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TANGENTIALLY CUBIC SUBMANIFOLDS OF $\mathbb{E}^m$

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Abstract. In the present study we consider the submanifold M of $\mathbb{E}^m$ satisfying the condition $\langle \Delta H, e_i \rangle = 0$, where H is the mean curvature of M and $e_i \in TM$. We call such submanifolds tangentially cubic. We proved that every null 2- type submanifold M of $\mathbb{E}^m$ is tangentially cubic. Further, we prove that the pointed helical geodesic surfaces of $\mathbb{E}^5$ with constant Gaussian curvature are tangentially cubic.

1. Introduction

Let $x : M \to \mathbb{E}^m$ be an isometric immersion from an n-dimensional connected manifold M into the Euclidean m-space $\mathbb{E}^m$. With respect to the Riemannian metric g on M induced from the Euclidean metric of the ambient space $\mathbb{E}^m$, M is a Riemannian manifold (M, g). Denote by Δ the Laplacian operator of the Riemannian manifold (M, g). One of the most important formulas in Differential Geometry of submanifolds is

$$\Delta x = -nH, \tag{1.1}$$

where H is the mean curvature vector field of the immersion, and x also denotes the position vector field of M in $\mathbb{E}^m$. Formula (1.1) implies that the immersion is minimal ($H = 0$) if and only if the immersion is harmonic, that is $\Delta x = 0$. An isometric immersion $x : M \to \mathbb{E}^m$ is called *biharmonic* if we have $\Delta^2 x = 0$, that is $\Delta H = 0$. It is obvious that minimal immersions are biharmonic [3].

M is said to be of null 2-type submanifold of $\mathbb{E}^m$ if each component of the position vector x has a finite spectral decomposition (see, [4])

$$x = x_0 + x_1,\ \Delta x_0 = 0,\ \Delta x_1 = cx_1, \tag{1.2}$$

for some non-constant vectors x_0 and x_1 on M, where c is a non-zero constant.

In [2] the present authors considered the differentiable curve γ in $\mathbb{E}^m$ satisfying the relation $\left\langle \Delta H, \gamma' \right\rangle = 0$. Such curves are called tangentially cubic, where H is the mean curvature vector of γ.

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In the present study we extend the results in [2] to the submanifolds of $\mathbb{E}^m$. The submanifolds satisfying the condition

$$\langle \Delta H, e_i \rangle = 0, \quad 1 \leq i \leq n, \ e_i \in TM \tag{1.3}$$

are called tangentially cubic ($T.C$ - submanifolds). We show that the hypercylinder over the tangentially cubic curves is also tangentially cubic. Further, we give some examples of $T.C$-submanifolds.

In [6] Y.H. Kim studied the submanifolds which has pointed helical geodesics with the same constant Frenet curvatures. We prove that the helical geodesics of $\mathbb{E}^5$ with constant Gaussian curvature are tangentially cubic surfaces.

2. Basic Concepts

Let $x : M \to \mathbb{E}^m$ be an isometric immersion from an n-dimensional, connected manifold M into the Euclidean m-space $\mathbb{E}^m$. Let ∇ and $\widetilde{\nabla}$ denote the covariant derivatives of M and $\mathbb{E}^m$ respectively. Thus $\widetilde{\nabla}_X$ is just the directional derivative in the direction X in $\mathbb{E}^m$. Then for tangent vector fields X, Y the *second fundamental form* h of the immersion x is defined by $h(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y$. For a vector field ξ normal to M we put $\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi$, where $-A_\xi X$ (resp. $D_X \xi$) denotes the tangential and normal component of $\widetilde{\nabla}_X \xi$ and D is the normal connection of M.

Let us choose a local field of orthonormal frame $\{e_1, e_2, ...e_n, e_{n+1}, ..., e_m\}$ in $\mathbb{E}^m$ such that, restricted to M, the vectors $e_1, e_2, ...e_n$ tangent to M and $e_{n+1}, ...e_m$ are normal to M. We denote by $\{w^1, w^2, ..., w^m\}$ the field of dual frames. The *structure equations* of $\mathbb{E}^m$ are given by (see [3])

$$\widetilde{\nabla}_{e_i} e_j = \sum_{k=1}^{n} w_j^k(e_i) e_k + \sum_{\alpha=n+1}^{m} w_j^\alpha(e_i) e_\alpha. \tag{2.1}$$

The *mean curvature vector* of M is

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{2.2}$$

If $H = 0$, then M is said to be *minimal*.

The Laplace operator Δ acting on a vector valued function V is given by

$$\Delta V = \sum_{i=1}^{n} \left[\widetilde{\nabla}_{\nabla_{e_i} e_i} V - \widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} V \right]. \tag{2.3}$$

We define the Laplacian Δ^D with respect to the normal connection D

$$\Delta^D H = \sum_{i=1}^{n} \left[D_{\nabla_{e_i} e_i} H - D_{e_i} D_{e_i} H \right]. \tag{2.4}$$

3. Main Results

Let M be a H-hypersurface in $\mathbb{E}^{n+1}$ then applying (2.3) to H , since $H = \alpha N$, we find

$$\Delta H = 2 A_N grad\alpha + n\alpha grad\alpha + (\Delta\alpha + S\alpha) N, \tag{3.1}$$

where α and S stand for the mean curvature and the square of the length of the second fundamental form, respectively. Suppose that the hypersurface M in the Euclidean space $\mathbb{E}^{n+1}$ is biharmonic. Then from (3.1) we have

$$2Agrad\alpha + n\alpha grad\alpha = 0 \tag{3.2}$$

and

$$\Delta\alpha + S\alpha = 0. \tag{3.3}$$

The relations (3.2) and (3.3) are necessary and sufficient conditions for M to be biharmonic. The hypersurfaces which satisfy (3.2) are called *H-hypersurfaces* [5].

First we prove the following result.

Proposition 3.1. *Every H-hypersurface is a trivial $T.C$-hypersurface.*

Proof. Let M be a H-hypersurface in $\mathbb{E}^{n+1}$ then using (3.2) with (3.1) we get

$$\Delta H = (\Delta\alpha + S\alpha)N. \tag{3.4}$$

So by the use of (3.4) we get

$$\langle \Delta H, e_i \rangle = 0,$$

which completes the proof. □

Proposition 3.2. *Every biharmonic submanifold of $\mathbb{E}^m$ is trivial $T.C$-submanifold.*

Proof. Let M be an n-dimensional connected submanifold of $\mathbb{E}^m$. Then by the Beltrami formula (1.1) we get

$$\langle \Delta H, e_i \rangle = -\frac{1}{n} < \Delta^2 x, e_i >, \quad 1 \leq i \leq n, \tag{3.5}$$

which completes the proof. □

Lemma 3.1. [4] *Let M be an n-dimensional submanifold of an Euclidean space $\mathbb{E}^m$. If there is a constant $c \neq 0$ such that $\Delta H = cH$, then M is either of 1-type or of null 2-type.*

Proposition 3.3. [3] *Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\mathbb{E}^m$. Let $e_{n+1}, ..., e_m$ be mutually orthogonal unit normal vector fields of M in $\mathbb{E}^m$ such that e_{n+1} is parallel to the mean curvature vector H of M in $\mathbb{E}^m$ then*

$$\Delta H = \Delta^D H + \|A_{n+1}\|^2 H + a(H) + tr(\widetilde{\nabla} A_H), \tag{3.6}$$

where

$$a(H) = \sum_{r=n+2}^{m} tr(A_H A_r)e_r, \;\; A_r = A_{e_r}, n+2 \leq r \leq m, \tag{3.7}$$

$$\|A_{n+1}\|^2 = tr(A_{n+1}A_{n+1}),$$

and

$$tr(\widetilde{\nabla} A_H) = \sum_{i=1}^{n} [(\nabla_{e_i} A_H)e_i + A_{D_{e_i}H} e_i]. \tag{3.8}$$

Lemma 3.2. [4] *Let M be an $n-$dimensional submanifold of an Euclidean space $\mathbb{E}^m$ such that M is not of 1-type. Then M is of null 2-type if and only if we have*

$$tr(\widetilde{\nabla}A_H) = 0 \tag{3.9}$$

and

$$\Delta H = \Delta^D H + \|A_{n+1}\|^2 H + a(H). \tag{3.10}$$

Consequently we have the following result.

Proposition 3.4. *Let M be an n-dimensional submanifold of an Euclidean space $\mathbb{E}^m$. If M is of null 2-type (i.e. not of 1-type) then M is a T.C-submanifold.*

Proof. If M is of null 2-type then (3.9) and (3.10) are full filled. So using (3.10) we get

$$< \Delta H, e_i >= 0,$$

which completes the proof. □

Definition 3.1. Consider the case when $M = M_1 \times M_2$ is a product submanifold. That is, there exist isometric embeddings

$$f_1 : M_1 \to \mathbb{E}^{m_1+d_1}\ ,\ f_2 : M_2 \to \mathbb{E}^{m_2+d_2}. \tag{3.11}$$

We put $m = m_1 + m_2, d = d_1 + d_2$ so that $\mathbb{E}^{m+d} = \mathbb{E}^{m_1+d_1} + \mathbb{E}^{m_2+d_2}$. Then the function $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ defines an embedding $f : M \to \mathbb{E}^{m+d}$ which is called the product immersion of f_1, f_2 (see, [7]).

Theorem 3.1. [1] *Let $f_1 : M_1 \to \mathbb{E}^{m_1+d_1}$ and $f_2 : M_2 \to \mathbb{E}^{m_2+d_2}$ be two isometric immersions of closed manifolds and Δ, Δ_1 and Δ_2 be the Laplacian of the submanifolds $M = M_1 \times M_2$, M_1 and M_2 respectively. Then*

$$\Delta = \Delta_1 + \Delta_2.$$

Theorem 3.2. *Let γ be a differentiable curve in $\mathbb{E}^m$. If γ is a T.C-curve then the cylinder over γ is also a T.C-surface.*

Proof. Let $\gamma(s) = (\gamma_1(s), \gamma_2(s), ..., \gamma_n(s))$ be the curve in $\mathbb{E}^m$. The cylinder over γ will have the parametrization

$$x = (s, u_1, u_2, ..., u_{n-1}) = (\gamma(s), u_1, u_2, ..., u_{n-1}).$$

Let $\gamma'(s) = v_1, v_2, ..., v_n$ be the oriented frame field of γ. We chose an orthonormal tangent frame of the cylinder by $\left\{x_s, x_{u_1}, x_{u_2}, ..., x_{u_{n-1}}\right\}$, where

$$\begin{aligned} x_s &= (v_1, 0, ..., 0) \\ x_{u_j} &= (0, 0, ..., 1, ..., 0), 1 \leq j \leq n-1. \end{aligned}$$

A simple calculation gives

$$\nabla_{x_s} x_s = 0,\ \nabla_{x_s} x_{u_j} = 0 = \nabla_{x_{u_j}} x_s = 0,\ \nabla_{x_{u_j}} x_{u_k} = 0$$

and

$$\begin{aligned} h(x_s, x_s) &= (\gamma_1''(s), \gamma_2''(s), ..., \gamma_n''(s), 0, 0, ..., 0), \\ h(x_s, x_{u_j}) &= h(x_{u_j}, x_{u_k}) = 0. \end{aligned}$$

So the mean curvature vector of the cylinder will become

$$\begin{aligned} H &= \frac{1}{n}\sum_{i=1}^{n-1}\{h(x_s,x_s)+h(x_{u_i},x_{u_i})\} \\ &= h(x_s,x_s), \end{aligned}$$

which is equal to the second derivative of γ with $n-1$ zeros will be added. If γ is a $T.C$- curve then the cylinder $\gamma \times \mathbb{E}^{n-1}$ will be a $T.C$-surface. □

We give the following examples.

Example 3.1. The helix in $\mathbb{S}^3 \subset \mathbb{E}^4$ given by the parametrization

$$\gamma(s) = (\cos\phi\cos(as), \cos\phi\sin(as), \sin\phi\cos(bs), \sin\phi\sin(bs)). \tag{3.12}$$

is a $T.C$-curve in $\mathbb{S}^3 \subset \mathbb{E}^4$(see, [2]). Hence, the cylinder M over γ given with the parametrization

$$x(s,t) = (\cos\phi\cos(as), \cos\phi\sin(as), \sin\phi\cos(bs), \sin\phi\sin(bs), t) \tag{3.13}$$

is a $T.C$-surface.

Example 3.2. The product manifold of Catenoid with the circle $S^1(b)$ is given by the parametrization

$$x(s,u_1,u_2) = (b\cos s, b\sin s, a\cosh u_1\cos u_2, a\cosh u_1\sin u_2, au_1), \tag{3.14}$$

In [1] it has been shown that the product immersion $x(s,u_1,u_2)$ is of null 2-type. So, by Theorem 3.2 the product submanifold given with the parametrization (3.14) is a $T.C$-submanifold.

In [6] Y.H. Kim studied the submanifolds which has pointed helical geodesics with the same constant Frenet curvatures. He proved the following result.

Proposition 3.5. [6] *Let $M \subset \mathbb{E}^5$ be a compact connected surface fully lies in $\mathbb{E}^5$. If M has pointed helical geodesics with the same constant Frenet curvatures then it has the parametrization*

$$\begin{aligned} x(s,\theta) &= (\frac{1}{k}\sin ks\cos\theta, \frac{1}{k}\sin ks\sin\theta, \frac{1}{k^2}(1-\cos ks)\left(k-\frac{2a^2}{k}\sin^2\theta\right), \\ &\qquad \frac{a}{k^2}(1-\cos ks)\sin 2\theta, \frac{b}{k^2}(1-\cos ks)\sin^2\theta) \end{aligned} \tag{3.15}$$

where k is the Frenet curvature of the helical geodesic on M and

$$a = \|h(e_1,e_2)\|, b^2 = k^2 - \frac{(k^2-2a^2)^2}{k^2}.$$

Proposition 3.6. *Let $M \subset \mathbb{E}^5$ be a compact connected surface fully lies in $\mathbb{E}^5$. If M has pointed helical geodesics with the same constant Frenet curvatures and has constant Gaussian curvature then it is a $T.C$-surface.*

Proof. Let M be a proper surface of $\mathbb{E}^5$. If M has pointed helical geodesics with the same constant Frenet curvatures then by Proposition 3.5 it has the parametrization of the form (3.15). Further, we assume that the Gaussian curvature of M is constant. So by Lemma 2.14 of [6] the Laplacian operator Δ of M is given by

$$\Delta = -\left(\frac{\partial^2}{\partial s^2} + \frac{1}{G}\frac{\partial^2}{\partial\theta^2}\right) - \frac{1}{2}\frac{\partial}{\partial s}(\log G)\frac{\partial}{\partial s} \tag{3.16}$$

where

$$G = \frac{1}{k^2}\sin^2 ks + \frac{1}{k^2}(1-\cos ks)^2.$$

Using Beltrami formula (1.1) and computing H where the means of (3.16), we obtain the following

$$\Delta H - \frac{3}{2}k^2 H = 0. \tag{3.17}$$

So, by the use of Lemma 3.1 and Proposition 3.4 M becomes a $T.C$-surface. □

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