metric connection vanishes, then the manifold is projectively flat.*

*Proof.* If $\widetilde{R}(X,Y)Z = 0$, the from (4.4) we have

$$P(X,Y)Z = 0,$$

which completes the proof. □

**Theorem 4.3.** *An LP-Sasakian manifold with vanishing Ricci tensor $\widetilde{S}$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ is projectively flat if and only if the curvature tensor $\widetilde{R}$ with respect to $\widetilde{\nabla}$ vanishes.*

*Proof.* From theorem (4.1) and theorem (4.2), we have the assertion of the theorem. □

In the theory of the projective transformations of connections the Weyl projective tensor plays an important role. The $v$-Weyl projective tensor $P^v$ is defined by [14]

$$P^v(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}. \tag{4.5}$$

**Theorem 4.4.** *Let $M$ be an $n$-dimensional LP-Sasakian manifold. Then the $v$-Weyl projective tensor $\widetilde{P^v}$ of $M$ with respect to the semi-symmetric non-metric connection is equal to the $v$-Weyl projective tensor $P^v$ of the manifold.*

*Proof.* Let $\widetilde{P^v}$ and $P^v$ denote the $v$-Weyl projective tensors of $M$ with respect to the semi-symmetric non-metric connection and the Levi-Civita connection, respectively. From the definition of $v$-Weyl projective tensor, the $v$-Weyl projective tensor of $M$ with respect to the semi-symmetric non-metric connection is

$$\widetilde{P^v}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{n-1}\{\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y\}.$$

Then by using (3.2), (3.18) and (4.5), we have

$$\begin{aligned}\widetilde{P^v}(X,Y)Z &= R(X,Y)Z-\alpha(Y,Z)X+\alpha(X,Z)Y\\ &\quad -\frac{1}{n-1}\{S(Y,Z)X-(n-1)\alpha(Y,Z)X\\ &\quad -S(X,Z)Y+(n-1)\alpha(X,Z)Y\}\end{aligned}$$

which gives $\widetilde{P^v}(X,Y)Z=P^v(X,Y)Z$. □

**Lemma 4.1.** *In an $n$-dimensional LP-Sasakian manifold, the $v$-Weyl projective tensor $\widetilde{P^v}$ of the manifold with respect to the semi-symmetric non-metric connection satisfies the followings:*

$$\begin{aligned}\widetilde{P^v}(X,Y)Z+\widetilde{P^v}(Y,Z)X+\widetilde{P^v}(Z,X)Y &= 0,\\ \widetilde{P^v}(X,Y)Z+\widetilde{P^v}(Y,X)Z &= 0.\end{aligned}$$

## 5. Concircular Curvature Tensor

Let $M$ be an $n$-dimensional LP-Sasakian manifold. The concircular curvature tensor of $M$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ is given by

$$\widetilde{V}(X,Y)Z=\widetilde{R}(X,Y)Z-\frac{\widetilde{r}}{n(n-1)}\{g(Y,Z)X-g(X,Z)Y\} \tag{5.1}$$

where $\widetilde{R}$ and $\widetilde{r}$ are the curvature tensor and the scalar curvature of $M$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$, respectively. The concircular curvature tensor can be thought as a measure of the failure of a Riemannian manifold to be of constant curvature.

**Theorem 5.1.** *The concircular curvature tensors $V$ and $\widetilde{V}$ of the semi-symmetric non-metric connection $\widetilde{\nabla}$ and of the LP-Sasakian manifold are related by*

$$\begin{aligned}\widetilde{V}(X,Y)Z &= V(X,Y)Z-\{\alpha(Y,Z)X-\alpha(X,Z)Y\}\\ &\quad +\frac{trace(\alpha)}{n}\{g(Y,Z)X-g(X,Z)Y\}.\end{aligned} \tag{5.2}$$

*Proof.* By using (3.2) and (3.19) in (5.1), we have

$$\begin{aligned}\widetilde{V}(X,Y)Z &= R(X,Y)Z-\alpha(Y,Z)X+\alpha(X,Z)Y\\ &\quad -\frac{r-(n-1)trace(\alpha)}{n(n-1)}\{g(Y,Z)X-g(X,Z)Y\}\end{aligned}$$

which gives (2.18). □

## 6. Conformal Curvature Tensor

Let $M$ be an $n$-dimensional LP-Sasakian manifold. The conformal curvature tensor of $M$ with respect to semi-symmetric non-metric connection $\widetilde{\nabla}$ is defined by

[22]

$$
\begin{aligned}
\widetilde{C}(X,Y,Z,U) &= \widetilde{K}(X,Y,Z,U) \\
&\quad -\frac{1}{n-2}\{g(Y,Z)\widetilde{S}(X,U)-g(X,Z)\widetilde{S}(Y,U) \\
&\quad +\widetilde{S}(Y,Z)g(X,U)-\widetilde{S}(X,Z)g(Y,U)\} \\
&\quad +\frac{\widetilde{r}}{(n-1)(n-2)}\{g(Y,Z)g(X,U)-g(X,Z)g(Y,U)\}.
\end{aligned} \tag{6.1}
$$

By using (3.8), (3.18) and (3.19) in (6.1), we get

$$
\begin{aligned}
\widetilde{C}(X,Y,Z,U) &= C(X,Y,Z,U)+\frac{1}{n-2}\{\alpha(Y,Z)g(X,U)-\alpha(X,Z)g(Y,U)\} \\
&\quad +\frac{n-1}{n-2}\{\alpha(X,U)g(Y,Z)-\alpha(Y,U)g(X,Z)\} \\
&\quad -\frac{trace(\alpha)}{n-2}\{g(X,U)g(Y,Z)-g(Y,U)g(X,Z)\}
\end{aligned}
$$

## 7. Quasi-Conformal Curvature Tensor

The notion of the quasi-conformal curvature tensor was introduced by Yano and Sawaki [20]. They define a quasi-conformal curvature tensor by

$$
\begin{aligned}
W(X,Y)Z &= aR(X,Y)Z+b\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX \\
&\quad -g(X,Z)QY\}-\frac{r}{n}[\frac{a}{n-1}+2b]\{g(Y,Z)X-g(X,Z)Y\},
\end{aligned} \tag{7.1}
$$

where $a$ and $b$ are constants such that $ab \neq 0$, $R$ is the Riemannian curvature tensor, $S$ is the Ricci tensor, $Q$ is the Ricci operator defined by $g(QX,Y)=S(X,Y)$ and $r$ is the scalar curvature of the manifold. If $a=1$ and $b=-\frac{1}{n-2}$, then (7.1) takes the form

$$
\begin{aligned}
W(X,Y)Z &= R(X,Y)Z-\frac{1}{n-2}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX \\
&\quad -g(X,Z)QY]+\frac{r}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y] \\
&= C(X,Y)Z,
\end{aligned}
$$

where $C$ is conformal curvature tensor [4]. Thus the conformal curvature tensor $C$ is a particular case of the quasi-conformal curvature tensor $W$. An $n$- dimensional ($n>3$) manifold is called quasi-conformally flat if the quasi-conformal curvature tensor $W$ vanishes.

A quasi-conformal curvature tensor $\widetilde{W}$ with respect to a semi-symmetric non-metric connection in an $n$-dimensional Lorentzian para-Sasakian manifold is defined by

$$
\begin{aligned}
\widetilde{W}(X,Y)Z &= a\widetilde{R}(X,Y)Z+b\{\widetilde{S}(Y,Z)X-\widetilde{S}(X,Z)Y+g(Y,Z)\widetilde{Q}X \\
&\quad -g(X,Z)\widetilde{Q}Y\}-\frac{\widetilde{r}}{n}[\frac{a}{n-1}+2b]\{g(Y,Z)X-g(X,Z)Y\},
\end{aligned} \tag{7.2}
$$

where $a$ and $b$ arbitrary constans such that $a$ and $b$ are not zero simultaneously, $\widetilde{Q}$ is the Ricci operator with respect to the semi-symmetric non-metric connection i.e., $g(\widetilde{Q}X,Y)=\widetilde{S}(X,Y)$ and $\widetilde{r}$ is the scalar curvature of the manifold with respect to the semi-symmetric non-metric connection. Let us denote

$$\widetilde{W}(X,Y,Z,U)=g(\widetilde{W}(X,Y)Z,U). \tag{7.3}$$

Then if we use (3.2), (3.8), (3.18), (3.19), (7.3) in (7.2) we have

$$\begin{aligned}\widetilde{W}(X,Y,Z,U) &= W(X,Y,Z,U)\\ &-[a+(n-1)b]\{\alpha(Y,Z)g(X,U)-\alpha(X,Z)g(Y,U)\}\\ &-[(n-1)b]\{\alpha(X,U)g(Y,Z)-\alpha(Y,U)g(X,Z)\}\\ &+[\frac{(n-1)trace(\alpha)}{n}(\frac{a}{n-1}+2b)]\{g(X,U)g(Y,Z)\\ &-g(Y,U)g(X,Z)\}.\end{aligned}$$

If $a=(1-n)b$, then

$$\begin{aligned}\widetilde{W}(X,Y,Z,U) &= W(X,Y,Z,U)+a\{\alpha(X,U)g(Y,Z)-\alpha(Y,U)g(X,Z)\}\\ &-\frac{a}{n}trace(\alpha)\{g(X,U)g(Y,Z)-g(Y,U)g(X,Z)\}\end{aligned}$$

## 8. Deformation Algebra $II(M,\nabla,\widetilde{\nabla})$

Let $\nabla^1$ and $\nabla^2$ be two linear connections in a Riemannian manifold. We define the product of two vector fields $X$ and $Y$ by

$$X\circ Y=\nabla^2_XY-\nabla^1_XY. \tag{8.1}$$

The module $\chi(M)$ of all differentiable vector fields becomes an algebra over the ring $F(M)$ all real functions over $M$. This algebra is called the deformation algebra [19] of the ordered pair of connection $(\nabla^1,\nabla^2)$ and is denoted by $II(M,\nabla,\widetilde{\nabla})$. An element $X\in\chi(M)$ is called a characteristic vector field if there is a $\lambda\in F(M)$ such that

$$X\circ X=\lambda X. \tag{8.2}$$

Here by using (8.1), we have

$$\begin{aligned}X\circ X &= \widetilde{\nabla}_XX-\nabla_XX\\ &= \eta(X)X.\end{aligned}$$

Thus we can state the following theorem

**Theorem 8.1.** *In the deformation algebra $II(M,\nabla,\widetilde{\nabla})$, every element is a characteristic vector field.*

## References

[1] Agashe, N. S. and Chafle, M. R., A semi-symmetric non-metric connection in a Riemannian manifold, Indian J. Pure Appl. Math. 23 (1992), 399-409.
[2] Biswas, S. C. and De, U. C., On a type of semi-symmetric non-metric connection on a Riemannian manifold, Ganita. 48 (1997), 91-94.
[3] De, U. C. and Kamilya, D., Hypersurfaces of a Riemannian manifold with semi-symmetric non-metric connection, J. Indian Inst. Sci. 75 (1995), 707-710.
[4] Eisenhart, L. P., Riemannian Geometry, Princeton University Press, Princeton, N. J., 1949.
[5] Friedmann, A. and Schouten, J. A., Über die Geometric der holbsymmetrischen Übertragurgen. Math. Z. 21 (1924), 211-233.
[6] Hayden, H. A., Subspaces of space with torsion, Proc. London Math. Soc. 34 (1932), 27-50.

[7] Matsumoto, K. and Mihai, I., On a certain transformation in a Lorentzian para-Sasakian manifold, Tensor (N. S.) 47 (1988), 189-197.
[8] Matsumoto, K., On Lorentzian Paracontact manifolds, Bull. Yamagata Univ. Natur. Sci. 12(2) (1989), 151-156.
[9] Mihai, I. and Rosca, R., On Lorentzian P-Sasakian manifolds, Classical Analysis, World Scientific Publi., Signapore, (1992), 155-169.
[10] Mihai, I., Shaikh, A. A. and De, U. C., On Lorentzian para-Sasakian manifolds,Rendicontidel Seminario Matematico di Messina, Serie II, 1999.
[11] Mishra, R. S., Structures on a differentiable manifold and their applications, Chandrama Prakashan, Allahabad, India, 1984.
[12] Sengupta, J., De, U. C. and Binh, T. Q., On a type of semi-symmetric non-metric connection on a Riemannian manifold, Indian J. Pure Appl. Math. 31(12) (2000), 1659-1670.
[13] Shaikh,A. A. and Biswas, S., On LP-Sasakian manifolds, Bull. of the Malaysian Math. Sci. Soc. 27,(2004), 17-26.
[14] Tigaeru, C., v-projective symmetries of fibred manifolds, Arch. Math. 34, (1998), 347-352.
[15] Tripathi, M. M., On a semi-symmetric metric connection in a Kenmotsu manifold, J. Pure Math. 16 (1999), 67-71.
[16] Tripathi, M. M. and Nakkar, N., On a semi-symmetric non-metric connection in a Kenmotsu manifold, Bull. Cal. Math. Soc. 16 (2001), no.4, 323-330.
[17] Tripathi, M. M. and De, U. C., Lorentzian Almost Paracontact Manifolds and their submanifolds, Journal of the Korean Society of Mathematical Education, 2 (2001), 101-125.
[18] Tripathi, M. M., A new connection in a Riemannian manifold, International Electronic Journal of Geometry. vol. 1 (2008), no.1, 15-24.
[19] Vaisman, I., Sur quelques formulas du calcul de Ricci global, Comm. Math. Helvv. 4 (1966-67), 73-87.
[20] Yano, K. and Sawaki, S., Riemannian manifolds admitting a conformal transformation group, J. Differential Geometry 2 (1968), 161-184.
[21] Yano, K., On semi-symmetric metric connections, Revue Roumaina De Math. Pures Appl. 15 (1970), 1579-1586.
[22] Yano, K. and Kon, M., Structures on Manifolds, Series in Pure Mathematics, vol.3, World Scientific, Singapore, 1984.

İnönü University, Faculty of Arts and Sciences, Department of Mathematics, 44280, Malatya/TURKEY.

*E-mail address*: selcenyuksel@inonu.edu.tr, ekilic@inonu.edu.tr, skeles@inonu.edu.tr