International Electronic Journal of Geometry
Volume 3 No. 2 pp. 108 - 111 (2010) ©IEJG

THE AFFINE EQUIVALENCE PROBLEM OF RULED SURFACES

ÖMER PEKŞEN

(Communicated by H. Hilmi HACISALIHOĞLU)

Abstract. Let $GL(n, R)$be the general linear group of all $n \times n$ real regular matrices. $GL(n, R)$ acts by $(g, x) \to gx$ on R^n where gx is the multiplication of a matrix g and a column vector $x \in R^n$. Let $Af(n)$ denote the group of all transformations $F : R^n \to R^n$ such that $F(x) = gx + b$ for $g \in GL(n, R)$ and $b \in R^n$. Definitions of C^∞-ruled surface in R^n, invariant differential rational functions of a C^∞-ruled surface and the affine equivalence of C^∞-ruled surfaces are introduced. A generating system of the differential field of invariant differential rational functions of a C^∞-ruled surface is described. Conditions of affine equivalence of ruled surfaces are given in terms of the generating differential invariants.

1. Introduction

This paper is concerned with the problem of equivalence of affine ruled surfaces. The first comprehensive treatment of affine ruled surfaces is given in the work of Blaschke [1]. For further developments of the subject we refer the reader [2], and the more modern texts ([3], [4], [5]), and survey paper [6]. The applications of invariant theory to differential geometry are given by Khadjiev [7]. Global integral and differential invariants of n-dimensional equi-affine curves are given in [8] and global integral and differential invariants of n-dimensional centro-affine curves are given in [9].

Ruled surfaces are investigated first by Monge [10]. The investigations of Mayer on centro-affine geometry of ruled surfaces [11] were continued by Magazinnikov [12]. Ruled surfaces in R^3 are widely applied in civil engineering and architecture [13].

Let R be the field of real numbers, and $(0, 1)$ be the open interval of R and $n \geq 1$.

Definition 1.1. A C^∞-map $f : (0, 1) \to R^n$ will be called a parametric curve in R^n.

2000 *Mathematics Subject Classification.* 53A15, 53A55.
Key words and phrases. affine differential invariant, ruled surface.
This article is the written version of author's plenary talk delivered on July 2, 2008 at the VIth National Geometry Symposium held at Bursa, Turkey. The author is supported by TUBITAK. Project number: 107T049.

Definition 1.2. a C^∞-map $x : (0,1)\times(0,1) \to R^n$ will be called a C^∞ -parametric ruled surface if $x(u,v) = x_1(u) + vx_2(u)$, where $x_1 : (0,1) \to R^n$, $x_2 : (0,1) \to R^n$ are parametric curves.

Let $GL(n,R)$ be the general linear group of all $n \times n$ real regular matrices. We denote the group

$$\{F : R^n \to R^n F(x) = gx + b,\ g \in GL(n,R),\ b \in R^n\}$$

of all affine transformations of R^n by $Af(n)$, where gx is the multiplication of a matrix g and a column vector $x \in R^n$.

Definition 1.3. Two ruled surfaces $x(u,v)$ and $y(u,v)$ in R^n will be called affine equivalent and written $x \overset{Af(n)}{\sim} y$ if there exists $g \in GL(n,R)$, $b \in R^n$ such that

$$y(u,v) = gx(u,v) + b$$

for all $u, v \in (0,1)$.

Let x_1 and x_2 be parametric curves inR^n.

Definition 1.4. A polynomial, $P\left(x_1, x_2, x_1', x_2', ..., x_1^{(m)}, x_2^{(m)}\right)$, which depends on x_1, x_2 and a finite number of their derivatives with the coefficients from R, will be called a differential polynomial of x_1, x_2. It will be denoted by $P\{x_1, x_2\}$.

The derivative $P'\{x_1, x_2\}$ of $P\{x_1, x_2\}$ can be get from the rules

$$x_i^{(0)} = x_i, (x_i^{(m-1)})' = x_i^{(m)}, i = 1, 2.$$

Definition 1.5. The quotient, $Q\langle x_1, x_2\rangle = \frac{P_1\{x_1,x_2\}}{P_2\{x_1,x_2\}}$, where P_1 and P_2 are differential polynomials of both x_1, x_2, $P_2\{x_1, x_2\} \neq 0$, is called differential rational function of x_1 and x_2.

Definition 1.6. A differential rational function Q is called affine invariant if

$$Q\langle gx_1 + b, gx_2 + b\rangle = Q\langle x_1, x_2\rangle$$

for all $g \in GL(n,R)$ and $b \in R^n$.

Affine invariant differential polynomial is defined with the same equation replacing Q by P.

There is no affine invariant polynomial except constant polynomial. But there is an affine invariant rational function which is not constant. For example, for $n = 1$, $R^n = R$ and $GL(1,R) = \{\textit{all real numbers except zero}\}$. We define $x_1 : (0,1) \to R$, $x_2 : (0,1) \to R$ as parametric curves in 1-dimensional space R, then $Q\langle x_1, x_2\rangle = \frac{x_1'}{x_2'}$ is an invariant differential rational function.

Let us denote the set of all differential rational functions of x_1 and x_2 by $R\langle x_1, x_2\rangle$. On this set for all $Q_1, Q_2 \in R\langle x_1, x_2\rangle$ and $\lambda \in R$ we define the sum $Q_1 + Q_2$, multiplication by a scalar λQ_1, the multiplication Q_1Q_2 and for all $Q_2 \neq 0$ the quotient $\frac{Q_1}{Q_2}$. Therefore $R\langle x_1, x_2\rangle$ is a field. Furthermore $R\langle x_1, x_2\rangle$ is an algebra over R. The derivative of $Q = \frac{P_1}{P_2} \in R\langle x_1, x_2\rangle$ is defined by

$$Q' = \left(\frac{P_1}{P_2}\right)' = \frac{P_1'P_2 - P_1P_2'}{(P_2)^2}.$$

Accordingly, $\forall Q_1, Q_2 \in R\langle x_1, x_2\rangle$ and $\lambda \in R$ the following properties are satisfied:

$$(Q_1+Q_2)^{'} = Q_1^{'} + Q_2^{'},$$

$$(\lambda Q_1)^{'} = \lambda Q_1^{'},$$

$$(Q_1 Q_2)^{'} = Q_1^{'} Q_2 + Q_1 Q_2^{'},$$

$$\left(\frac{Q_1}{Q_2}\right)^{'} = \frac{Q_1^{'} Q_2 - Q_1 Q_2^{'}}{(Q_2)^2}.$$

Therefore $R\langle x_1, x_2\rangle$ is a differential field and algebra.

Let $G = Af(n)$ and let us denote the set of all affine invariant differential rational functions of x_1 and x_2 by $R\langle x_1, x_2\rangle^G$. $R\langle x_1, x_2\rangle^G$ is a differential subfield and subalgebra of $R\langle x_1, x_2\rangle$.

Definition 1.7. A subset $S = \{Q_1, ..., Q_k\}$ of $R\langle x_1, x_2\rangle^G$ will be called a generating system of $R\langle x_1, x_2\rangle^G$ if the smallest differential subfield and subalgebra containing S is $R\langle x_1, x_2\rangle^G$.

2. Differential Invariants of Ruled Surfaces

For $a_1, ..., a_n \in R^n$ the determinant, $\begin{vmatrix} a_{11} & ... & a_{n1} \\ : & : & : \\ a_{1n} & ... & a_{nn} \end{vmatrix}$, where a_{ki} are the coordinates of a_k, will be denoted by simply $[a_1...a_n]$.

Theorem 2.1. *The following system*

$$w_i(x) = \frac{\left[x_1'...x_1^{(i-1)} x_1^{(n+1)} x_1^{(i+1)}...x_1^{(n)}\right]}{\left[x_1'...x_1^{(n)}\right]}, i = 1, ..., n$$

and

$$z_i(x) = \frac{\left[x_1'...x_1^{(i-1)} x_2 - x_1 x_1^{(i+1)}...x_1^{(n)}\right]}{\left[x_1'...x_1^{(n)}\right]}, i = 1, ..., n$$

is a generating system of $R\langle x_1, x_2\rangle^G$.

There are $2n$ elements in this system.

3. The Problem of Affine Equivalence

Definition 3.1. A parametric ruled surface $x(u,v) = x_1(u) + vx_2(u)$ is called regular parametric ruled surface if $\left[x_1'...x_1^{(n)}\right] \neq 0$ for all $u \in (0,1)$.

Let $x(u,v)$ and $y(u,v)$ be two ruled surfaces and $f \in R\langle x_1, x_2\rangle^G$, where $G = Af(n)$. If $x \overset{G}{\sim} y$ then $f(x) = f(y)$ always satisfied. Conversely, is it always possible to say that $x \overset{G}{\sim} y$, when $f(x) = f(y)$?. There are examples to show that this is not true.

Theorem 3.1. *Let x and y be two regular parametric ruled surfaces. If*

$$w_i(x) = w_i(y), i = 1, ..., n$$

and

$$z_i(x) = z_i(y), i = 1, ..., n$$

then $x \overset{G}{\sim} y$.

References

[1] Blaschke, W., Affine Differentialgeometrie, Berlin, 1923.
[2] Salkovski, E., Affine Differentialgeometrie, W. De Gruyter, Berlin, 1934.
[3] Schirokov, P.A. & Schirokov, A. P., Affine Differentialgeometrie, Teubner, Leipzig, 1962.
[4] Li, A.M., Simon, U., Zhao, G., Global Affine Differential Geometry of Hypersurfaces, Walter de Gruyter, Berlin, New York, 1993.
[5] Nomizu, K. & Sasaki, T., Affine Differential Geometry: Geometry of Affine Immersions, Cambridge Univ. Press, 1994.
[6] Simon, U., Recent developments in afine differential geometry, Diff. Geo. And its App., Proc. Conf. Dubrovnik, Yugosl. (1988), 327-347.
[7] Khadjiev, Dj., An Application of the Invariant Theory to Differential Geometry (Russian),Fan, Tashkent, 1988.
[8] Khadjiev, Dj. & Pekşen, Ö., The complete system of global differential and integral invariants for equi-affine curves, Dif. Geo. and its App. 20, (2004), 167-175.
[9] Pekşen, Ö. & Khadjiev, Dj., On invariants of curves in centro-affine geometry, J. Math. Kyoto Univ. 44-3, (2004), 603-613.
[10] Monge, G., Application de l'analyse a'la geometrie, Paris, 1807 at 1809.
[11] Mayer, O., Geometrie centro-affine diferentielle des surfaces, Ann. Sci. Univ., Jassy, 21, (1934), 1-77.
[12] Magazinnikov, L.I., Centro-affine theory of ruled surfaces (Russian), Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161,** (1962), 101–110.
[13] Katou, M., Center maps of afine minimal ruled hypersurfaces, Interdisiplinary Information Sciences, 12-1, (2006), 53-56.

Department of Mathematics, Karadeniz Technical University,Trabzon-TURKEY
E-mail address: `peksen@ktu.edu.tr`