

## ON THE COMPLETE LIFT DISTRIBUTIONS AND THEIR APPLICATIONS TO SEMI-RIEMANNIAN GEOMETRY

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**ABSTRACT.** In this paper, we investigate the prolongations of the semi-Riemannian distributions to tangent bundle. To achieve this, we define the complete lift of some geometrical objects defined on a given distribution. In addition, we give some general results for the complete lift distribution.

### 1. INTRODUCTION

The theory of distributions was studied by many authors, such as S. Ianus [3], J. A. Schouten [5], G. Vranceanu [8], A. G. Walker [9], A. Bejancu and H. R. Farran [1], and B. Şahin and R. Güneş [6]. These papers in above references are dealt with the geometry of distributions and some properties of submanifold.

In fact distributions on a manifold  $M$  are related with some submanifolds. Besides, each distribution on  $M$  determines a vector subbundle of  $TM$ . Using vector subbundle structure of distributions, we can define induced connections and metric tensors on them. This case enables us to characterize the geometry of submanifolds. Especially, in lightlike geometry, the structure of a degenerate submanifold is being made up using screen, radical and transversal distributions.

In this paper our aim is the prolongation of a distribution on a manifold  $M$  to its tangent bundle  $TM$  and investigating geometrical relations between original distribution and prolonged distribution.

We have to mention here vertical and complete lifts of differentiable elements defined on  $M$  to tangent bundle  $TM$ . The notion of vertical and complete lift was given, firstly, by K. Yano [11]. These lifts are based on in the present paper.

In addition we also know that when a metric tensor  $G$  is given on  $M$ ,  $TM$  becomes a semi-Riemannian manifold with respect to the complete lift  $G^c$  of  $G$ . Then we can argue the geometry of  $D^c$  with respect to semi-Riemannian structure  $(TM, G^c)$ .

Of course, for these arguments we need to some information about degenerate and nondegenerate subspaces. About these informations, we refer to K. L. Duggal and A. Bejancu [2] and B. O'neill [4].

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After preliminaries in section 2, we give some results about the complete lifts of distributions in section 3. In section 4, we discuss the semi-Riemannian structure of the complete lift of  $D$ . Finally, in section 5 we define the complete lift of induced geometrical objects on semi-Riemannian distributions and using these lifts we prolonge the semi-Riemannian distributions to  $TM$ .

## 2. PRELIMINARIES

For any differentiable manifold  $N$ , we denote by  $TN$  its tangent bundle with the projection  $\pi_N : TN \rightarrow N$  and by  $T_p(N)$  its tangent space at a point  $p$  of  $N$ .  $\mathfrak{S}_s^r(N)$  is the space of tensor fields of class  $C^\infty$  and of type  $(r, s)$ . An element of  $\mathfrak{S}_0^0(N)$  is a  $C^\infty$  function defined on  $N$ . We denote by  $\mathfrak{S}(N)$  the tensor algebra on  $N$ .

Let  $M$  be an  $m$ - dimensional paracompact differentiable manifold and  $V$  be a coordinate neighborhood in  $M$  and  $(x^A)$ , where  $1 \leq A \leq m$ , certain local coordinates defined in  $V$ . We shall take a system of coordinates  $(x^A, y^A)$  in  $\pi_M^{-1}(V)$  such that  $(y^A)$  are cartesian coordinates in each tangent space  $T_p(M)$ ,  $p$  being an arbitrary point of  $V$ , with respect to the natural frame  $(\frac{\partial}{\partial x^A})$  of local coordinates  $(x^A)$ . We call  $(x^A, y^A)$  the coordinates induced in  $\pi_M^{-1}(V)$  from  $(x^A)$ . We suppose that all the used maps belong to the class  $C^\infty$  and we shall adopt the Einstein summation convention through this paper. In addition, we assume the range for indices as in following.

$$\begin{aligned} A, B, C \in \{1, \dots, m\}, \quad \alpha, \beta \in \{k+1, \dots, m\}, \quad i, j \in \{1, \dots, k\}, \\ a, b \in \{1, \dots, q\} \quad \bar{a}, \bar{b} \in \{q+1, \dots, k\}. \end{aligned}$$

An  $k$ - dimensional distribution on a manifold  $M$  of dimension  $m$  is a function  $D$  in  $M$  such that  $D_p$  is a  $k$ - dimensional subspace of  $T_pM$  (where  $0 < k \leq n$ ) and which satisfies the following differentiability condition. Each point  $M$  in the domain of  $D$  has a neighbourhood  $V$  on which vector fields  $X_1, \dots, X_k$  are defined so that  $D_q$  is spanned by  $X_{1q}, \dots, X_{kq}$  if  $q \in V$ . We call such a set of vector fields a basis for  $D$  at  $p$ . In the sequel, by a distribution we shall always mean a differentiable distribution and we shall write  $\dim D$  instead of dimension of  $D$ .

A chart  $x = (x^A)$  of  $M$  is said to be flat with respect to a distribution  $D$  in  $M$  if the vector fields  $\frac{\partial}{\partial x^i}$  ( $i = 1, \dots, k$ ) form a basis for  $\Gamma(D)$ . A distribution on  $M$  is integrable if each point of  $M$  lies in the domain of a flat chart.

A vector field  $X$  is said to belong to a distribution  $D$  (and we write  $X \in \Gamma(D)$ ) if  $X_p \in D_p$  for each point  $p$  in the domain of  $X$ . We define the distribution to be involutive if  $[X, Y] \in \Gamma(D)$  whenever  $X, Y \in \Gamma(D)$ .

From Frobenius theorem we know that a distribution is involutive iff it is integrable.  $M'$  is said to be an integral manifold of  $D$  if it is submanifold of  $M$  such that for all  $p \in M'$ ,

$$D_p = j_*(T_pM')$$

where  $j : M' \rightarrow M$  is the natural injection.

If  $M'$  is a connected integral manifold of  $D$  and there exists no connected integral manifold  $\bar{N}$ , with immersion  $\bar{f} : \bar{N} \rightarrow M$ , such that  $f(M') \subset \bar{f}(\bar{N})$ , we say that  $M'$  is a maximal integral manifold or a *leaf* of  $D$ . The leaves of  $D$  determine a foliation on  $M$  of dimension  $k$ , that is,  $M$  is disjoint union of connected subsets  $\{L_t\}_{t \in I}$  and each point  $p$  of  $M$  has a coordinate system  $(U, x^A)$  such that  $L_t \cap U$  is locally given by equations

$$x^\alpha = c^\alpha \quad , \quad k+1 \leq \alpha \leq n$$

where  $c^\alpha$  are real constants and  $(x^i), 1 \leq i \leq k$  are local coordinates on the leaf  $L_t$ . We say that the foliation defined by  $D$  is totally geodesic, totally umbilical or minimal, if any leaf of  $D$  is totally geodesic, totally umbilical or minimal, respectively [1].

Now suppose  $\bar{D}$  is a complementary distribution to the integrable distribution  $D$  on  $M$  that is, we have

$$TM = D \oplus \bar{D}$$

we call  $\bar{D}$  a transversal distribution to  $D$  in  $TM$ . The existence of  $\bar{D}$  is guaranteed by paracompactness of  $M$  [1].

Let  $D$  be a distribution on  $M$  and  $\hat{\nabla}$  be a linear connection on  $M$ . We say that  $D$  is parallel with respect to  $\hat{\nabla}$ , if for each  $X \in \Gamma(D)$  and  $Y \in \Gamma(TM)$ ,  $\hat{\nabla}_Y X$  is an element of  $\Gamma(D)$  [1].

Let  $(M, G)$  be a semi-Riemannian manifold and  $D$  be a distribution on  $M$ . If for each  $p \in \text{dom}D$ ,  $\text{ind}D_p = q$ , then we say that  $D$  has index  $q$ . Consider a distribution  $D^\perp$  which assigns to each point  $p$  of  $M$  a linear subspace orthogonal to  $D_p$ . Such a distribution  $D^\perp$  is called orthogonal to  $D$  and for any point  $p \in M$  we write

$$D_p^\perp = \{w_p \in T_pM \mid g(w_p, v_p) = 0 \forall v_p \in D_p\}.$$

The subspace  $D_p^\perp \cap D_p$  is known to be *radical* subspace of  $D_p$  and denoted by  $\text{Rad} D_p$ . If for all  $p \in M$  the dimension of  $\text{Rad} D_p$  is same, the dimension of  $\text{Rad} D_p$  is called the nullity of  $D$  and denoted by  $\text{null}D$ . If  $\text{null}D \neq 0$ , it is called degenerate (or lightlike) distribution on  $M$ . In this case  $D_p$  is decomposable as in follows,

$$(2.1) \quad D_p = \text{Rad}D_p \perp S(D_p)$$

where  $S(D_p)$  is non-degenerate subspace of  $D_p$ .  $S(D_p)$  is known as a *screen subspace* of  $D_p$ . By virtue of (2.1) we see that  $D$  has subdistributions. Thus,  $D$  is written as independent from point in following form,

$$D = \text{Rad}D \perp S(D).$$

If  $\text{Rad}D = \{0\}$ ,  $D$  is called a nondegenerate distribution [1]

Now we must recall the definition of vertical and complete lifts of differentiable elements defined on  $M$ . Let  $f$ ,  $X$ ,  $w$ ,  $F$  and  $G$  be a function, a vector field, a 1-form, a tensor field of type  $(1, 1)$  and a tensor field of type  $(0, 2)$  respectively. We denote by  $f^v$ ,  $X^v$ ,  $w^v$ ,  $F^v$  and  $G^v$  their vertical lifts and by  $f^c$ ,  $X^c$ ,  $w^c$ ,  $F^c$  and  $G^c$  their complete lifts, respectively. For a function  $f : M \rightarrow \mathbb{R}$ , we have

$$(2.2) \quad \begin{aligned} f^v &= f \circ \pi_M \\ f^c &= y^A \frac{\partial f}{\partial x^A} \end{aligned}$$

with respect to induced coordinates  $(x^A, y^A)$  on  $TM$ . Moreover these lifts have the following properties:

$$\begin{aligned}
& (fX)^v = f^v X^v, \quad (fX)^c = f^v X^c + f^c X^v, \\
& X^v f^v = 0, \quad X^c f^c = (Xf)^c, \quad X^v f^c = X^c f^v = (Xf)^v, \\
& w^v(X^v) = 0, \quad w^c(X^c) = (w(X))^c, \quad w^v(X^c) = w^c X^v = (w(X))^v, \\
(2.3) \quad & [X^v, Y^v] = 0, \quad [X, Y]^c = [X^c, Y^c], \quad [X, Y]^v = [X^v, Y^c] = [X^c, Y^v], \\
& G^c(X^v, Y^v) = 0, \quad G^c(X^c, Y^c) = (G(X, Y))^c, \\
& G^c(X^v, Y^c) = G^c(X^c, Y^v) = (G(X, Y))^v \\
& F^c X^c = (FX)^c \quad F^v X^c = F^c X^v = (FX)^v
\end{aligned}$$

(see [11]). In special case,  $(\frac{\partial}{\partial x^A})^c = \frac{\partial}{\partial x^A}$  and  $(\frac{\partial}{\partial x^A})^v = \frac{\partial}{\partial y^A}$

Let  $M$  be  $m$ - dimensional manifold and  $D$  be  $k$ - dimensional distribution on  $M$ . Suppose that  $D$  is determined by a projection tensor  $P \in \mathfrak{S}_1^1(M)$ , i.e., for all  $p \in M$ ,  $P(T_p M) = D_p$ . Since  $P$  is a projection tensor,  $P^2 = P$  and so, we have  $(P^2)^c = (P^c)^2 = P^c$ . Thus if  $P$  is a projection tensor on  $M$  then  $P^c$  is so on  $TM$  [11].

**Definition 2.1.** [11] Let  $D$  be a distribution on  $M$  determined by  $P$ . The distribution determined by  $P^c$  on  $TM$  is called the complete lift of  $D$  and denoted by  $D^c$ .

From (2.3) we conclude that if a local basis of  $\Gamma(D)$  is the set  $\{X_1, \dots, X_k\}$  then a local basis of  $\Gamma(D^c)$  is the set  $\{X_1^c, \dots, X_k^c, X_1^v, \dots, X_k^v\}$ .

*Remark 2.1.* From definition of the complete lift of a vector field on  $M$ , it is clear that if  $X \in \Gamma(D)$ , then  $X^c \in \Gamma(D^c)$ .

**Theorem 2.1.** [11] *The complete lift  $D^c$  of a distribution  $D$  on  $M$  is integrable if and only if  $D$  is so on  $M$ .*

Now, let us consider semi-Riemannian distributions on a semi-Riemannian manifold  $(M, G)$ . If  $D$  is a distribution on  $M$ , we can write following decomposition

$$(2.4) \quad TM = D \oplus D^\perp$$

where,  $D^\perp$  is orthogonal distribution to  $D$  in  $M$ .

Denote the induced semi-Riemannian metrics on  $D$  and  $D^\perp$  by  $g$  and  $g^\perp$ , respectively. And denote by  $P$  and  $Q$  projection tensors in  $\Gamma(TM)$  on  $D$  and  $D^\perp$  respectively.

Let  $\hat{\nabla}$  be a Levi-Civita connection on  $(M, G)$ . Then according to (2.4) we write

$$(2.5) \quad \tilde{\nabla}_X P Y = \nabla_X P Y + B(X, P Y)$$

and

$$(2.6) \quad \tilde{\nabla}_X Q Y = \nabla_X^\perp Q Y + B^\perp(X, Q Y),$$

where we put

$$(2.7) \quad a) \nabla_X PY = P\hat{\nabla}_X PY \quad b) \nabla_X^\perp QY = Q\hat{\nabla}_X QY$$

and

$$(2.8) \quad a) B(X, PY) = Q(\hat{\nabla}_X PY) \quad b) B^\perp(X, QY) = P\hat{\nabla}_X QY$$

for any  $X, Y \in \Gamma(TM)$  (see for details [1]).

It is easily seen that  $\nabla$  and  $\nabla^\perp$  are linear connections on  $D$  and  $D^\perp$  respectively. In addition, in (2.5) and (2.6)  $B$  and  $B^\perp$  are  $\mathfrak{S}(M)$ -bilinear mapping:

$$\begin{aligned} B & : \Gamma(TM) \times \Gamma(D) \longrightarrow \Gamma(D^\perp) \\ B^\perp & : \Gamma(TM) \times \Gamma(D^\perp) \longrightarrow \Gamma(D). \end{aligned}$$

$\nabla$  and  $\nabla^\perp$  are called induced connections by  $\hat{\nabla}$  on  $(D, g)$  and  $(D^\perp, g^\perp)$  respectively. Also the restrictions  $B|_{\Gamma(D) \times \Gamma(D)}$  and  $B^\perp|_{\Gamma(D^\perp) \times \Gamma(D^\perp)}$  are called the second fundamental forms of  $D$  and  $D^\perp$  respectively. From (2.8) we can easily see that

$$(2.9) \quad a) B(PX, PY) = Q(\hat{\nabla}_{PX} PY) \quad b) B^\perp(QX, QY) = P(\hat{\nabla}_{QX} QY)$$

The shape operators of  $D$  and  $D^\perp$  are defined by

$$A_{QX} : \Gamma(D) \longrightarrow \Gamma(D), \quad A_{QX} PY = -B^\perp(PY, QX)$$

and

$$A_{PX}^\perp : \Gamma(D^\perp) \longrightarrow \Gamma(D^\perp), \quad A_{PX}^\perp QY = -B(QY, PX)$$

respectively, (see [1]).

Since  $G$  is parallel with respect to  $\hat{\nabla}$  we infer that the second fundamental forms and shape operators are related by

$$(2.10) \quad G(B(PX, PY), QZ) = G(A_{QZ} PX, PY)$$

$$(2.11) \quad G(B^\perp(QX, QY), PZ) = G(A_{PZ}^\perp QX, QY)$$

Finally, we write (2.5) and (2.6) by using second fundamental forms and shape operator as follows

$$(2.12) \quad \left. \begin{aligned} a) \hat{\nabla}_{PX} PY &= \nabla_{PX} PY + B(PX, PY) \\ b) \hat{\nabla}_{QX} PY &= -A_{PY}^\perp QX + \nabla_{QX} PY \end{aligned} \right\}$$

$$(2.13) \quad \left. \begin{aligned} a) \hat{\nabla}_{QX} QY &= \nabla_{QX}^\perp QY + B^\perp(QX, QY) \\ b) \hat{\nabla}_{PX} QY &= -A_{QY}^\perp PX + \nabla_{PX}^\perp QY \end{aligned} \right\}$$

### 3. SOME RESULTS FOR $D^c$

**Theorem 3.1.** *Let  $D$  be an integrable distribution on  $M$ . If  $S$  is an integral manifold of  $D$  then  $TS$  is an integral manifold of  $D^c$ .*

*Proof.* Suppose  $S$  be integral manifold of  $D$  and  $j : S \rightarrow M$  be natural injection. Then for each  $p \in S$ ,  $D_p = j_{*p}(T_p S)$ , that is, if  $X \in \Gamma(D)$  then  $X$  is tangent vector field to  $S$ . From [7]  $X^c$  is tangent to  $TS$ . By definition of  $D^c$ , we get, for each  $u \in TS$ ,  $\hat{j}_{*u}(T_u TS) = D_u^c$ , where  $\hat{j}$  is the differential mapping of  $j$ .  $\square$

**Theorem 3.2.** *If  $D$  is parallel distribution on  $M$  with respect to  $\hat{\nabla}$ , then  $D^c$  is also parallel with respect to  $\hat{\nabla}^c$ .*

*Proof.* Suppose  $D$  be parallel distribution on  $M$  with respect to  $\hat{\nabla}$  and  $\{X_1, \dots, X_k\}$  be a local basis for  $\Gamma(D)$ . Consider the equalities in follows;

$$(3.1) \quad \begin{aligned} \hat{\nabla}_{\frac{\partial}{\partial x^A}}^c X_j^c &= (\hat{\nabla}_{\frac{\partial}{\partial x^A}} X_j)^c \\ \hat{\nabla}_{\frac{\partial}{\partial y^A}}^c X_j^c &= \hat{\nabla}_{\frac{\partial}{\partial x^A}}^c X_j^v = (\hat{\nabla}_{\frac{\partial}{\partial x^A}} X_j)^v \\ \hat{\nabla}_{\frac{\partial}{\partial y^A}}^c X_j^v &= 0 \end{aligned}$$

where  $(x^1, \dots, x^m, y^1, \dots, y^m)$  is the induced local coordinate system on  $TM$ . From definition of  $D^c$ , the vector fields in right hand side of (3.1) are elements of  $\Gamma(D^c)$ . This is sufficient to prove the assertion.  $\square$

We know from literature that if  $M$  is paracompact then  $TM$  is also paracompact. From this fact, we can discuss the existence of a transversal distribution to  $D^c$ .

**Theorem 3.3.** *Let  $D_1$  and  $D_2$  be  $k_1$  and  $k_2$  - dimensional distributions on  $M$ , respectively. Suppose that  $\text{dom}D_1 \cap \text{dom}D_2 \neq \emptyset$ . Then we get,*

$$(D_1 \oplus D_2)^c = D_1^c \oplus D_2^c$$

*Proof.*  $\varphi_1 = \{X_1, \dots, X_{k_1}\}$  and  $\varphi_2 = \{Y_1, \dots, Y_{k_2}\}$  be a local basis for  $\Gamma(D_1)$  and  $\Gamma(D_2)$  respectively. In this case the set  $\varphi = \{X_1, \dots, X_{k_1}, Y_1, \dots, Y_{k_2}\}$  is linearly independent and spans  $D_1 + D_2$ . Thus,  $\bar{\varphi} = \{X_i^c, Y_a^c, X_i^v, Y_a^v\}$  is a locally basis for  $\Gamma(D_1 + D_2)^c$ .

On the other hand, since  $\varphi_1$  and  $\varphi_2$  are locally basis,  $\bar{\varphi}_1 = \{X_i^c, X_i^v\}$  and  $\bar{\varphi}_2 = \{Y_a^c, Y_a^v\}$  are locally basis for  $\Gamma(D_1^c)$  and  $\Gamma(D_2^c)$  respectively. In that case  $\bar{\varphi}_1 \cup \bar{\varphi}_2$  is a basis for  $D_1^c \oplus D_2^c$ . Since  $\text{Span}\{X_i^c, Y_a^c, X_i^v, Y_a^v\} = \text{Span}\{X_i^c, X_i^v, Y_a^c, Y_a^v\}$ , we have  $(D_1 \oplus D_2)^c = D_1^c \oplus D_2^c$ .  $\square$

In above theorem for shortness we denoted  $\{X_1^c, \dots, X_k^c, X_1^v, \dots, X_k^v\}$  by  $\{X_i^c, X_i^v\}$  and similarly the others.

**Corollary 3.1.** *Let  $D$  be a distribution on  $M$  and  $D^t$  be a transversal distribution to  $D$  on  $M$ . Then there exists a transversal distribution  $D'$  to  $D^c$  on  $TM$  and  $D' = (D^t)^c$ .*

#### 4. APPLICATIONS TO SEMI-RIEMANNIAN GEOMETRY

In this section, in literature well known notions for a distribution is discussed for  $D$  and  $D^c$ . In addition, we shall make some comparisons concerned with these notions.

**Theorem 4.1.** *Let  $(M, G)$  be  $m$ -dimensional semi-Riemannian manifold and  $D$  be  $k$ -dimensional non-degenerate distribution on  $M$ . If the index of  $D$  is of  $q$ , then  $D^c$  has index of  $k$ .*

*Proof.* Let  $\Phi = \{X_1, \dots, X_k\}$  be an orthonormal basis of  $\Gamma(D)$ . We know that the local basis of  $D^c$  is  $\tilde{\Phi} = \{X_1^c, \dots, X_k^c, X_1^v, \dots, X_k^v\}$ . It easily seen that  $\tilde{\Phi}$  consists of lightlike vector fields. In this case we consider following vector fields,

$$E_i = \frac{1}{\sqrt{2}}(X_i^c - X_i^v), \quad E_i^* = \frac{1}{\sqrt{2}}(X_i^c + X_i^v) \quad (1 \leq i \leq k)$$

Hence we get

$$\begin{aligned} G^c(E_i, E_j) &= -(G(X_i, X_j))^v \\ &= -\varepsilon_i \delta_{ij} \\ G^c(E_i^*, E_j^*) &= (G(X_i, X_j))^v \\ &= \varepsilon_i \delta_{ij} \end{aligned}$$

Here, for each  $\bar{a} \in \{q+1, \dots, k\}$ ,  $G^c(E_{\bar{a}}, E_{\bar{a}}) = -1$  and  $a \in \{1, \dots, q\}$ ,  $G^c(E_a^*, E_a^*) = -1$ . Thus we get,

$$\begin{aligned} \text{ind}D^c &= k - q + q \\ &= k \end{aligned}$$

□

In [11] the index of TM with respect to  $G^c$  was given by K. Yano and it is determined to be  $\dim M$ . In special case of  $D = TM$  we get  $\text{ind}D^c = m$ , that is,  $(TM, G^c)$  is a semi-Riemannian manifold of index  $m = \dim M$ .

**Theorem 4.2.** *Let  $D$  be  $k$ -dimensional distribution on semi-Riemannian manifold  $(M, G)$ . Then  $D$  is lightlike if and only if  $D^c$  is lightlike distribution on  $(TM, G^c)$  and if  $\text{null}D = r$  then  $\text{null}D^c = 2r$ .*

*Proof.* Suppose  $D$  be a lightlike distribution. Then we can write following decomposition,

$$(4.1) \quad D = \text{Rad}(D) \perp S(D).$$

Let a basis of  $D$  adapted to this decomposition be the set

$$\Phi = \{X_1, \dots, X_r, X_{r+1}, \dots, X_k\},$$

where  $X_1, \dots, X_r \in \Gamma(\text{Rad}(D))$ .

Since

$$\tilde{\Phi} = \{X_1^c, \dots, X_r^c, X_1^v, \dots, X_r^v, X_{r+1}^c, \dots, X_k^c, X_{r+1}^v, \dots, X_k^v\}$$

is a basis of  $D^c$ , we get the following equalities,

$$(4.2) \quad \begin{aligned} G^c(X_i^c, X_j^c) &= G^c(X_i^c, X_j^v) = 0 \quad , \quad 1 \leq i, j \leq r \\ G^c(X_i^c, X_a^c) &= G^c(X_i^c, X_a^v) = 0 \quad , \quad 1 \leq i \leq r, \quad r+1 \leq a \leq k \\ G^c(X_i^v, X_j^v) &= 0 \quad , \quad 1 \leq i, j \leq r \\ G^c(X_i^v, X_a^c) &= G^c(X_i^v, X_a^v) = 0 \quad , \quad 1 \leq i \leq r, \quad r+1 \leq a \leq k. \end{aligned}$$

From (4.2) and definition of lightlike distribution,  $D^c$  is a lightlike distribution on  $TM$  and we can write  $\Gamma(\text{Rad}(D^c)) = \text{Span}\{X_1^c, \dots, X_r^c, X_1^v, \dots, X_r^v\}$ .

Conversely, let  $D^c$  be a lightlike distribution and  $Z \in \Gamma(\text{Rad}(D^c))$ . Locally  $Z$  can be written as in follows

$$Z = h^i X_i^c + \bar{h}^i X_i^v,$$

where  $\{X_1, \dots, X_k\}$  is a local basis for  $\Gamma(D)$ . Since  $D^c$  is lightlike for all  $i = 1, \dots, k$

$$G^c(Z, X_i^c) = 0 \quad \text{and} \quad G^c(Z, X_i^v) = 0,$$

are obtained. Hence we get the following homogeneous system of linear equations

$$(4.3) \quad \left. \begin{aligned} h^i G^c(X_i^c, X_j^c) + \bar{h}^i G^c(X_i^v, X_j^c) &= 0 \\ h^i G^c(X_i^c, X_j^v) + \bar{h}^i G^c(X_i^v, X_j^v) &= 0 \end{aligned} \right\}$$

Since  $G^c(X_i^v, X_j^v) = 0$  the system (4.3) is equivalent to following system

$$(4.4) \quad \left. \begin{aligned} h^i G^c(X_i^c, X_j^c) + \bar{h}^i G^c(X_i^v, X_j^c) &= 0 \\ h^i G^c(X_i^c, X_j^v) &= 0 \end{aligned} \right\}.$$

We know from linear algebra that, for existence of a non-trivial solution of (4.4) the determinant of coefficients matrix of the system (4.4) must be zero. That is,

$$\det \begin{pmatrix} G^c(X_i^c, X_j^c) & G^c(X_i^v, X_j^c) \\ G^c(X_i^c, X_j^v) & 0 \end{pmatrix} = 0$$

or equivalently to this

$$\begin{aligned} \det \begin{pmatrix} G^c(X_i^v, X_j^c) & G^c(X_i^c, X_j^c) \\ 0 & G^c(X_i^c, X_j^v) \end{pmatrix} &= \det(G^c(X_i^v, X_j^c)) \cdot \det(G^c(X_i^c, X_j^v)) \\ &= (\det(G(X_i, X_j)))^2 \\ &= 0 \end{aligned}$$

is obtained. To satisfied this,  $\det(G(X_i, X_j))$  must be zero. This means that  $D$  is a lightlike distribution. The proof is complete.  $\square$

**Corollary 4.1.** *If  $D$  is an  $k$ - dimensional degenerate distribution and  $\text{null}D = r$ , then  $\text{ind}D^c = k - r$ .*

**Corollary 4.2.** *If  $D$  is a non-degenerate distribution on semi- Riemannian manifold  $(M, G)$  then  $D^c$  is also non-degenerate distribution on  $TM$ .*

**Lemma 4.1.** *Let  $(M, G)$  be an  $m$ - dimensional semi-Riemannian manifold and  $D$  be a distribution on  $M$ . Then  $(D^c)^\perp = (D^\perp)^c$ , where  $D^\perp$  is complementary orthogonal distribution to  $D$ .*

*Proof.* Since  $D^\perp$  is complementary orthogonal distribution to  $D$  we can write

$$TM = D \perp D^\perp.$$

From Theorem 3.3 we get  $TTM = D^c \perp (D^\perp)^c$ . Because of  $G$  is non-degenerate  $(D^\perp)^c$  must be equal to  $(D^c)^\perp$ .  $\square$

**Theorem 4.3.** *If  $D$  is  $k$ -dimensional ( $0 < k < m$ ) non-degenerate distribution on semi- Riemannian manifold  $(M, G)$ , then there exists a lightlike distribution on  $TM$  such that it includes  $D^c$  as a screen subdistribution. In addition, the transversal distribution to this is a subdistribution of  $(D^c)^\perp$ .*

*Proof.* Since  $D$  is non-degenerate distribution, we can write

$$TM = D \perp D^\perp.$$

Assume that  $\Gamma(D) = \text{Span}\{X_1, \dots, X_k\}$  and  $\Gamma(D^\perp) = \text{Span}\{\xi_{k+1}, \dots, \xi_m\}$ , where  $\{\xi_{k+1}, \dots, \xi_m\}$  is an orthonormal basis of  $\Gamma(D^\perp)$ . Then for each  $\alpha \in \{k+1, \dots, m\}$ , we have  $\xi_\alpha^c$  and  $\xi_\alpha^v$  are orthogonal to  $D^c$ . If we put

$$\tilde{D}_\alpha = \text{Span}\{X_1^c, \dots, X_k^c, X_1^v, \dots, X_k^v, \xi_\alpha^c\}$$

and

$$\hat{D}_\alpha = \text{Span}\{X_1^c, \dots, X_k^c, X_1^v, \dots, X_k^v, \xi_\alpha^v\}.$$



It is easily seen that  $\tilde{D}_\alpha$  and  $\hat{D}_\alpha$  are 1- lightlike distributions on  $TM$  and moreover  $D^c$  is a screen subdistribution in both of  $\tilde{D}_\alpha$  and  $\hat{D}_\alpha$ .

Consider two distributions such that for all  $p \in \text{dom}(D)$

$$(\tilde{\mathbf{D}}_\alpha)_p = \text{Span}\{\xi_{k+1}^c|_p, \dots, \hat{\xi}_\alpha^c|_p, \dots, \xi_m^c|_p, \xi_{k+1}^v|_p, \dots, \xi_\alpha^v|_p, \dots, \xi_m^v|_p\}$$

and

$$(\bar{\mathbf{D}}_\alpha)_p = \text{Span}\{\xi_{k+1}^c|_p, \dots, \xi_\alpha^c|_p, \dots, \xi_m^c|_p, \xi_{k+1}^v|_p, \dots, \hat{\xi}_\alpha^v|_p, \dots, \xi_m^v|_p\}$$

where the entries  $\hat{\xi}_\alpha^c$  and  $\hat{\xi}_\alpha^v$  are to be deleted. Hence we see that,

$$\begin{aligned} G^c(X_i^c, \xi_\beta^c) &= G^c(X_i^v, \xi_\beta^c) = 0, \\ G^c(X_i^c, \xi_\beta^v) &= G^c(X_i^v, \xi_\beta^v) = 0, \\ G^c(\xi_\alpha^c, \xi_\beta^c) &= 0 \\ G^c(\xi_\alpha^c, \xi_\beta^v) &= \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \end{aligned}$$

and thus we have the lightlike transversal distributions to  $\tilde{D}_\alpha$  and  $\hat{D}_\alpha$  are to be  $\text{span}\{\hat{\xi}_\alpha^v\}$  and  $\text{span}\{\hat{\xi}_\alpha^c\}$ , respectively. Thus the proof is complete.  $\square$

Now, we will investigate whether  $\tilde{D}_\alpha$  and  $\hat{D}_\alpha$  are integrable distributions or not. Suppose that  $D$  be an  $k$ - dimensional integrable distribution on  $M$  and  $\{X_1, \dots, X_k\}$  be a local basis for  $\Gamma(D)$ . From definition of integrability we write,

$$[X_i, X_j] = P_{ij}^l X_l \quad \text{for } 1 \leq i, j, l \leq k$$

Hence, by considering the equalities in (2.3), we get the followings,

$$(4.5) \quad [X_i, X_j]^c = (P_{ij}^l)^v X_l^c + (P_{ij}^l)^c X_l^v$$

$$(4.6) \quad [X_i, Y_j]^v = (P_{ij}^l)^v X_l^v$$

$$(4.7) \quad [\xi_\alpha^c, \xi_\alpha^c] = 0$$

According to (4.5), (4.6) and (4.7), whether  $\tilde{D}_\alpha$  is integrable or not depends on the vector fields  $[X_i^c, \xi_\alpha^c]$  are included in  $\Gamma(\tilde{D}_\alpha)$  for any  $i \in \{1, \dots, k\}$  and  $\alpha \in \{k+1, \dots, m\}$ . From equalities in (2.3) we know that the vector field  $[X_i^c, \xi_\alpha^c]$  is the complete lift of  $[X_i, \xi_\alpha]$ . Since  $[X_i, \xi_\alpha]$  is an element of  $\mathfrak{S}_0^1(M)$ , it is written to be

$$[X_i, \xi_\alpha] = K_{i\alpha}^l X_l + \bar{K}_{i\alpha}^\beta \xi_\beta.$$

By virtue of (2.3) we have

$$(4.8) \quad [X_i, \xi_\alpha]^c = (K_{i\alpha}^l)^v X_l^c + (K_{i\alpha}^l)^c X_l^v + (\bar{K}_{i\alpha}^\beta)^v \xi_\beta^c + (\bar{K}_{i\alpha}^\beta)^c \xi_\beta^v$$

Thus we can state the following theorem.

**Theorem 4.4.** *Let  $(M, G)$  be a semi-Riemannian manifold and  $\hat{\nabla}$  be a Levi-Civita connection on  $M$ . Suppose that  $D$  be an integrable distribution on  $M$ ,  $\{X_1, \dots, X_k\}$  and  $\{\xi_{k+1}, \dots, \xi_m\}$  be local basis for  $\Gamma(D)$  and  $\Gamma(D^\perp)$ , respectively. Then for each  $\alpha \in \{k+1, \dots, m\}$ ,  $\tilde{D}_\alpha$ , which is spanned by  $\{X_1^c, \dots, X_k^c, X_1^v, \dots, X_k^v, \xi_\alpha^c\}$ , is also integrable if and only if  $G(A_{X_i}^\perp \xi_\alpha, \xi_\alpha)$  is a constant real number, where  $A_{X_i}^\perp$  is shape operator of  $D^\perp$  with respect to  $X_i$ .*

*Proof.* Considering (4.5) - (4.8) we say that  $\tilde{D}_\alpha$  is integrable if and only if

$$\begin{aligned}\bar{K}_{i\alpha}^\beta &= 0, \quad \text{if } \beta \neq \alpha \\ \bar{K}_{i\alpha}^\beta &= a \text{ real constant, if } \beta = \alpha.\end{aligned}$$

In this case, from (4.8) we obtain  $G([X_i, \xi_\alpha], \xi_\alpha)$  must be a real constant. Taking into account that  $\hat{\nabla}$  is torsion-free and metric connection we deduce that

$$\begin{aligned}G([X_i, \xi_\alpha], \xi_\alpha) &= G(\hat{\nabla}_{X_i} \xi_\alpha - \hat{\nabla}_{\xi_\alpha} X_i, \xi_\alpha) \\ &= -G(\hat{\nabla}_{\xi_\alpha} X_i, \xi_\alpha) \\ &= G(A_{X_i}^\perp \xi_\alpha, \xi_\alpha) \\ &= \lambda_\alpha \text{ (a constant real number)}\end{aligned}$$

□

**Corollary 4.3.** *If  $D$  is an  $k$ -dimensional ( $0 < k < m$ ) integrable non-degenerate distribution on semi-Riemannian manifold  $(M, G)$ , then there exists a lightlike submanifold of  $TM$  such that it includes the tangent bundle of integral manifold of  $D$  as a non-degenerate submanifold.*

Now we give an example satisfies Theorem 8.

**Example 4.1.** Let us consider a submersion  $f : M \rightarrow \mathbb{R}$  and a distribution  $D$  on  $M$  such that for each  $p \in \text{Dom} f$ ,  $D_p = \{v_p \in T_p M \mid f_{*p}(v_p) = 0\}$ . If we put  $\xi = \frac{\hat{\nabla} f}{|\hat{\nabla} f|}$ , we can write  $\Gamma(D^\perp) = \text{Span}\{\xi\}$ . We know from [12] that the complete lift  $f^c$  is also a submersion from  $TM$  to  $\mathbb{R}$ . Then we obtain a distribution  $\tilde{D}$  on  $TM$  such that  $\hat{D}_u = \{w_u \in T_u TM \mid (f^c)_{*p}(w_u) = 0\}$ . If the set  $\{X_1, \dots, X_{n-1}\}$  is a local basis for  $\Gamma(D)$  then the set  $\{X_1^c, \dots, X_{n-1}^c, X_1^v, \dots, X_{n-1}^v\}$  and  $\{\xi^c, \xi^v\}$  are local basis for  $\Gamma(D^c)$  and  $\Gamma((D^\perp)^c)$ , respectively. From Corollary 2 in [12] we have  $\Gamma(\tilde{D}) = \text{Span}\{X_1^c, \dots, X_{n-1}^c, X_1^v, \dots, X_{n-1}^v, Z\}$ , where  $Z = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{|\hat{\nabla} f|^c}{|\hat{\nabla} f|^v}} \xi^v - \sqrt{\frac{|\hat{\nabla} f|^v}{|\hat{\nabla} f|^c}} \xi^c \right)$ . Thus we see that  $D^c \subset \hat{D}$ . In case of  $|\hat{\nabla} f|$  is a constant real number,  $Z = \xi^c$  is obtained and so  $\tilde{D}$  is a lightlike distribution. It is easily seen that radical and screen subspaces of  $\tilde{D}$  are  $\text{span}\{\xi^c\}$  and  $\Gamma(D^c)$ , respectively. In addition the lightlike transversal distribution to  $\tilde{D}$  is  $\text{span}\{\xi^v\}$ .

On the other hand, consider a level hypersurface  $\bar{N} = (f^c)^{-1}(0)$ , then we see that for a point  $u \in \bar{N}$ ,  $T_u \bar{N} = \hat{D}_u$ . Thus  $\bar{N}$  is an integral manifold of  $\tilde{D}$ . Moreover  $\bar{N}$  includes tangent bundles of level hypersurfaces of  $f$  as submanifolds. It is clear that if  $\tilde{D}$  is lightlike distribution then  $\bar{N}$  is a lightlike (degenerate) hypersurface of  $TM$  and each of tangent bundles of level hypersurfaces of  $f$  is a nondegenerate submanifold of  $\bar{N}$ .

## 5. PROLONGATIONS OF SEMI-RIEMANNIAN DISTRIBUTIONS TO TANGENT BUNDLE

Now we shall define the complete lifts of  $\nabla$ ,  $\nabla^\perp$ ,  $B$  and  $B^\perp$ . We know from [11] that the complete lift of an afin connection  $\hat{\nabla}$  on  $M$  is defined by

$$(5.1) \quad \hat{\nabla}_{X^c}^c Y^c = (\hat{\nabla}_X Y)^c,$$

for  $X, Y \in \Gamma(TM)$ . By using Theorem 3.3 and (2.4) we write

$$(5.2) \quad TTM = D^c \oplus (D^c)^\perp$$

Hence, we get

$$\hat{\nabla}_{X^c}^c P^c Y^c = P^c(\hat{\nabla}_{X^c}^c P^c Y^c) + Q^c(\hat{\nabla}_{X^c}^c P^c Y^c)$$

$$\hat{\nabla}_{X^c}^c Q^c Y^c = P^c(\hat{\nabla}_{X^c}^c Q^c Y^c) + Q^c(\hat{\nabla}_{X^c}^c Q^c Y^c)$$

for  $X, Y \in \Gamma(TM)$ . If we put

$$(5.3) \quad \bar{\nabla}_{X^c} P^c Y^c = P^c(\hat{\nabla}_{X^c}^c P^c Y^c)$$

$$(5.4) \quad \bar{\nabla}_{X^c}^\perp Q^c Y^c = Q^c(\hat{\nabla}_{X^c}^c Q^c Y^c)$$

we see that  $\bar{\nabla}$  and  $\bar{\nabla}^\perp$  are linear connections on  $D^c$  and  $(D^c)^\perp$ , respectively.

**Theorem 5.1.** *For all  $X, Y \in \Gamma(TM)$  we have the following properties for  $\bar{\nabla}$  and  $\bar{\nabla}^\perp$ .*

$$\begin{array}{lll} a) \quad \bar{\nabla}_{X^c} P^c Y^c = (\nabla_X P Y)^v & b) \quad \bar{\nabla}_{X^c} P^c Y^c = (\nabla_X P Y)^v & c) \quad \bar{\nabla}_{X^c} P^c Y^c = 0 \\ a') \quad \bar{\nabla}_{X^c}^\perp Q^c Y^c = (\nabla_X^\perp Q Y)^v & b') \quad \bar{\nabla}_{X^c}^\perp Q^c Y^c = (\nabla_X^\perp Q Y)^v & c') \quad \bar{\nabla}_{X^c}^\perp Q^c Y^c = 0. \end{array}$$

*Proof.* By virtue of (5.3), (5.4), definition of complete lift of a tensor type (1,1) and Proposition 6.3 in [11] (p.p. 43) the proof is complete.  $\square$

By using Theorem 5.1 we can find the components of  $\bar{\nabla}$  and  $\bar{\nabla}^\perp$ . Thus linear connections  $\bar{\nabla}$  and  $\bar{\nabla}^\perp$ , obtained in this way, are unique.

**Definition 5.1.** We call the above linear connections  $\bar{\nabla}$  and  $\bar{\nabla}^\perp$  the complete lifts of  $\nabla$  and  $\nabla^\perp$  respectively, and denote by  $\nabla^c$  and  $(\nabla^\perp)^c$ .

**Theorem 5.2.** *Let  $(M, G)$  be a semi-Riemannian manifold and  $D$  be a distribution on  $M$ . If  $\hat{\nabla}$  induces  $\nabla$  and  $\nabla^\perp$  on  $D$  and  $D^\perp$  respectively, then  $\hat{\nabla}^c$  induces the linear connections  $\nabla^c$  and  $(\nabla^\perp)^c$  on  $D^c$  and  $(D^c)^\perp$ , respectively.*

*In particular, if  $D = TM$  then  $\hat{\nabla}^c = \nabla^c$ .*

By considering (2.12) and (2.13) we obtain the followings

$$(5.5) \quad \left. \begin{array}{l} a) \quad \hat{\nabla}_{P^c X^c}^c P^c Y^c = \nabla_{P^c X^c}^c P^c Y^c + \bar{B}(P^c X^c, P^c Y^c) \\ b) \quad \hat{\nabla}_{Q^c X^c}^c P^c Y^c = \nabla_{Q^c X^c}^c P^c Y^c - \bar{A}_{P^c X^c}^\perp Q^c X^c \end{array} \right\}$$

and

$$(5.6) \quad \left. \begin{array}{l} a) \quad \hat{\nabla}_{Q^c X^c}^c Q^c Y^c = (\nabla^\perp)^c_{Q X} Q Y + \bar{B}^\perp(Q^c X^c, Q^c Y^c) \\ b) \quad \hat{\nabla}_{P^c X^c}^c Q^c Y^c = -\bar{A}_{Q^c Y^c} P^c X^c + (\nabla^\perp)^c_{P^c X^c} Q^c Y^c, \end{array} \right\}$$

where  $\bar{B}$  and  $\bar{B}^\perp$  the second fundamental forms of  $D^c$  and  $(D^c)^\perp$ , respectively. From (5.5), (5.6) and (5.1) we infer that

$$\begin{aligned} \bar{B}(P^c X^c, P^c Y^c) &= (B(PX, PY))^c \\ \bar{B}^\perp(P^c X^c, P^c Y^c) &= (B^\perp(QX, QY))^c. \end{aligned}$$

Since  $B$  and  $B^\perp$  are tensor fields, by considering the definition of complete lift of a tensor field we can say that  $\bar{B}$  and  $\bar{B}^\perp$  are complete lift of  $B$  and  $B^\perp$ , respectively. That is,  $\bar{B} = B^c$  and  $\bar{B}^\perp = (B^\perp)^c$ . In similar way, we have  $\bar{A}_{Q^c X^c} = (A_{QX})^c$  and  $\bar{A}_{P^c X^c}^\perp = (A_{PX}^\perp)^c$ , for all  $X \in \Gamma(TM)$ .

**Theorem 5.3.** *If  $B$  and  $B^\perp$  are second fundamental forms of  $D$  and  $D^\perp$  respectively, then  $B^c$  and  $(B^\perp)^c$  are fundamental forms of  $D^c$  and  $(D^c)^\perp$ , respectively.*

By using Theorem 5.3, we obtain the followings,

$$(5.7) \quad \left. \begin{aligned} -(B^\perp)^c(P^cY^c, Q^cX^c) &= -(B^\perp(PY, QX))^c \\ &= (A_{QX}PY)^c \\ &= (A_{QX})^cP^cY^c \end{aligned} \right\}$$

$$(5.8) \quad \left. \begin{aligned} -(B^\perp)^c(P^cY^c, Q^cX^v) &= -(B^\perp(PY, QX))^v \\ &= (A_{QX}PY)^v \\ &= (A_{QX})^vP^cY^c \end{aligned} \right\}$$

$$(5.9) \quad \left. \begin{aligned} -(B)^c(Q^cY^c, P^cX^c) &= -(B(QY, PX))^c \\ &= (A_{PX}^\perp QY)^c \\ &= (A_{PX}^\perp)^cQ^cY^c \end{aligned} \right\}$$

$$(5.10) \quad \left. \begin{aligned} -(B)^c(Q^cY^c, P^cX^v) &= -(B(QY, PX))^v \\ &= (A_{PX}^\perp QY)^v \\ &= (A_{PX}^\perp)^vQ^cY^c \end{aligned} \right\}$$

By virtue of (5.7)-(5.10) we prove the following.

**Theorem 5.4.** *Let  $(M, G)$  be a semi-Riemannian manifold and  $(D, g)$  be a semi-Riemannian distribution on  $M$ . Then the shape operators of  $D$  with respect to  $Q^cX^c$  and  $Q^cX^v$  are the complete and vertical lifts of the shape operator  $A_{QX}$  respectively. For  $D^\perp$  similar statements are true.*

Let  $g$  be an induced semi- Riemannian metric on  $D$  by  $G$ . That is, for  $\forall X, Y \in \Gamma(TM)$

$$G(PX, PY) = g(PX, PY).$$

Now we obtain a semi- Riemannian metric on  $D^c$  and  $(D^c)^\perp$ . By virtue of [11]

$$G^c(P^cX^c, P^cY^c) = (G(PX, PY))^c$$

and so we get

$$G^c(P^cX^c, P^cY^c) = (g(PX, PY))^c.$$

If we put  $\bar{g}(P^cX^c, P^cY^c) = (g(PX, PY))^c$ ,  $\bar{g}$  defines a bilinear mapping on  $\Gamma(D^c)$ , and moreover  $\bar{g}$  is unique.

We shall call  $\bar{g}$  the complete lift of  $g$ . By virtue of Theorem 4.1, index of  $\bar{g}$  is dimension of  $D$ . We denote by  $g^c$  the complete lift of  $g$ . In similar way, for the complete lift of  $g^\perp$

$$(g^\perp)^c(Q^cX^c, Q^cY^c) = G^c(Q^cX^c, Q^cY^c)$$

is obtained.

**Theorem 5.5.** *If the induced metrics from  $G$  on  $D$  and  $D^\perp$  are  $g$  and  $g^\perp$ , respectively then  $G^c$  induces  $g^c$  and  $(g^\perp)^c$  on  $D^c$  and  $(D^c)^\perp$ , respectively.*

We know from [11] that if  $D$  is integrable then  $D^c$  is so. Moreover, from Theorem 3.1 the each leaf of  $D^c$  are tangent bundle of each leaf of  $D$ . More precisely, if  $L$  is a leaf of  $D$  then  $TL$  is a leaf of  $D^c$ . In addition, in case of  $M$  is locally a Riemannian product of leaves of  $D$  and  $D^\perp$ ,  $TM$  is also locally a Riemannian product of these leaves, that is, of leaves of  $D^c$  and  $(D^c)^\perp$ .

Let  $\mathbb{D}$  and  $\mathbb{D}^\perp$  be Levi-Civita connections on  $D$  and  $D^\perp$  respectively. Then, (using Corollary 2.2 in [1]), we can prove the following.

**Theorem 5.6.** *If the induced linear connections  $\nabla$  and  $\nabla^\perp$  coincide with the Levi-Civita connections  $\mathbb{D}$  and  $\mathbb{D}^\perp$  respectively, then  $\nabla^c$  and  $(\nabla^\perp)^c$  also coincide with  $\mathbb{D}^c$  and  $(\mathbb{D}^\perp)^c$ , respectively.*

**Conclusion** *In case of  $D$  is an integrable  $n - 1$  dimensional distribution, the leaves of  $D$  are hypersurfaces of  $M$ . In [7], M. Tani, using the complete and vertical lifts, prolonged a hypersurface  $S$  of a Riemannian manifold  $M$  to  $TM$  and obtained some relations between the geometry of  $S$  in  $M$  and  $TS$  in  $TM$ . For example, if  $G$  is a Riemannian metric on  $M$  and  $g$  is induced metric on  $S$  by  $G$ , then the induced metric on  $TS$  by  $G^c$  is the complete lift of  $g$  to  $TS$ . Similarly if  $\hat{\nabla}$  is Levi-Civita connection on  $M$ , the induced Levi-Civita connection on  $TS$  by  $\hat{\nabla}^c$  is the complete lift of induced connection  $\nabla$  by  $\hat{\nabla}$ . Using these facts M. Tani established the geometry of  $(TS, g^c)$  in  $(TM, G^c)$ .*

*The present paper not only prolongs the semi-Riemannian distributions to tangent bundle but also generalizes the results obtained by M. Tani in meaning of distribution. Indeed, in case of  $D$  is integrable, the induced geometrical objects on leaves have same properties with induced geometrical objects on  $D$ .*

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