International Electronic Journal of Geometry
Volume 3 No. 2 pp. 40 - 48 (2010) ©IEJG

# SEIBERG-WITTEN EQUATIONS WITH NEGATIVE SING

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(Communicated by Kazım İLARSLAN )

Abstract. In this work we write down Seiberg-Witten equations with negative sign. We give some explicit solutions to these equations on $\mathbb{R}^4$ which are related with the famous Dirac monopole. We also point out a relationship between Seiberg-Witten equations with negative sign and Freud equations which are stated on Minkowski space $\mathbb{R}^{1,3}$.

## 1. Introduction

Seiberg-Witten equations which are stated for $4-$dimensional $spin^c-$manifolds firstly appeared in 1994 ([12]). The solution space of these equations contains informations about the topological structure of underlying manifolds whose scalar curvature are negative ([8]). There are some modifications ([7]) and some generalizations to higher dimensions ([2, 3]). In the present paper we propose similar equations to Seiberg-Witten equations for $4-$dimensional manifolds which are meaningful for $4-$manifolds whose scalar curvature is positive.

## 2. Basic Definitions

Most detailed form of the prerequisites below can be found in [11].

**Definition 2.1.** A spin$^c-$structure on a 2n-dimensional oriented real Hilbert space $V$ is a pair $(W, \Gamma)$ where $W$ is a $2^n-$dimensional complex Hermitian vector space and $\Gamma : V \longrightarrow End(W)$ is a linear map which satisfies

$$\Gamma(v)^* + \Gamma(v) = 0, \qquad \Gamma(v)^* \Gamma(v) = |v|^2 1$$

for every $v \in V$.

Note that, because of the universal property of the Clifford algebra $Cl(V)$ on $V$, $\Gamma$ can be extended to an algebra homomorphism from $Cl(V)$ to $End(W)$.

Let $(W, \Gamma)$ be a spin$^c$ structure on $V$. There is a natural splitting of $W$. Fix an orientation of $V$ and denote by

$$\varepsilon = e_{2n}...e_2e_1 \in Cl(V),$$

2000 *Mathematics Subject Classification.* 57R15, 53C27.
*Key words and phrases.* Spin$^c$ manifolds, spinors, Dirac operator, Seiberg-Witten equations.

the unique element of $Cl(V)$ which has degree $2n$ and is generated by a positively oriented orthonormal basis $\{e_1, ..., e_{2n}\}$ . Then $\varepsilon^2 = (-1)^n$ and hence

$$W = W^+ \oplus W^-,$$

where the $W^\pm$ are the eigenspaces of $\Gamma(\varepsilon)$ given by

$$W^\pm = \{w \in W \mid \Gamma(\varepsilon) w = \pm i^n w\}.$$

Note that $\Gamma(v) W^+ \subset W^-$ and $\Gamma(v) W^- \subset W^+$ for every $v \in V$. So the restriction of $\Gamma(v)$ to $W^+$ for any $v \in V$ determines a linear map $\gamma : V \longrightarrow Hom(W^+, W^-)$ satisfying

$$\gamma(v)^* \gamma(v) = |v|^2 1$$

for every $v \in V$ .

Let $(W, \Gamma)$ be a spin$^c$ structure on $V$. Such a structure gives an action of the space of 2−forms $\Lambda^2 V$ on $W$. This action is defined by the following:

Firstly, identify $\Lambda^2 V$ with the space of second order elements of Clifford algebra $C_2(V)$ via the map

$$\Lambda^2 V \longrightarrow C_2(V) \; ; \; \eta = \sum_{i<j} \eta_{ij} e_i \wedge e_j \longmapsto \sum_{i<j} \eta_{ij} e_i e_j.$$

Compose this map with $\Gamma$ to obtain a map $\rho : \Lambda^2 V \longrightarrow End(W)$ given by

$$\rho\left(\sum_{i<j} \eta_{ij} e_i \wedge e_j\right) = \sum_{i<j} \eta_{ij} \Gamma(e_i) \Gamma(e_j)$$

for any orthonormal basis $\{e_1, ..., e_{2n}\}$ of $V$. This map is independent of the choice of the orthonormal basis $\{e_1, ..., e_{2n}\}$. The spaces $W^\pm$ are invariant under $\rho(\eta)$ for every 2−form $\eta \in \Lambda^2 V$. Therefore we can define

$$\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$$

for $\eta \in \Lambda^2 V$. If $V$ is a 4-dimensional, then $\rho^+(\eta) = \rho^+(\eta^+)$ for every 2−form $\eta \in \Lambda^2 V$, where $\eta^+$ is the self-dual part of $\eta$. The map $\rho$ extends to a map

$$\rho : \Lambda^2 V \otimes \mathbb{C} \longrightarrow End(W)$$

on the space of complex valued 2−forms. If $\eta$ is a real valued 2−form then $\rho(\eta)$ is skew-Hermitian and if $\eta$ is imaginary valued then $\rho(\eta)$ is Hermitian.

Under certain conditions over the 2n-dimensional oriented manifold $M$, a global version of the map $\Gamma$ can be defined. A *spin$^c$* structure is defined by the map $\Gamma : TM \longrightarrow End(W)$, $W$ being a $2^n$-dimensional complex Hermitian vector bundle on $M$. Such a structure exists iff $w_2(M)$ has an integral lift (see [6]). $\Gamma$ extends to an isomorphism between the complex Clifford algebra bundle $\mathbb{C}l(TM)$ and $End(W)$. There is a natural splitting $W = W^+ \oplus W^-$ into the $\pm i^n$ eigenspaces of $\Gamma(e_{2n} e_{2n-1} ... e_1)$ where $\{e_1, e_2, ..., e_{2n}\}$ is any positively oriented local orthonormal frame of $TM$.

A Hermitian connection $\nabla$ on $W$ is called a *spin$^c$* connection (compatible with the Levi-Civita connection) if

$$\nabla_v(\Gamma(w)\Phi) = \Gamma(w)\nabla_v\Phi + \Gamma(\nabla_v w)\Phi,$$

where $\Phi$ is a spinor (section of $W$ ), $v$ and $w$ are vector fields on $M$ and $\nabla_v w$ is the Levi-Civita connection on $M$. $\nabla$ preserves the subbundles $W^\pm$.

There is a principal $Spin^c(2n)$-bundle $P$ on $M$ such that the bundle $W$ of spinors and the tangent bundle $TM$ can be recovered as the associated bundles

$$W = P \times_{Spin^c(2n)} \mathbb{C}^{2n} \ , \quad TM = P \times_{Ad} \mathbb{R}^{2n}$$

where $Ad$ is being the adjoint action of

$$Spin^c(2n) = \left\{ e^{i\theta}x : \theta \in \mathbb{R}, x \in Spin(2n) \right\} \subset \mathbb{C}l_{2n}$$

on $\mathbb{R}^{2n}$. In addition a complex line bundle $L_\Gamma = P \times_\delta \mathbb{C}$ can be obtained from the principal $Spin^c(2n)$-bundle $P$ where $\delta : Spin^c(2n) \to S^1$ defined by $\delta\left(e^{i\theta}x\right) = e^{2i\theta}$.

There is a one-to-one correspondence between $spin^c$ connections on $W$ and $spin^c(2n) = Lie(Spin^c(2n)) = spin(2n) \oplus i\mathbb{R}$-valued connection 1-forms $\widehat{A} \in A(P) \subset \Omega^1(P, spin^c(2n))$ on $P$. Hence every $spin^c$ connection $\widehat{A}$ decomposes as

$$\widehat{A} = \widehat{A}_0 + \frac{1}{2^n} trace(\widehat{A}),$$

where $\widehat{A}_0$ is the traceless part of $\widehat{A}$. Let $A = \frac{1}{2^n} trace(\widehat{A})$ which is an imaginary valued 1-form in $\Omega^1(P, i\mathbb{R})$ that satisfies

$$A_{pg}(vg) = A_p(v), \qquad A_p(p.\xi) = \frac{1}{2^n} trace(\xi) \tag{2.1}$$

for $v \in T_pP$, $g \in Spin^c(2n)$, and $\xi \in spin^c(2n)$. Let

$$\mathcal{A}(\Gamma) = \left\{ A \in \Omega^1(P, i\mathbb{R}) : A \text{ satisfies } (2.1) \right\}.$$

There is a one-to-one correspondence between the elements of $\mathcal{A}(\Gamma)$ and $spin^c$ connections on $W$. Let $\nabla_A$ be the $spin^c$ connection corresponding to $A \in \mathcal{A}(\Gamma)$. $\mathcal{A}(\Gamma)$ is an affine space with the parallel vector space $\Omega^1(M, i\mathbb{R})$. Let $F_A \in \Omega^2(P, i\mathbb{R})$ be the curvature of the 1-form $A$ and $D_A$ denote the Dirac operator corresponding to $A \in \mathcal{A}(\Gamma)$,

$$D_A : C^\infty\left(M, W^+\right) \longrightarrow C^\infty\left(M, W^-\right)$$

defined by

$$D_A(\Phi) = \sum_{i=1}^{2n} \Gamma(e_i) \nabla_{A,e_i}(\Phi)$$

where $\Phi \in C^\infty(M, W^+)$ and $\{e_1, e_2, ..., e_{2n}\}$ is any local orthonormal frame.

## 3. Seiberg-Witten equations

The Seiberg-Witten equations on a 4−dimensional $spin^c$−manifold $M$ can be expressed as follows:

Let $\Gamma : TM \longrightarrow End(W)$ be a fixed $spin^c$ structure on $M$ and consider the pair $(A, \Phi) \in \mathcal{A}(\Gamma) \times C^\infty(M, W^+)$. The Seiberg-Witten equations read

$$D_A(\Phi) = 0, \tag{3.1}$$

$$\rho^+(F_A) = (\Phi\Phi^*)_0, \tag{3.2}$$

where $(\Phi\Phi^*)_0 \in C^\infty(M, End(W^+))$ is defined by $(\Phi\Phi^*)(\tau) = \langle \Phi, \tau \rangle \Phi$ for $\tau \in C^\infty(M, W^+)$ and $(\Phi\Phi^*)_0$ is the traceless part of $(\Phi\Phi^*)$. Hence the Seiberg-Witten equations have been obtained on 4−dimensional $spin^c$-manifolds.

Now let us consider these equations on flat space $\mathbb{R}^4$. The $spin^c$ connection $\nabla = \nabla^A$ on $\mathbb{R}^4$ is given by

$$\nabla_j \Phi = \frac{\partial \Phi}{\partial x_j} + A_j \Phi,$$

where $A_j : \mathbb{R}^4 \longrightarrow i\mathbb{R}$ and $\Phi : \mathbb{R}^4 \longrightarrow \mathbb{C}^2$ . Then the associated connection on the line bundle $L_\Gamma = \mathbb{R}^4 \times \mathbb{C}$ is the connection 1−form

$$A = \sum_{i=1}^{4} A_i dx_i \in \Omega^1\left(\mathbb{R}^4, i\mathbb{R}\right)$$

and its curvature 2-form is given by

$$F_A = dA = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2\left(\mathbb{R}^4, i\mathbb{R}\right),$$

where $F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$ for $i, j = 1, ..., 4$ .

Let $\Gamma : \mathbb{R}^4 \longrightarrow End(\mathbb{C}^4)$ be the classical $spin^c$ structure which is given by

$$\Gamma(w) = \begin{bmatrix} 0 & \gamma(w) \\ -\gamma(w)^* & 0 \end{bmatrix}$$

where $\gamma : \mathbb{R}^4 \longrightarrow End(\mathbb{C}^2)$ is defined on generators $e_0, e_1, e_2, e_3$ by the followings:

$$\gamma(e_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \gamma(e_1) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \gamma(e_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \gamma(e_3) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Note that in the definition of $\gamma$, the $2\times2$ identity matrix and $i$ multiples of the well-known Pauli matrices $\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ and $\sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are used. In many works, the classical $spin^c$−structure has been used (see for instance [11], [10], [5]).

Note that $\Gamma(e_3 e_2 e_1 e_0) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and the eigenspaces of $\Gamma(e_3 e_2 e_1 e_0)$ are

$$\begin{aligned} W_1^+ &= \left\{(\phi_1, \phi_2, 0, 0) \mid \phi_1, \phi_2 \in C^\infty\left(\mathbb{R}^4, \mathbb{C}\right)\right\}, \\ W_1^- &= \left\{(0, 0, \phi_3, \phi_4) \mid \phi_3, \phi_4 \in C^\infty\left(\mathbb{R}^4, \mathbb{C}\right)\right\}. \end{aligned}$$

According to the above data Seiberg-Witten equations on $\mathbb{R}^4$, i. e. equations (3.1) and (3.2), are as follows (see [11], [10] ):

The first of these equations, $D_A\Phi = 0$, can be expressed as

$$-\nabla_0\Phi + i\sigma_1\nabla_1\Phi + i\sigma_2\nabla_2\Phi + i\sigma_3\nabla_3\Phi = 0,$$

or more explicitly

$$\begin{aligned} &\frac{\partial\phi_1}{\partial x_0} + A_0\phi_1 = i\left(\frac{\partial\phi_1}{\partial x_1} + A_1\phi_1\right) + \left(\frac{\partial\phi_2}{\partial x_2} + A_2\phi_2\right) + i\left(\frac{\partial\phi_2}{\partial x_3} + A_3\phi_2\right) \\ &\frac{\partial\phi_2}{\partial x_0} + A_0\phi_2 = -i\left(\frac{\partial\phi_2}{\partial x_1} + A_1\phi_2\right) - \left(\frac{\partial\phi_1}{\partial x_2} + A_2\phi_1\right) + i\left(\frac{\partial\phi_1}{\partial x_3} + A_3\phi_1\right) \end{aligned} \tag{3.3}$$

where $\Phi = (\phi_1, \phi_2, 0, 0)$. The second one is

$$\rho^+(F_A) = (\Phi\Phi^*)_0$$

and this equation can be expressed explicitly

$$\begin{array}{rcl} F_{01}+F_{23} & = & -\frac{i}{2}\left(\phi_1\overline{\phi}_1-\phi_2\overline{\phi}_2\right), \\ F_{02}-F_{13} & = & \frac{1}{2}\left(\phi_1\overline{\phi}_2-\phi_2\overline{\phi}_1\right), \\ F_{03}+F_{12} & = & -\frac{i}{2}\left(\phi_1\overline{\phi}_2+\phi_2\overline{\phi}_1\right). \end{array} \tag{3.4}$$

where $F_A = dA$ hence $F_A = F_{01}dx_0 \wedge dx_1 + F_{02}dx_0 \wedge dx_2 + F_{03}dx_0 \wedge dx_3 + F_{12}dx_1 \wedge dx_2 + F_{13}dx_1 \wedge dx_3 + F_{23}dx_2 \wedge dx_3$.

## 4. Seiberg-Witten Equations with Negative Sign

Now we change the second equation of Seiberg-Witten equations by multiplying $(-1)$ its right hand side. Then the Seiberg-Witten equations with negative sign on a $4-$dimensional *spin*$^c$ manifold can be expressed as follow:

$$D_A(\Phi) = 0, \tag{4.1}$$

$$\rho^+(F_A) = -(\Phi\Phi^*)_0 . \tag{4.2}$$

Now we consider the Seiberg-Witten Equations with negative sign on $\mathbb{R}^4$. The second equation of the Seiberg-Witten equations with negative sign on $\mathbb{R}^4$ is

$$\rho^+(F_A) = -(\Phi\Phi^*)_0$$

and this equation can be expressed explicitly as follows:

$$\begin{array}{rcl} F_{01}+F_{23} & = & \frac{i}{2}\left(\phi_1\overline{\phi}_1-\phi_2\overline{\phi}_2\right), \\ F_{02}-F_{13} & = & -\frac{1}{2}\left(\phi_1\overline{\phi}_2-\phi_2\overline{\phi}_1\right), \\ F_{03}+F_{12} & = & \frac{i}{2}\left(\phi_1\overline{\phi}_2+\phi_2\overline{\phi}_1\right). \end{array} \tag{4.3}$$

## 5. Seiberg-Witten Equations on Minkowski Space

When one take 4-dimensional Euclidean space with the metric

$$\eta(x,y) = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3,$$

where $x = (x_0, x_1, x_2, x_3), y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, this space is called Minkowski space and it's denoted by $\mathbb{R}^{1,3}$. A spin$^c-$structure on $\mathbb{R}^{1,3}$ can be defined similar to the Euclidean case as follows:

**Definition 5.1.** A spin$^c-$structure on $\mathbb{R}^{1,3}$ is a pair $(W, \Gamma)$ where $W$ is a $4-$dimensional complex vector space and $\Gamma : \mathbb{R}^{1,3} \longrightarrow End(W)$ is a linear map which satisfies

$$\Gamma(v)^2 = -\eta(v,v)\mathbf{1}$$

for every $v \in V$.

Due to the universal properties of Clifford algebras, the map $\Gamma$ can be extended to an algebra isomorphism from complex Clifford algebra $\mathbb{C}l_4 \cong Cl_{1,3} \otimes \mathbb{C}$ to $End(W)$ which is still denoted by $\Gamma$ where $Cl_{1,3}$ is the real Clifford algebra on $\mathbb{R}^{1,3}$.

An explicit spin$^c-$structure on $\mathbb{R}^{1,3}$ can be given by using Pauli matrices:

$$\Gamma(e_0) = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \quad \Gamma(e_1) = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$$
$$\Gamma(e_2) = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \Gamma(e_3) = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix},$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that $\Gamma(e_0)^2 = -I_4$, $\Gamma(e_1)^2 = \Gamma(e_2)^2 = \Gamma(e_3)^2 = I_4$, where $I_4$ is $4\times 4$ identity matrix. As in the Euclidean case, $\Gamma$ gives rise to an action of the space $\Lambda^2\left(\mathbb{R}^{1,3}\right)$ on $W$ which is induced by Clifford multiplication:

$$\rho : \Lambda^2\left(\mathbb{R}^{1,3}\right) \to End\left(W\right), \quad \rho\left(\sum_{i<j} \eta_{ij} dx_i \wedge dx_j\right) = \sum_{i<j} \eta_{ij}\Gamma(e_i)\Gamma(j).$$

The map $\rho$ extends to a map

$$\rho : \Lambda^2 V \otimes \mathbb{C} \longrightarrow End(W)$$

on the space of complex valued $2-$forms. The representation space $W = \mathbb{C}^4$ is called spinor space. This space has the following natural decomposition:

$$W = W^+ \oplus W^-$$

where

$$W^+ = \left\{(\psi_1, \psi_2, 0, 0) | \psi_i \in \mathbb{C}\right\},$$
$$W^- = \left\{(0, 0, \psi_3, \psi_4) | \psi_i \in \mathbb{C}\right\}.$$

These subspaces are invariant under the action $\rho$. Hence we get the new maps $\rho^\pm$ by restrictions:

$$\rho^+(\eta) = \rho(\eta)|_{W^+}; \quad \rho^-(\eta) = \rho(\eta)|_{W^-}.$$

Generally Dirac operator $D_A : C^\infty\left(\mathbb{R}^{1,3}, W\right) \longrightarrow C^\infty\left(\mathbb{R}^{1,3}, W\right)$ on $\mathbb{R}^{1,3}$ associated to spin$^c-$structure $\Gamma$ is defined by

$$D_A\left(\Phi\right) = -\Gamma\left(e_0\right)\nabla^A_{e_0}\left(\Phi\right) + \Gamma\left(e_1\right)\nabla^A_{e_1}\left(\Phi\right) + \Gamma\left(e_2\right)\nabla^A_{e_2}\left(\Phi\right) + \Gamma\left(e_3\right)\nabla^A_{e_3}\left(\Phi\right)$$

where $\Phi \in C^\infty\left(\mathbb{R}^{1,3}, W\right)$ and $\{e_0, e_1, e_2, e_3\}$ is any orthonormal frame on $\mathbb{R}^{1,3}$. Note that $\nabla^A$ preserves subbundles $W^\pm$ and the Clifford multiplication by vectors interchanges these subbundles. Hence we get the following decomposition

$$D_A^\pm : C^\infty\left(\mathbb{R}^{1,3}, W^\pm\right) \longrightarrow C^\infty\left(\mathbb{R}^{1,3}, W^\mp\right).$$

From now on we consider the Dirac operator $D_A^+$ and denote it by $D_A$, explicitly

$$D_A\left(\Phi\right) = -\sigma_0\nabla^A_{e_0}\left(\Phi\right) + \sigma_1\nabla^A_{e_1}\left(\Phi\right) + \sigma_2\nabla^A_{e_2}\left(\Phi\right) + \sigma_3\nabla^A_{e_3}\left(\Phi\right).$$

Now we can state Seiberg-Witten equations on $\mathbb{R}^{1,3}$:

$$D_A\left(\Phi\right) = 0$$
$$\rho^+(F_A) = \left(\Phi\Phi^*\right)_0.$$

The explicit form of the first equation is

$$\begin{aligned} \tfrac{\partial\phi_1}{\partial x_0} + A_0\phi_1 &= \tfrac{\partial\phi_2}{\partial x_1} + A_1\phi_2 - i\left(\tfrac{\partial\phi_2}{\partial x_2} + A_2\phi_2\right) + \tfrac{\partial\phi_1}{\partial x_3} + A_3\phi_1, \\ \tfrac{\partial\phi_2}{\partial x_0} + A_0\phi_2 &= \tfrac{\partial\phi_1}{\partial x_1} + A_1\phi_1 + i\left(\tfrac{\partial\phi_1}{\partial x_2} + A_2\phi_1\right) - \left(\tfrac{\partial\phi_2}{\partial x_3} + A_3\phi_2\right). \end{aligned}$$

The left side of the second equation can be written by

$$\rho^+(F_A) = \begin{pmatrix} F_{03} + iF_{12} & F_{01} + iF_{23} - i\left(F_{02} - iF_{13}\right) \\ F_{01} + iF_{23} + i\left(F_{02} - iF_{13}\right) & -\left(F_{03} + iF_{12}\right) \end{pmatrix}.$$

On the other hand the traceless part of the endomorphism $(\Phi\Phi^*)$ is

$$(\Phi\Phi^*)_0 = \begin{pmatrix} \frac{\phi_1\overline{\phi}_1-\phi_2\overline{\phi}_2}{2} & \phi_1\overline{\phi}_2 \\ \overline{\phi}_1\phi_2 & \frac{\phi_2\overline{\phi}_2-\phi_1\overline{\phi}_1}{2} \end{pmatrix}.$$

Then the second equation is

$$\begin{aligned} F_{03}+iF_{12} &= \tfrac{\phi_1\overline{\phi}_1-\phi_2\overline{\phi}_2}{2}, \\ F_{01}+iF_{23}-i\left(F_{02}-iF_{13}\right) &= \phi_1\overline{\phi}_2, \\ F_{01}+iF_{23}+i\left(F_{02}-iF_{13}\right) &= \overline{\phi}_1\phi_2. \end{aligned}$$

If we rearrange these equations, we obtain

$$\begin{aligned} F_{03}+iF_{12} &= \tfrac{1}{2}(\phi_1\overline{\phi}_1-\phi_2\overline{\phi}_2), \\ F_{01}+iF_{23} &= \tfrac{1}{2}\left(\phi_1\overline{\phi}_2+\overline{\phi}_1\phi_2\right), \\ F_{02}-iF_{13} &= \tfrac{i}{2}\left(\phi_1\overline{\phi}_2-\overline{\phi}_1\phi_2\right). \end{aligned}$$

These equations are also known as Freund equations, because in [4] Freund gave the following explicit solutions to these equations:

$$A_0 = A_3 = 0, A_1 = \frac{-ix_2}{2r\left(r-x_3\right)}, A_2 = \frac{ix_1}{2r\left(r-x_3\right)}$$

and

$$\phi_1 = \frac{1}{\sqrt{2}r}\frac{x_1-ix_2}{\sqrt{r\left(r-x_3\right)}}, \ \phi_2 = \frac{1}{\sqrt{2}r}\sqrt{\frac{r-x3}{r}}.$$

Also see [1, 9, 10] for some discussion of these solutions.

We can produce a solution for the Seiberg-Witten equations with negative sign by using the Freund's solution. Firstly, in the above solution of Freund, change the coordinate $(x_0,x_1,x_2,x_3)$ to $(x_0,x_3,x_2,x_1)$, then the pair $(A,\Phi)$ become the pair $(A',\Phi')$ with the following components:

$$A_0' = A_3' = 0, A_1' = \frac{-ix_2}{2r\left(r-x_1\right)}, A_2' = \frac{ix_3}{2r\left(r-x_1\right)}$$

and

$$\phi_1' = \frac{1}{\sqrt{2}r}\frac{x_3-ix_2}{\sqrt{r\left(r-x_1\right)}}, \ \phi_2' = \frac{1}{\sqrt{2}r}\sqrt{\frac{r-x_1}{r}}.$$

Then the pair $(B,\Psi)$ is a solution to the Seiberg-Witten equations with negative sign by $B_0 = A_0'$, $B_1 = A_3'$, $B_2 = A_2'$, $B_3 = A_1'$ and $\psi_1 = \phi_1'$, $\psi_2 = \phi_2'$.

Hence the solution space of Seiberg-Witten equations with negative sign is non-empty. One can produce infinitely many solutions by using the above special solution:

The group $\mathcal{G} = Map(M,S^1)$ acts on the space $A\left(\Gamma\right)\times C^{\infty}\left(M,W^+\right)$ via

$$u^*\left(B,\Psi\right) = \left(B+u^{-1}du, u^{-1}\Psi\right)$$

for $u\in Map(M,S^1)$, $B\in A\left(\Gamma\right)$ and $\Psi\in C^{\infty}\left(M,W^+\right)$. It can be checked that

$$D_{u^*B}\left(u^{-1}\Psi\right) = u^{-1}D_B\Psi, \ F_{u^*B} = F_B$$

and thus $(B,\Psi)$ satisfies the Seiberg-Witten equations with negative sign if and only if the pair $\left(u^*B, u^{-1}\Psi\right)$ satisfies these equations.

Let us consider the smooth map $u:\mathbb{R}^4\to S^1$, $u(x) = e^{if(x)}$ where $f:\mathbb{R}^4\to\mathbb{R}$ is a smooth map. Then the pair $\left(B+idf, e^{if}\Psi\right)$ is a new solution to the Seiberg-Witten equations with negative sign. More explicitly

$$\widetilde{B} = (B_0 + i\frac{\partial f}{\partial x_0})dx_0 + (B_1 + i\frac{\partial f}{\partial x_1})dx_1 + (B_2 + i\frac{\partial f}{\partial x_2})dx_2 + (B_3 + i\frac{\partial f}{\partial x_3})dx_3$$

$$\widetilde{\Psi} = \left(e^{if}\psi_1, e^{if}\psi_2\right).$$

Putting above special solution in this last expressions:

$$\widetilde{B} = (i\frac{\partial f}{\partial x_0})dx_0 + (i\frac{\partial f}{\partial x_1})dx_1 + (\frac{ix_3}{2r\,(r+x_1)} + i\frac{\partial f}{\partial x_2})dx_2 + (\frac{-ix_2}{2r\,(r+x_1)} + i\frac{\partial f}{\partial x_3})dx_3$$

$$\widetilde{\Psi} = \left(e^{if}\frac{1}{\sqrt{2}r}\frac{-x_3+ix_2}{\sqrt{r\,(r+x_1)}}, e^{if}\frac{1}{\sqrt{2}r}\sqrt{\frac{r+x_1}{r}}\right),$$

where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

It is known that if $(A, \Phi)$ is a solution to the Seiberg-Witten equation on a 4-dimensional compact manifold, then it has the global bound

$$\|\Phi(x)\|^2 \leq -s.$$

The proof of the compacness of the solution set of the Seiberg-Witten equations based on this fact. Unfortunately similar situation doesn't hold for the Seiberg-Witten equations with negative sign.

**Lemma 5.1.** *Let $(A, \Phi)$ be a solution to the Seiberg-Witten equation with negative sign on a compact manifold. Then the following inequality*

$$\frac{1}{2}s \leq \|\Phi(x)\|^2$$

*holds.*

*Proof.* The proof relies on the identity

$$\Delta_g\|\Phi\|^2 = -2\|\nabla_A\Phi\|^2 + 2\mathrm{Re}\,\langle\Phi, \nabla_A^*\nabla_A\Phi\rangle$$

where $\Delta_g = d^*d$ denotes the positive definite Laplace operator of the metric $g$ ( see [11]). From the last equation and Weitzenbock formula,

$$\begin{aligned}
\Delta_g\|\Phi\|^2 &= -2\|\nabla_A\Phi\|^2 + 2\mathrm{Re}\,\langle\Phi, \nabla_A^*\nabla_A\Phi\rangle \\
&\leq 2\mathrm{Re}\,\langle\Phi, \nabla_A^*\nabla_A\Phi\rangle \\
&= -2\,\langle\Phi, \rho^+(F_A)\Phi\rangle - \tfrac{1}{2}s\|\Phi\|^2 \\
&= 2\,\langle\Phi, (\Phi\Phi^*)_0\,\Phi\rangle - \tfrac{1}{2}s\|\Phi\|^2 \\
&= \|\Phi\|^4 - \tfrac{1}{2}s\|\Phi\|^2.
\end{aligned}$$

Since $\Delta_g\|\Phi\|^2 \geq 0$, $\|\Phi\|^4 - \frac{1}{2}s\|\Phi\|^2 \geq 0$, hence we obtain

$$\frac{1}{2}s \leq \|\Phi\|^2.$$

□

Note that if the manifold $M$ has positive scalar curvature $s$ , then the spinor $\Phi = 0$ is not a solution to the Seiberg-Witten equation with negative sign on $M$.

**Conclusion 1.** *We have proposed the Seiberg-Witten equations with negative sign, and we have observed that these equations have some solutions which are related with the Dirac monopole. But we have not discussed structure of the solution space.*

*It is known that if $(A, \Phi)$ is a solution of Seiberg-Witten equations over a compact 4-dimensional Riemannian manifold $M$ , then*

$$\|\Phi(x)\|^2 \leq -s$$

*at each point, where $s$ is the scalar curvature of $M$. Compacness of the modulo space of Seiberg-Witten equations is proved by using this property and familar Sobolev imbedding theorems (see* [8]*). On the other hand there is no a priory pointwise bound to the size of the spinor field of any solution to the Seiberg-Witten equations with negative sign as we shown above.*

*The Seiberg-Witten equations give invariants for 4-dimensional compact Riemannian manifolds with negative scalar curvature. These equations are meaningless for 4-dimensional compact Riemannian manifolds with positive scalar curvature. At this point a question arise: Is it possible to define similar invariants for 4-dimensional compact Riemannian manifolds with positive scalar curvature by considering the Seiberg-Witten equations with negative sign?*

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