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EQUITORSION HOLOMORPHICALLY PROJECTIVE MAPPINGS OF GENERALIZED KÄHLERIAN SPACE OF THE SECOND KIND

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Abstract. Starting from the definition of generalized Riemannian space (GR_N) [5], in which a non-symmetric basic tensor g_{ij} is introduced, in the present paper a generalized Kählerian space $G\underset{2}{K}_N$ of the second kind is defined, as a GR_N with almost complex structure F_i^h, that is covariantly constant with respect to the second kind of covariant derivative (equation (2.3)).

We observe hollomorphically projective mapping of the spaces $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ with invariant complex structure. Also, we consider equitorsion geodesic mapping between these two spaces, and for them we find invariant geometric objects.

1. Introduction

A generalized Riemannian space GR_N in the sense of Eisenhart's definition [5] is a differentiable N-dimensional manifold, equipped with a non-symmetric basic tensor g_{ij}. Connection coefficients of this space are generalized Christoffel's symbols of the second kind. Generally, $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. More about GR_N: [5, 14, 15, 16, 21, 31].

The use of non-symmetric basic tensor and non-symmetric connection became especially topical after the appearance of the papers of A. Einstein [1]-[4] related to the creation of the Unified Field Theory (UFT). We remark that in UFT the symmetric part $g_{\underline{ij}}$ of the basic tensor g_{ij} is related to the gravitation, and the antisymmetric one $g_{\underset{\vee}{ij}}$ is related to the electromagnetism.

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In a generalized Riemannian space one can define four kinds of covariant derivatives [14, 15]. For example, for a tensor a^i_j in GR_N we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p, \qquad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p, \tag{1.1}$$

$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p, \qquad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p. \tag{1.2}$$

In the case of the space GR_N we have five independent curvature tensors [16] :

$$\underset{1}{R}{}^i{}_{jmn} = \Gamma^i_{j[m,n]} + \Gamma^p_{j[m}\Gamma^i_{pn]}, \tag{1.3}$$

$$\underset{2}{R}{}^i{}_{jmn} = \Gamma^i_{[mj,n]} + \Gamma^p_{[mj}\Gamma^i_{n]p}, \tag{1.4}$$

$$\underset{3}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}\Gamma^i_{[pj]}, \tag{1.5}$$

$$\underset{4}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn}\Gamma^i_{[pj]}, \tag{1.6}$$

$$\underset{5}{R}{}^i{}_{jmn} = \frac{1}{2}(\Gamma^i_{j[m,n]} + \Gamma^i_{[mj,n]} + \Gamma^p_{jm}\Gamma^i_{pn} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}), \tag{1.7}$$

where $[i \dots j]$ denotes an antisymmetrization without division with respect to the indices i, j, and also $(i \dots j)$ denotes a symmetrization without division with respect to indices i, j.

Kählerian spaces and their mappings were investigated by many authors, for example K. Yano [33, 34], M. Prvanović [22], T. Otsuki [19], N. S. Sinyukov [28], J. Mikeš [11, 12], N. Pušić [23]-[26] and many others.

In [17, 29] we defined a generalized Kählerian space GK_N as a generalized N-dimensional Riemannian space with a (non-symmetric) metric tensor g_{ij} and an almost complex structure F^i_j such that

$$\begin{gathered} F^h_p(x)F^p_i(x) = -\delta^h_i\,, \\ g_{\underline{pq}}\,F^p_iF^q_j = g_{\underline{ij}}\,, \quad g^{\underline{ij}} = g^{\underline{pq}}F^i_pF^j_q, \\ F^h_{i\underset{\theta}{|}j} = 0, \quad (\theta = 1, 2), \end{gathered} \tag{1.8}$$

where $\underset{\theta}{|}$ denotes the covariant derivative of the kind θ with respect to the metric tensor g_{ij}.

In [30] we defined a generalized Kählerian space of the first kind $\underset{1}{GK}_N$ as a generalized N-dimensional Riemannian space with a (non-symmetric) metric tensor g_{ij} and an almost complex structure F^i_j such that

$$\begin{gathered} F^h_p(x)F^p_i(x) = -\delta^h_i\,, \\ g_{\underline{pq}}\,F^p_iF^q_j = g_{\underline{ij}}\,, \quad g^{\underline{ij}} = g^{\underline{pq}}F^i_pF^j_q, \\ F^h_{i\underset{1}{|}j} = 0, \end{gathered} \tag{1.9}$$

where $\underset{1}{|}$ denotes the covariant derivative of the first kind with respect to the metric tensor g_{ij}.

2. Generalized Kählerian spaces of the second kind

A generalized N-dimensional Riemannian space with (non-symmetric) metric tensor g_{ij}, is a generalized Kählerian space of the second kind $\underset{2}{GK}_N$ if there exists an almost complex structure $F^i_j(x)$, such that

$$F^h_p(x)F^p_i(x) = -\delta^h_i\,, \tag{2.1}$$

$$g_{\underline{pq}}F^p_iF^q_j = g_{\underline{ij}}\,,\quad g^{\underline{ij}} = g^{\underline{pq}}F^i_pF^j_q, \tag{2.2}$$

$$F^h_{i\underset{2}{|}j} = 0, \tag{2.3}$$

where $\underset{2}{|}$ denotes the covariant derivative of the second kind with respect to the metric tensor g_{ij}. From (2.2), using (2.1), we get $F_{ij} = -F_{ji},\ F^{ij} = -F^{ji}$, where we denote $F_{ji} = F^p_j g_{\underline{pi}},\quad F^{ji} = F^j_p g^{\underline{pi}}$.

Theorem 2.1. *For the almost complex structure F^i_j of a generalized Kählerian space of the second kind the next relations*

$$F^h_{i\underset{1}{|}j} = 2(F^p_i\Gamma^h_{\underset{\vee}{pj}} + F^h_p\Gamma^p_{\underset{\vee}{ji}}),\quad F^h_{i\underset{3}{|}j} = 2F^p_i\Gamma^h_{\underset{\vee}{pj}},\quad F^h_{i\underset{4}{|}j} = 2F^h_p\Gamma^p_{\underset{\vee}{ji}} \tag{2.4}$$

are valid, where $\Gamma^h_{\underset{\vee}{ij}}$ is a torsion tensor.

Proof. We get the relations (2.4) by using the condition (2.3) and the covariant derivative (1.1), (1.2). □

In [32] several theorems are proved. These theorems are generalizations of the corresponding theorems relating to K_N. The relations between F^h_i and four curvature tensors from GR_N are obtained. From here we state these theorems

Theorem 2.2. *For the Ricci tensor R_{ij}, given by $g_{\underline{ij}}$, the relation*

$$R_{hk} = F^p_hF^q_kR_{pq} - g^{\underline{pq}}F^s_h\mathcal{D}_{(s.pqk)} \tag{2.5}$$

is valid, where

$$\mathcal{D}^h{}_{ijk} = F^p_{i;[k}\Gamma^h_{\underset{\vee}{j]p}} + F^p_i\Gamma^h_{\underset{\vee}{[jp;k]}} + F^h_{p;[k}\Gamma^p_{\underset{\vee}{ij]}} + F^h_p\Gamma^p_{\underset{\vee}{i[j;k]}}, \tag{2.6}$$

$\mathcal{D}_{h.ijk} = g^{\underline{ph}}\mathcal{D}^p{}_{ijk}$, and (;) is a covariant derivative with respect to symmetric connection $\Gamma^h_{\underline{ij}}$.

Theorem 2.3. *The Ricci tensors $\underset{\theta}{R}{}_{jm}$ $(\theta = 1,\cdots,5)$ of the space $G\underset{2}{K}_N$ satisfy the relations*

$$\begin{aligned}
&\underset{\alpha}{R}_{(pq)}F^p_jF^q_m = \underset{\alpha}{R}_{(jm)} - 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}}F^r_jF^s_m + 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}} + 2g^{\underline{pq}}F^s_h\mathcal{D}_{(s.pqk)},\\
&\qquad\qquad \alpha = 1,2,3,\\
&\underset{4}{R}_{(pq)}F^p_jF^q_m = \underset{4}{R}_{(jm)} + 6\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}}F^r_jF^s_m - 6\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}} + 2g^{\underline{pq}}F^s_h\mathcal{D}_{(s.pqk)},\\
&\underset{5}{R}_{(pq)}F^p_jF^q_m = \underset{5}{R}_{(jm)} + 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}}F^r_jF^s_m - 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}} + 2g^{\underline{pq}}F^s_h\mathcal{D}_{(s.pqk)},
\end{aligned} \tag{2.7}$$

where (jm) denotes the symmetrization without division with respect to the indices j, m.

3. Holomorphically projective mappings of generalized Kählerian space of the second kind which preserve complex structure

By generalizing the notion of analytic planar curve of Kählerian space [19, 28] we come to an analogous notion of generalized Kählerian spaces of the second kind.

Definition 3.1. A $G\underset{2}{K}_N$ space curve, which is, in parametric form, given by equation

$$x^h = x^h(t), \quad (h = 1, 2, \cdots, N) \tag{3.1}$$

will be called planar if:

$$\lambda^h{}_{\underset{\theta}{|}p}\lambda^p = a(t)\lambda^h + b(t)F^h_p\lambda^p, \quad (\theta = 1, 2) \tag{3.2}$$

where $\lambda^h = dx^h/dt$, also $a(t)$ and $b(t)$ are the functions of the parameter t.

Considering that

$$\lambda^h{}_{\underset{1}{|}p}\lambda^p = \frac{d\lambda^h}{dt} + \Gamma^h_{pq}\lambda^p\lambda^q = \lambda^h{}_{\underset{2}{|}p}\lambda^p,$$

we conclude that the expression on the left-hand side in (3.2) is the same with respect to both kinds of covariant derivative, so we can define analytic planar curve in the space $G\underset{2}{K}_N$ by the following relation:

$$\frac{d\lambda^h}{dt} + \Gamma^h_{pq}\lambda^p\lambda^q = a(t)\lambda^h + b(t)F^h_p\lambda^p. \tag{3.3}$$

We can consider two N-dimensional generalized Kählerian spaces of second kind $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ with complex structures F^h_i and $\overline{F}^h_i$, where:

$$F^h_i = \overline{F}^h_i \tag{3.4}$$

in the same local coordinate system, defined by the map $f : G\underset{2}{K}_N \to G\underset{2}{\overline{K}}_N$.

Definition 3.2. A diffeomorphism $f : G\underset{2}{K}_N \to G\underset{2}{\overline{K}}_N$ will be called *holomorphically projective* or analytic planar if it maps analytic planar curves of the space $G\underset{2}{K}_N$ into analytic planar curves of the space $G\underset{2}{\overline{K}}_N$.

We can denote

$$P^h_{ij} = \overline{\Gamma}^h_{ij} - \Gamma^h_{ij} \tag{3.5}$$

the deformation tensor of connection under an analytic planar mapping. Here Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ are the second kind Christoffel's symbols of the spaces $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$, respectively. Analytic planar curves of the space $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ are given by the following relations, respectively:

$$\frac{d\lambda^h}{dt} + \Gamma^h_{pq}\lambda^p\lambda^q = a(t)\lambda^h + b(t)F^h_p\lambda^p, \qquad \frac{d\lambda^h}{dt} + \overline{\Gamma}^h_{pq}\lambda^p\lambda^q = \overline{a}(t)\lambda^h + \overline{b}(t)F^h_p\lambda^p,$$

From the previous relations we have

$$(\overline{\Gamma}^h_{pq} - \Gamma^h_{pq})\lambda^p\lambda^q = \psi(t)\lambda^h + \sigma(t)F^h_p\lambda^p,$$

where we denote $\psi(t) = \overline{a}(t) - a(t),\ \sigma(t) = \overline{b}(t) - b(t)$. We can now put: $\psi(t) = \psi_p\lambda^p,\ \sigma(t) = \sigma_q\lambda^q$. So we have

$$(\overline{\Gamma}^h_{pq} - \Gamma^h_{pq} - \psi_p\delta^h_q - \sigma_p F^h_q)\lambda^p\lambda^q = 0,$$

wherefrom we can conclude that:

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} + \sigma_{(i}F^h_{j)} + \xi^h_{ij}, \tag{3.6}$$

where ξ^h_{ij} is an arbitrary anti-symmetric tensor. In (3.6) we can select the vector σ_i so that $\sigma_i = -\psi_p F^p_i$. Because of that we have:

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} - \psi_p F^p_{(i}F^h_{j)} + \xi^h_{ij}. \tag{3.7}$$

Contracting over the indices h, i in (3.7) and using $F^p_p = 0,\ \xi^p_{pj} = 0$, we get:

$$\overline{\Gamma}^p_{pj} - \Gamma^p_{pj} = (N+2)\psi_j. \tag{3.8}$$

Thus from (3.8) we can see that ψ_j is obviously a gradient vector. If we substitute from (3.8) into (3.7) we have

$$\begin{aligned}\overline{\Gamma}^h_{ij} - \frac{1}{N+2}(\overline{\Gamma}^p_{p(i}\delta^h_{j)} - \overline{\Gamma}^q_{qp}\overline{F}^p_{(i}\overline{F}^h_{j)}) - \overline{\Gamma}^h_{\underset{\vee}{ij}}\\ = \Gamma^h_{ij} - \frac{1}{N+2}(\Gamma^p_{p(i}\delta^h_{j)} - \Gamma^q_{qp}F^p_{(i}F^h_{j)}) - \Gamma^h_{\underset{\vee}{ij}}.\end{aligned} \tag{3.9}$$

Denoting

$$HT^h_{ij} = \Gamma^h_{\underline{ij}} - \frac{1}{N+2}(\Gamma^p_{p(i}\delta^h_{j)} - \Gamma^q_{qp}F^p_{(i}F^h_{j)}). \tag{3.10}$$

we can present (3.9) in the form:

$$H\overline{T}^h_{ij} = HT^h_{ij}, \tag{3.11}$$

where by $H\overline{T}^h_{ij}$ we denoted the object of the form (3.10) for $G\underset{2}{\overline{K}}_N$. The magnitude HT^h_{ij} is not a tensor. We will call it **holomorphically projective parameter of the type of Tomas's projective parameter**. This way, based on the fact above we have proved:

Theorem 3.1. *The quantities* (3.10) *represent invariants of holomorphically projective mapping of generalized Kählerian space of the second kind with the equals complex structures.* □

4. Holomorphically projective parameters of generalized Kählerian space of the second kind

If $f : G\underset{2}{K}_N \to G\underset{2}{\overline{K}}_N$ is holomorphically projective mappings, and if the torsion tensors of the spaces $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ satisfy

$$\overline{\Gamma}^h_{\underset{\vee}{ij}} = \Gamma^h_{\underset{\vee}{ij}}, \tag{4.1}$$

then we can tell that:

$$\xi^h_{ij} = 0. \tag{4.2}$$

4.1. **Holomorphically projective parameters of the first kind.** The relation between the curvature tensors $\underset{1}{R}$ and $\underset{1}{\overline{R}}$ of the $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ spaces is given by:

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + P^i_{j[m\underset{1}{|}n]} + P^p_{j[m}P^i_{pn]} + 2\underset{\vee}{\Gamma}{}^p_{mn}P^i_{jp}. \tag{4.3}$$

Substituting (3.5), (3.7) and (4.2) in (4.3) we get

$$\begin{aligned}\underset{1}{\overline{R}}{}^i_{jmn} =& \underset{1}{R}{}^i{}_{jmn} + \delta^i_{[m}\underset{1}{\psi}_{jn]} + \delta^i_j\underset{1}{\psi}_{[mn]} + F^{(p}_j F^{i)}_{[n}\underset{1}{\psi}_{pm]} + 2\underset{\vee}{\Gamma}{}^i_{mn}\psi_j \\ &+ 2\underset{\vee}{\Gamma}{}^p_{mn}\psi_p\delta^i_j + 2\underset{\vee}{\Gamma}{}^{(p}_{[nq}\psi_p F^q_{(m]}F^{i)}_{j)} - 2\underset{\vee}{\Gamma}{}^p_{nm}\psi_q F^q_{(p}F^i_{j)} - 2\underset{\vee}{\Gamma}{}^p_{[nj}\psi_q F^{(i}_p F^{q)}_{m]}\end{aligned} \tag{4.4}$$

where we denoted

$$\underset{1}{\psi}_{ij} = \psi_{i\underset{1}{|}j} - \psi_i\psi_j + \psi_p F^p_i \psi_q F^q_j. \tag{4.5}$$

Contracting with respect to indices i, n in (4.4) we get

$$\underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} + \underset{1}{\psi}_{[mj]} - N\underset{1}{\psi}_{jm} - F^p_j F^q_m \underset{1}{\psi}_{(pq)} - 2\underset{\vee}{\Gamma}{}^p_{jm}\psi_p - 2\underset{\vee}{\Gamma}{}^p_{rj}\psi_q F^r_{(p}F^q_{m)}. \tag{4.6}$$

Anti-symmetrizing without division in (4.6) with respect to indices j, m gives:

$$(N+2)\underset{1}{\psi}_{[jm]} = \underset{1}{R}_{[jm]} - \underset{1}{\overline{R}}_{[jm]} + 4\underset{\vee}{\Gamma}{}^p_{mj}\psi_p + 2\underset{\vee}{\Gamma}{}^p_{[jr}\psi_q F^r_{(p}F^q_{m)]}. \tag{4.7}$$

By symmetrization without division in (4.6) with respect to indices j, m we obtain:

$$\underset{1}{\overline{R}}_{(jm)} = \underset{1}{R}_{(jm)} - N\underset{1}{\psi}_{(jm)} - 2F^p_j F^q_m \underset{1}{\psi}_{(pq)} + 2\underset{\vee}{\Gamma}{}^p_{(jr}\psi_q F^r_{(p}F^q_{m))}. \tag{4.8}$$

The relation analogous to relation (2.7) for $\underset{1}{R}$ in the $G\underset{2}{\overline{K}}_N$ space is valid.

By composition with $F^j_p F^m_q$, contraction with respect to j, m, and by use of the conditions (2.7) for $\underset{1}{R}$ and $\underset{1}{\overline{R}}$ in $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ from (4.8) we get

$$\underset{1}{\overline{R}}_{(jm)} = \underset{1}{R}_{(jm)} - N\underset{1}{\psi}_{(pq)}F^p_j F^q_m - 2\underset{1}{\psi}_{(jm)} + 2\underset{\vee}{\Gamma}{}^p_{rq}\psi_{(m}F^r_p F^q_{j)} + 2\underset{\vee}{\Gamma}{}^p_{(mr}\psi_q F^q_p F^r_{j)}. \tag{4.9}$$

From (4.8) and (4.9) we get:

$$(N-2)F^p_j F^q_m \underset{1}{\psi}_{(pq)} = (N-2)\underset{1}{\psi}_{(jm)} - 2\underset{\vee}{\Gamma}{}^p_{(jr}\psi_q F^r_p F^q_{m)} + 2\underset{\vee}{\Gamma}{}^p_{rq}\psi_{(m}F^r_p F^q_{j)}. \tag{4.10}$$

Replacing (4.10) in (4.9) we get:

$$\begin{aligned}(N+2)\underset{1}{\psi}_{(jm)} =& \underset{1}{R}_{(jm)} - \underset{1}{\overline{R}}_{(jm)} + \frac{2}{N-2}(N\underset{\vee}{\Gamma}{}^p_{(jr}\psi_q F^r_p F^q_{m)} \\ &- 2\underset{\vee}{\Gamma}{}^p_{rq}\psi_{(m}F^r_p F^q_{j)}) + 2\underset{\vee}{\Gamma}{}^p_{(mr}\psi_q F^q_p F^r_{j)}.\end{aligned} \tag{4.11}$$

Using (4.7) and (4.11) we have:

$$\begin{aligned}(N+2)\underset{1}{\psi}_{jm} =& \underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} + 2\underset{\vee}{\Gamma}{}^p_{mj}\psi_p \\ &+ 2\underset{\vee}{\Gamma}{}^p_{jr}\psi_q F^r_m F^q_p + \frac{2N-2}{N-2}\underset{\vee}{\Gamma}{}^p_{jr}\psi_q F^r_p F^q_m \\ &+ \frac{2}{N-2}\underset{\vee}{\Gamma}{}^p_{mr}\psi_q F^r_p F^q_j - \frac{2}{N-2}\underset{\vee}{\Gamma}{}^p_{rq}\psi_{(m}F^r_p F^q_{j)}.\end{aligned} \tag{4.12}$$

Eliminating ψ_i and using the (3.8) condition the last equation becomes:

$$(N+2)\underset{1}{\psi}_{jm} = \underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} + \underset{1}{\overline{P}}_{jm} - \underset{1}{P}_{jm}, \tag{4.13}$$

where we denoted

$$\begin{aligned}\underset{1}{P}_{jm} =&\frac{2}{N+2}(\Gamma^p_{\underset{\vee}{mj}}\Gamma^q_{qp} + \Gamma^p_{\underset{\vee}{jr}}\Gamma^s_{sq}F^q_pF^r_m + \frac{N-1}{N-2}\Gamma^p_{\underset{\vee}{jr}}\Gamma^s_{sq}F^q_mF^r_p \\ &+ \frac{1}{N-2}\Gamma^p_{\underset{\vee}{mr}}\Gamma^s_{sq}F^q_jF^r_p - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{rq}}\Gamma^s_{s(m}F^r_pF^q_{j)}).\end{aligned} \tag{4.14}$$

In the same way the object $\underset{1}{\overline{P}}_{jm}$ of the $G\underset{2}{\overline{K}}_N$ space is defined. Eliminating $\underset{1}{\psi}_{jm}$ from (4.4) we get

$$HP\underset{1}{\overline{W}}^i{}_{jmn} = HP\underset{1}{W}^i{}_{jmn}, \tag{4.15}$$

where we denoted

$$\begin{aligned}HP\underset{1}{W}^i{}_{jmn} =&\underset{1}{R}^i{}_{jmn} + \frac{1}{N+2}[\delta^i_{[m}(\underset{1}{R}-\underset{1}{P})_{jn]} + \delta^i_j(\underset{1}{R}_{[mn]} - \underset{1}{P}_{[mn]}) \\ &+ F^{(p}_jF^{i)}_{[n}(\underset{1}{R}-\underset{1}{P})_{pm]} - 2\Gamma^i_{\underset{\vee}{mn}}\Gamma^q_{qj} - 2\delta^i_j\Gamma^p_{\underset{\vee}{mn}}\Gamma^q_{qp} \\ &+ 4\Gamma^p_{\underset{\vee}{[nq}}\Gamma^s_{sp}F^q_{(m]}F^i_{j)} - 2\Gamma^p_{\underset{\vee}{nm}}\Gamma^s_{sq}F^{(q}_pF^{i)}_j - 2\Gamma^p_{\underset{\vee}{[nj}}\Gamma^s_{sq}F^{(i}_pF^{q)}_{m]}\end{aligned} \tag{4.16}$$

is an object of the space $G\underset{2}{K}_N$. We denoted in last equation $(\underset{1}{R}-\underset{1}{P})_{jm} = (\underset{1}{R}_{jm} - \underset{1}{P}_{jm})$. We see that the quantity $HP\underset{1}{\overline{W}}^i{}_{jmn}$ is expressed in the same way as the quantity $HP\underset{1}{W}^i{}_{jmn}$. Obviously, the quantity $HP\underset{1}{W}^i{}_{jmn}$ is not a tensor, so we shall call it an **equitorsion holomorphically projective parameter of the first kind** of the space $G\underset{2}{K}_N$. Because of all those facts the following theorem is proved:

Theorem 4.1. *The equitorsion holomorphically projective parameter of the first kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{2}{K}_N$ *and* $G\underset{2}{\overline{K}}_N$. □

4.2. **Holomorphically projective parameter of the second kind.** The connection between the curvature tensors $\underset{2}{R}$ and $\underset{2}{\overline{R}}$ of the $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ spaces is given by:

$$\underset{2}{\overline{R}}^i_{jmn} = \underset{2}{R}^i{}_{jmn} + P^i_{[mj\underset{2}{|}n]} + P^p_{[mj}P^i_{n]p} + 2\Gamma^p_{\underset{\vee}{nm}}P^i_{pj}. \tag{4.17}$$

Replacing (3.5), (3.7) and (4.2) in (4.17) we have:

$$\begin{aligned}\underset{2}{\overline{R}}^i_{jmn} =&\underset{2}{R}^i{}_{jmn} + \delta^i_{[m}\underset{2}{\psi}_{jn]} + \delta^i_j\underset{2}{\psi}_{[mn]} + F^{(p}_jF^{i)}_{[n}\underset{2}{\psi}_{pm]} \\ &+ 2\Gamma^p_{\underset{\vee}{nm}}\psi_p\delta^i_j + 2\Gamma^i_{\underset{\vee}{nm}}\psi_j - 2\Gamma^p_{\underset{\vee}{nm}}\psi_qF^q_{(p}F^i_{j)},\end{aligned} \tag{4.18}$$

where we denoted

$$\underset{2}{\psi}_{ij} = \psi_{i\underset{2}{|}j} - \psi_i\psi_j + \psi_pF^p_i\psi_qF^q_j. \tag{4.19}$$

Contracting with respect to indices i, n in (4.18) we get

$$\underset{2}{\overline{R}}_{jm} =\underset{2}{R}_{jm} + \underset{2}{\psi}_{[mj]} - N\underset{2}{\psi}_{jm} - F^p_jF^q_m\underset{2}{\psi}_{(pq)} + 2\Gamma^p_{\underset{\vee}{jm}}\psi_p - 2\Gamma^p_{\underset{\vee}{rm}}\psi_qF^q_{(p}F^r_{j)} \tag{4.20}$$

Anti-symmetrizing without division in (4.20) with respect to indices j, m gives:

$$(N+2)\underset{2}{\psi}_{[jm]} = \underset{2}{R}_{[jm]} - \underset{2}{\overline{R}}_{[jm]} + 4\Gamma^p_{\underset{\vee}{jm}}\psi_p + 2\Gamma^p_{\underset{\vee}{[mr}}\psi_q F^q_{(p}F^r_{j)]}. \tag{4.21}$$

By symmetrization without division in (4.20) with respect to indices j, m gives:

$$\underset{2}{\overline{R}}_{(jm)} = \underset{2}{R}_{(jm)} - N\underset{2}{\psi}_{(jm)} - 2F^p_jF^q_m\underset{2}{\psi}_{(pq)} - 2\Gamma^p_{\underset{\vee}{rm}}\psi_qF^q_{(p}F^r_{j)} - 2\Gamma^p_{\underset{\vee}{rj}}\psi_qF^q_{(p}F^r_{m)}. \tag{4.22}$$

The relation analogous to relation (2.7) for $\underset{2}{R}$ in the $G\underset{2}{\overline{K}}_N$ space is valid.

By composition with $F^j_pF^m_q$, contraction with respect to j, m, and by use of the conditions (2.7) for $\underset{2}{R}$ and $\underset{2}{\overline{R}}$ in the $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ respectively, from (4.22) we get

$$\begin{aligned}\underset{2}{\overline{R}}_{(jm)} =& \underset{2}{R}_{(jm)} - N\underset{2}{\psi}_{(pq)}F^p_jF^q_m - 2\underset{2}{\psi}_{(jm)}\\ &+ 2\Gamma^p_{\underset{\vee}{(jr}}\psi_qF^q_pF^r_{m)} + 2\Gamma^p_{\underset{\vee}{rq}}\psi_{(j}F^r_pF^q_{m)}\end{aligned} \tag{4.23}$$

From (4.22) and (4.23) we get:

$$(N-2)F^p_jF^q_m\underset{2}{\psi}_{(pq)} = (N-2)\underset{2}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{r(m}}\psi_qF^q_{j)}F^r_p + 2\Gamma^p_{\underset{\vee}{rq}}\psi_{(j}F^r_pF^q_{m)}. \tag{4.24}$$

Replacing (4.24) in (4.23) we get

$$\begin{aligned}(N+2)\underset{2}{\psi}_{(jm)} =& \underset{2}{R}_{(jm)} - \underset{2}{\overline{R}}_{(jm)} - \frac{2}{N-2}(N\Gamma^p_{\underset{\vee}{r(m}}\psi_qF^q_{j)}F^r_p\\ &+ 2\Gamma^p_{\underset{\vee}{rq}}\psi_{(j}F^r_pF^q_{m)}) - 2\Gamma^p_{\underset{\vee}{r(m}}\psi_qF^q_pF^r_{j)}.\end{aligned} \tag{4.25}$$

Using (4.21) and (4.25) we have:

$$\begin{aligned}(N+2)\underset{2}{\psi}_{jm} =& \underset{2}{R}_{jm} - \underset{2}{\overline{R}}_{jm} + 2\Gamma^p_{\underset{\vee}{jm}}\psi_p\\ &- 2\Gamma^p_{\underset{\vee}{rm}}\psi_qF^q_pF^r_j - \frac{2N-2}{N-2}\Gamma^p_{\underset{\vee}{rm}}\psi_qF^q_jF^r_p\\ &- \frac{2}{N-2}\Gamma^p_{\underset{\vee}{rj}}\psi_qF^q_mF^r_p - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{rq}}\psi_{(j}F^r_pF^q_{m)}.\end{aligned} \tag{4.26}$$

Eliminating ψ_i and using the condition (3.8) the last equation becomes:

$$(N+2)\underset{2}{\psi}_{jm} = \underset{2}{R}_{jm} - \underset{2}{\overline{R}}_{jm} + \underset{2}{\overline{P}}_{jm} - \underset{2}{P}_{jm}, \tag{4.27}$$

where we denoted

$$\begin{aligned}\underset{2}{P}_{jm} =& \frac{2}{N+2}(\Gamma^p_{\underset{\vee}{jm}}\Gamma^q_{qp} - \Gamma^p_{\underset{\vee}{rm}}\Gamma^s_{sq}F^q_pF^r_j - \frac{N-1}{N-2}\Gamma^p_{\underset{\vee}{rm}}\Gamma^s_{sq}F^q_jF^r_p\\ &- \frac{1}{N-2}\Gamma^p_{\underset{\vee}{rj}}\Gamma^s_{sq}F^q_mF^r_p - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{rq}}\Gamma^s_{s(j}F^r_pF^q_{m)}.\end{aligned} \tag{4.28}$$

In the same way the object $\underset{2}{\overline{P}}_{jm}$ of the space $G\underset{2}{\overline{K}}_N$ is defined. Eliminating $\underset{2}{\psi}_{jm}$ from (4.18) we get

$$HP\underset{2}{\overline{W}}{}^i{}_{jmn} = HP\underset{2}{W}{}^i{}_{jmn}, \tag{4.29}$$

where we denoted

$$
\begin{aligned}
\underset{2}{HPW}{}^{i}{}_{jmn} =& \underset{2}{R}{}^{i}{}_{jmn} + \frac{1}{N+2}[\delta^i_{[m}\,(\underset{2}{R} - \underset{2}{P})_{jn]} + \delta^i_j\,(\underset{2}{R}_{[mn]} - \underset{2}{P}_{[mn]}) \\
&+ F_j^{(p}F^{i)}_{[n}(\underset{2}{R} - \underset{2}{P})_{pm]} - 2\delta^i_j\,\Gamma^p_{\underset{\vee}{nm}}\Gamma^q_{qp} \\
&- 2\Gamma^i_{\underset{\vee}{nm}}\Gamma^q_{qj} + 2\Gamma^p_{\underset{\vee}{nm}}\Gamma^s_{sq}F^q_{(p}F^i_{j)}].
\end{aligned} \tag{4.30}
$$

It is easy to prove that magnitude $\underset{2}{HPW}{}^{i}{}_{jmn}$ is not a tensor, so we shall call it an **equitorsion holomorphically projective parameter of the second kind** of the space $G\underset{2}{K}_N$. The next theorem is valid:

Theorem 4.2. *The equitorsion holomorphically projective parameter of the second kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{2}{K}_N$ *and* $G\underset{2}{\overline{K}}_N$*.* □

4.3. **Holomorphically projective parameters of the third kind.** The connection between the curvature tensors $\underset{3}{R}$ and $\underset{3}{\overline{R}}$ of the $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ spaces is given by:

$$
\begin{aligned}
\underset{3}{\overline{R}}{}^{i}_{jmn} =& \underset{3}{R}{}^{i}{}_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{np} - P^p_{nj}P^i_{pm} \\
&+ 2P^p_{nm}\Gamma^i_{\underset{\vee}{pj}} + 2\Gamma^p_{\underset{\vee}{nm}}P^i_{\underset{\vee}{pj}}.
\end{aligned} \tag{4.31}
$$

Replacing (3.5), (3.7) and (4.2) in (4.31) we have:

$$
\begin{aligned}
\underset{3}{\overline{R}}{}^{i}_{jmn} =& \underset{3}{R}{}^{i}{}_{jmn} + \delta^i_{[m}\,\underset{2}{\psi}_{jn]} + \delta^i_j\,(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + F^p_j(F^i_n\underset{1}{\psi}_{pm} - F^i_m\underset{2}{\psi}_{pn}) \\
&+ F^i_j(F^p_n\underset{1}{\psi}_{pm} - F^p_m\underset{2}{\psi}_{pn}) + 2\Gamma^i_{\underset{\vee}{mj}}\psi_n + 2\Gamma^i_{\underset{\vee}{nj}}\psi_m \\
&- 2\Gamma^i_{\underset{\vee}{pj}}\psi_q F^q_{(n}F^p_{m)} + 2\Gamma^p_{\underset{\vee}{qm}}\psi_p F^q_{(n}F^i_{j)} + 2\Gamma^p_{\underset{\vee}{mn}}\psi_q F^q_{(p}F^i_{j)} \\
&+ 2\Gamma^p_{\underset{\vee}{mj}}\psi_q F^i_{(p}F^q_{n)} + 2\Gamma^p_{\underset{\vee}{qm}}\psi_p F^q_j F^i_n + 2\Gamma^i_{\underset{\vee}{qn}}\psi_p F^q_n F^p_j,
\end{aligned} \tag{4.32}
$$

where we use the notation

$$
\underset{\theta}{\psi}_{ij} = \psi_{i\underset{\theta}{|}j} - \psi_i\psi_j + \psi_p F^p_i\psi_q F^q_j, \quad (\theta = 1, 2). \tag{4.33}
$$

It is easy to prove

$$
\underset{2}{\psi}_{[mn]} = \underset{1}{\psi}_{[mn]} + 2\Gamma^p_{\underset{\vee}{mn}}\psi_p.
$$

The equation (4.32) becomes

$$
\begin{aligned}
\underset{3}{\overline{R}}{}^{i}_{jmn} =& \underset{3}{R}{}^{i}{}_{jmn} + \delta^i_{[m}\,\underset{1}{\psi}_{jn]} + \delta^i_j\,\underset{1}{\psi}_{[mn]} + F^{(p}_j F^{i)}_{[n}\underset{1}{\psi}_{pm]} \\
&+ 2\Gamma^p_{\underset{\vee}{(jn}}\psi_p\delta^i_{m)} - 2\Gamma^p_{\underset{\vee}{qn}}\psi_p F^q_{(j}F^i_{m)} + 2\Gamma^i_{\underset{\vee}{(mj}}\psi_{n)} \\
&- 2\Gamma^i_{\underset{\vee}{pj}}\psi_q F^q_{(n}F^p_{m)} + 2\Gamma^p_{\underset{\vee}{qm}}\psi_p F^q_{(n}F^i_{j)} \\
&+ 2\Gamma^p_{\underset{\vee}{mn}}\psi_q F^q_{(p}F^i_{j)} + 2\Gamma^i_{\underset{\vee}{qm}}\psi_p F^q_{(j}F^p_{n)} + 2\Gamma^p_{\underset{\vee}{mj}}\psi_q F^i_{(p}F^q_{n)}.
\end{aligned} \tag{4.34}
$$

Contracting with respect to indices i, n in (4.34) we get

$$
\underset{3}{\overline{R}}_{jm} = \underset{3}{R}_{jm} + \underset{1}{\psi}_{[mj]} - N\underset{1}{\psi}_{jm} - F^p_j F^q_m\underset{1}{\psi}_{(pq)} + 2\Gamma^p_{\underset{\vee}{mj}}\psi_p - 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^r_{(p}F^q_{m)}. \tag{4.35}
$$

Anti-symmetrizing without division in (4.35) with respect to indices j, m we obtain:

$$(4.36)\quad (N+2)\underset{1}{\psi}_{[jm]} = \underset{3}{R}_{[jm]} - \underset{3}{\overline{R}}_{[jm]} + 4\Gamma^p_{\underset{\vee}{mj}}\psi_p + 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^r_{(p}F^q_{j)} - 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^r_{(p}F^q_{m)}.$$

By symmetrization without division in (4.35) with respect to indices j, m, we get:

$$(4.37)\quad \underset{3}{\overline{R}}_{(jm)} = \underset{3}{R}_{(jm)} - N\underset{1}{\psi}_{(jm)} - 2F^p_jF^q_m\underset{1}{\psi}_{(pq)} - 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^r_{(p}F^q_{j)} - 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^r_{(p}F^q_{m)}.$$

By composition with $F^j_pF^m_q$, contraction with respect to j, m, and by use of the conditions (2.7) for $\underset{3}{R}$ and $\underset{3}{\overline{R}}$ in $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ respectively, from (4.37) we get

$$(4.38)\quad \underset{3}{\overline{R}}_{(jm)} = \underset{3}{R}_{(jm)} - N\underset{1}{\psi}_{(pq)}F^p_jF^q_m - 2\underset{1}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{rq}}\psi_{(j}F^r_pF^q_{m)} + 2\Gamma^p_{\underset{\vee}{(jr}}\psi_qF^q_pF^r_{m)}.$$

From (4.37) and (4.38) we obtain

$$(4.39)\quad \begin{aligned}(N-2)F^p_jF^q_m\underset{1}{\psi}_{(pq)} = {} & (N-2)\underset{1}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{rm}}\psi_qF^q_jF^r_p \\ & + 2\Gamma^p_{\underset{\vee}{rj}}\psi_qF^q_mF^r_p + 2\Gamma^p_{\underset{\vee}{rq}}\psi_{(j}F^r_pF^q_{m)}.\end{aligned}$$

Replacing (4.39) in (4.38) we get

$$(4.40)\quad \begin{aligned}(N+2)\underset{1}{\psi}_{(jm)} = {} & \underset{3}{R}_{(jm)} - \underset{3}{\overline{R}}_{(jm)} - \frac{2}{N-2}(N\Gamma^p_{\underset{\vee}{r(m}}\psi_qF^q_{j)}F^r_p \\ & + 2\Gamma^p_{\underset{\vee}{rq}}\psi_{(j}F^r_pF^q_{m)}) - 2\Gamma^p_{\underset{\vee}{r(m}}\psi_qF^q_pF^r_{j)}.\end{aligned}$$

Using (4.36) and (4.40) one obtains:

$$(4.41)\quad \begin{aligned}(N+2)\underset{1}{\psi}_{jm} = {} & \underset{3}{R}_{jm} - \underset{3}{\overline{R}}_{jm} + 2\Gamma^p_{\underset{\vee}{mj}}\psi_p \\ & - \frac{2N-2}{N-2}\Gamma^p_{\underset{\vee}{rm}}\psi_qF^q_jF^r_p - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{rj}}\psi_qF^q_mF^r_p \\ & - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{rq}}\psi_{(j}F^r_pF^q_{m)} - 2\Gamma^p_{\underset{\vee}{rm}}\psi_qF^q_pF^r_j.\end{aligned}$$

Eliminating ψ_i and using the (3.8) condition the last equation becomes:

$$(4.42)\quad (N+2)\underset{1}{\psi}_{jm} = \underset{3}{R}_{jm} - \underset{3}{\overline{R}}_{jm} + \underset{3}{\overline{P}}_{jm} - \underset{3}{P}_{jm},$$

where we denoted

$$(4.43)\quad \begin{aligned}\underset{3}{P}_{jm} = {} & \frac{2}{N+2}(\Gamma^p_{\underset{\vee}{mj}}\Gamma^q_{qp} - \frac{N-1}{N-2}\Gamma^p_{\underset{\vee}{rm}}\Gamma^s_{sq}F^q_jF^r_p - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{rj}}\Gamma^s_{sq}F^q_mF^r_p \\ & - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{rq}}\Gamma^s_{s(j}F^r_pF^q_{m)} - \Gamma^p_{\underset{\vee}{rm}}\Gamma^s_{sq}F^q_pF^r_j).\end{aligned}$$

In the same way is given object $\underset{3}{\overline{P}}_{jm}$ in the space $G\underset{2}{\overline{K}}_N$. Eliminating $\underset{1}{\psi}_{jm}$ from (4.34) we get

$$(4.44)\quad HP\underset{3}{\overline{W}}{}^i{}_{jmn} = HP\underset{3}{W}{}^i{}_{jmn},$$

where

$$\begin{aligned}HP\underset{3}{W}{}^{i}{}_{jmn} =& \underset{3}{R}{}^{i}{}_{jmn} + \frac{1}{N+2}[\delta^{i}_{[m}\,(\underset{3}{R} - \underset{3}{P})_{jn]} + \delta^{i}_{j}\,(\underset{3}{R}_{[mn]} - \underset{3}{P}_{[mn]})\\ &+ F^{(p}_{j}F^{i)}_{[n}(\underset{3}{R} - \underset{3}{P})_{pm]} + 2\Gamma^{p}_{\underset{\vee}{(jn}}\Gamma^{s}_{sp}\delta^{i}_{m)} - 2\Gamma^{p}_{\underset{\vee}{q[n}}\Gamma^{s}_{sp}F^{q}_{(j}F^{i}_{m)]}\\ &+ 2\Gamma^{i}_{\underset{\vee}{(mj}}\Gamma^{s}_{sn)} - 2\Gamma^{i}_{\underset{\vee}{p[j}}\Gamma^{s}_{sq}F^{q}_{(n}F^{p}_{m)]} + 2\Gamma^{p}_{\underset{\vee}{m(n}}\Gamma^{s}_{sq}F^{q}_{(p}F^{i}_{j))}\end{aligned} \tag{4.45}$$

Of course, $HP\underset{3}{\overline{W}}{}^{i}{}_{jmn}$ is expressed by geometric objects of the space $G\underset{2}{\overline{K}}_{N}$. It is not a tensor, so we shall call it an **equitorsion holomorphically projective parameter of the third kind** of the space $G\underset{2}{K}_{N}$. Finally, the next theorem is proved:

Theorem 4.3. *The equitorsion holomorphically projective parameter of the third kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{2}{K}_{N}$ *and* $G\underset{2}{\overline{K}}_{N}$. □

4.4. **Holomorphically projective parameters of the fourth kind.** The connection between the curvature tensors $\underset{4}{R}$ and $\underset{4}{\overline{R}}$ of the $G\underset{2}{K}_{N}$ and $G\underset{2}{\overline{K}}_{N}$ spaces is given by:

$$\begin{aligned}\underset{4}{\overline{R}}{}^{i}_{jmn} =& \underset{4}{R}{}^{i}{}_{jmn} + P^{i}_{jm\underset{2}{|}n} - P^{i}_{nj\underset{1}{|}m} + P^{p}_{jm}P^{i}_{np} - P^{p}_{nj}P^{i}_{pm}\\ &+ 2P^{p}_{mn}\Gamma^{i}_{\underset{\vee}{pj}} + 2\Gamma^{p}_{\underset{\vee}{mn}}P^{i}_{\underset{\vee}{pj}},\end{aligned} \tag{4.46}$$

With the help of (4.1) and (4.2) we see that the tensor deformation (3.5) is symmetric, i.e. $P^{i}_{jk} = P^{i}_{kj}$. Now we can write

$$\underset{4}{\overline{R}}{}^{i}_{jmn} - \underset{4}{R}{}^{i}{}_{jmn} = \underset{3}{\overline{R}}{}^{i}_{jmn} - \underset{3}{R}{}^{i}{}_{jmn}. \tag{4.47}$$

From (4.47) we get

$$HP\underset{4}{\overline{W}}{}^{i}{}_{jmn} = HP\underset{4}{W}{}^{i}{}_{jmn} \tag{4.48}$$

where

$$\begin{aligned}HP\underset{4}{W}{}^{i}{}_{jmn} =& \underset{4}{R}{}^{i}{}_{jmn} + \frac{1}{N+2}[\delta^{i}_{[m}\,(\underset{4}{R} - \underset{4}{P})_{jn]} + \delta^{i}_{j}\,(\underset{4}{R}_{[mn]} - \underset{4}{P}_{[mn]})\\ &+ F^{(p}_{j}F^{i)}_{[n}(\underset{4}{R} - \underset{4}{P})_{pm]} + 2\Gamma^{p}_{\underset{\vee}{(jn}}\Gamma^{s}_{sp}\delta^{i}_{m)} - 2\Gamma^{p}_{\underset{\vee}{q[n}}\Gamma^{s}_{sp}F^{q}_{(j}F^{i}_{m)]}\\ &+ 2\Gamma^{i}_{\underset{\vee}{(mj}}\Gamma^{s}_{sn)} - 2\Gamma^{i}_{\underset{\vee}{p[j}}\Gamma^{s}_{sq}F^{q}_{(n}F^{p}_{m)]} + 2\Gamma^{p}_{\underset{\vee}{m(n}}\Gamma^{s}_{sq}F^{q}_{(p}F^{i}_{j))}\end{aligned} \tag{4.49}$$

This quantity is not a tensor, and we shall call it an **equitorsion holomorphically projective parameter of the fourth kind** of the space $G\underset{2}{K}_{N}$. Finally, the next theorem is proved:

Theorem 4.4. *The equitorsion holomorphically projective parameter of the fourth kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{2}{K}_{N}$ *and* $G\underset{2}{\overline{K}}_{N}$. □

4.5. **Holomorphically projective parameters of the fifth kind.** The connection between the curvature tensors $\underset{5}{R}$ and $\underset{5}{\overline{R}}$ of the $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ spaces is given by:

$$\begin{aligned}\underset{5}{\overline{R}}{}^i_{jmn} =& \underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}(P^i_{jm\underset{1}{|}n} - P^i_{jn\underset{2}{|}m} + P^i_{mj\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{pn} \\ &- P^p_{jn}P^i_{mp} + P^p_{mj}P^i_{np} - P^p_{nj}P^i_{pm} + 4\Gamma^p_{\underset{\vee}{jn}}P^i_{\underset{\vee}{pm}} + 4\Gamma^p_{\underset{\vee}{jm}}P^i_{\underset{\vee}{pn}}).\end{aligned} \tag{4.50}$$

Denote

$$\underset{12}{\psi}_{jm} = \frac{1}{2}(\psi_{j\underset{1}{|}m} + \psi_{j\underset{2}{|}m}) - \psi_j\psi_m + \psi_p F^p_j \psi_q F^q_m. \tag{4.51}$$

For the holomorphically projective parameters of the fifth kind we can do the same procedure that we used in the previous three cases, for the holomorphically projective parameters of the first, second and third kind. It is easy to prove that

$$(N+2)\underset{12}{\psi}_{jm} = \underset{5}{R}_{jm} - \underset{5}{\overline{R}}_{jm} + \underset{5}{\overline{P}}_{jm} - \underset{5}{P}_{jm}, \tag{4.52}$$

where

$$\begin{aligned}\underset{5}{P}_{jm} =& \frac{2}{N+2}(\Gamma^p_{\underset{\vee}{(mr}}\Gamma^s_{sq}F^q_pF^r_{j)} - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{rq}}\Gamma^s_{sm}F^r_pF^q_j \\ &- \frac{2}{N-2}\Gamma^p_{\underset{\vee}{qr}}\Gamma^s_{sj}F^q_pF^r_m - \frac{N}{N-2}\Gamma^p_{\underset{\vee}{(jr}}\Gamma^s_{sq}F^r_pF^q_{m)}.\end{aligned} \tag{4.53}$$

In the end for the fifth kind we get

$$HP\underset{5}{\overline{W}}{}^i{}_{jmn} = HP\underset{5}{W}{}^i{}_{jmn}, \tag{4.54}$$

where

$$\begin{aligned}HP\underset{5}{W}{}^i{}_{jmn} =& \underset{5}{R}{}^i_{jmn} + \frac{1}{N+2}[\delta^i_{[m}\underset{5}{R}_{jn]} + \delta^i_j\underset{5}{R}_{[mn]} + F^{(p}_jF^{i)}_{[n}(\underset{5}{R} - \underset{5}{P})_{pm]} \\ &- 2\Gamma^p_{\underset{\vee}{[nq}}\Gamma^s_{sp}F^q_{(m]}F^i_{j)} - 2\Gamma^p_{\underset{\vee}{mn}}\Gamma^s_{sq}F^q_{(p}F^i_{j)} - \Gamma^p_{\underset{\vee}{j[n}}\Gamma^s_{sq}F^i_{(p}F^q_{m)]}.\end{aligned} \tag{4.55}$$

The quantity $HP\underset{5}{\overline{W}}{}^i{}_{jmn}$ is not a tensor, so we shall call it an **equitorsion holomorphically projective parameter of the fifth kind** of the space $G\underset{2}{K}_N$. Finally, the next theorem is proved:

Theorem 4.5. *The equitorsion holomorphically projective parameter of the fifth kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{2}{K}_N$ *and* $G\underset{2}{\overline{K}}_N$. □

5. Concluding remarks

1. For $g_{ij}(x) = g_{ji}(x)$ GR_N reduces to the Riemannian space R_N. The curvature tensors $\underset{\theta}{R}$, $\theta = 1, \ldots, 5$ in generalized Riemannian space reduce to the single curvature tensor R in Riemannian space (in the symmetric case).

2. In the case of holomorphic mapping of the Kählerian spaces (in the symmetric case) $HP\underset{\theta}{W}{}^i{}_{jmn}$, $(\theta = 1, \cdots, 5)$, given by the formulas (4.16, 30, 45, 49, 55) reduce to the holomorphically projective curvature tensor [28]

$$HPW^i{}_{jmn} = R^i{}_{jmn} + \frac{1}{N+2}(R_{j[n}\delta^i_{m]} + F^p_j R_{p[m}F^i_{n]} + 2F^i_j F^p_n R_{pm}).$$

3. In this paper by using the condition (2.3), non-symmetric metric tensor and equal torsion tensors in the spaces $G\underset{2}{K}_N$ and $G\underset{2}{\overline{K}}_N$ we get new quantities $HP\underset{\theta}{W}^i{}_{jmn}$, $(\theta = 1, \cdots, 5)$ given by the formulas (4.16, 30, 45, 49, 55), and $\underset{\theta}{P}, (\theta = 1, \cdots, 5)$.

4. In the future work we can consider mappings between spaces GK_N, $G\underset{1}{K}_N$ and $G\underset{2}{K}_N$ and probably get new quantities. All these quantities are interesting in constructions of new mathematical and physical structures.

Also we can consider some connections of mentioned spaces.

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