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CLASSIFICATION OF FLAT LAGRANGIAN H-UMBILICAL SUBMANIFOLDS IN PARA-KÄHLER n-PLANE

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Abstract. Lagrangian submanifolds of Kähler manifolds have been studied extensively since early 1970s. On the other hand, the study of Lagrangian submanifolds of para-Kähler manifolds was initiated only recently by the author in [8]. In a subsequence paper [9] the author defines the notion of Lagrangian H-submanifolds of para-Kähler manifolds and classifies non-flat Lagrangian H-umbilical submanifolds of the para-Kähler n-plane $(\mathbb{E}^{2n}_n, g_0, P)$. The main purpose of this paper is thus to classify all flat Lagrangian H-umbilical submanifolds of $(\mathbb{E}^{2n}_n, g_0, P)$.

1. Introduction.

An almost para-Hermitian manifold is a manifold M endowed with an almost product structure $P \neq \pm I$ and a pseudo-Riemannian metric g such that

$$P^2 = I, \;\; g(PX, PY) = -g(X, Y) \tag{1.1}$$

for X, Y tangent to M, where I is the identity map. It follows from (1.1) that P maps space-like vectors into time-like vectors and verse visa. Consequently, the dimension of M is even and the signature of g is (n, n), where $\dim M = 2n$.

Let ∇ denote the Levi-Civita connection of M. An almost para-Hermitian manifold is called *para-Kähler* if it satisfies $\nabla P = 0$ identically.

Properties of para-Kähler manifolds were first studied by R. K. Rashevski in 1948 in which he considered a neutral metric of signature (n, n) defined from a potential function on a locally product $2n$-manifold [17]. He called such manifolds stratified space. Para-Kähler manifolds were explicitly defined by B. A. Rozenfeld in 1949 [18]. Rozenfeld compared Rashevskij's definition with Kähler's definition in the complex case and established the analogy between Kähler and para-Kähler ones. Such manifolds were also defined independently by H. S. Ruse in 1949 [19].

The Levi-Civita connection of a para-Kähler manifold (M, g, P) preserves P, equivalently, its holonomy group Hol_p, $p \in M$, preserves the eigenspace decomposition $T_pM = T^+_p \oplus T^-_p$. The parallel eigendistributions $T^{\pm}$ of P are g-isotropic integrable distributions. Moreover, they are Lagrangian distributions with respect

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to the Kähler form $\omega = g \circ P$, which is parallel and hence closed. The leaves of these distributions are totally geodesic submanifolds. Thus, from the standpoint of symplectic manifolds, a para-Kähler structure can be regarded as a pair of complementary integrable Lagrangian distributions (T^+, T^-) on a symplectic manifold (M, ω). Such a structure is often called a bi-Lagrangian structure or a Lagrangian 2-web (cf. [14]).

There exist many para-Kähler manifolds, for instance, a homogeneous manifold $M = G/H$ of a semisimple Lie group G admits an invariant para-Kähler structure (g, P) if and only if it is a covering of the adjoint orbit $\mathrm{Ad}_G h$ of a semisimple element h (see [15] for details). Para-Kähler manifolds have been applied in supersymmetric field theories as well as in string theory in recent years, see for instance, [11, 12, 13].

Analogous to totally real submanifolds in an almost Hermitian manifold (cf. [10]), we call a space-like submanifold N in an almost para-Hermitian manifold (M^{2m}_m, g, P) *totally real* if P maps each tangent space T_pN, $p \in N$, into the normal space $T_p^\perp N$. In particular, we call N *Lagrangian* if $P(T_pN) = T_p^\perp N$ for each $p \in N$.

Lagrangian submanifolds in Kähler manifolds have been studied extensively since early 1970s (see [6, 7] for surveys). In contrast, no results on Lagrangian submanifolds in para-Kähler manifolds are known (see [14, Section 5: Open Problems], in particular, see Open Problem (3)). This is exactly the reason the author initiated recently the study of Lagrangian submanifolds of para-Kähler manifolds in [8] in which two optimal inequalities for Lagrangian submanifolds in the para-Kähler n-plane $(\mathbb{E}^{2n}_n, g_0, P)$ were proved. Lagrangian submanifolds satisfying either equality are also completely classified in [8].

In another paper [9] the author defines the notion of Lagrangian H-submanifolds of para-Kähler manifolds and classifies non-flat Lagrangian H-umbilical submanifolds of the para-Kähler n-plane $(\mathbb{E}^{2n}_n, g_0, P)$. In this paper we classify all flat Lagrangian H-umbilical submanifolds of $(\mathbb{E}^{2n}_n, g_0, P)$.

2. Preliminaries.

Let $\psi : N \to M^m_s$ be an isometric immersion of a Riemannian n-manifold N into a pseudo-Riemannian m-manifold M^m_s with index s. Denote by ∇' and ∇ the Levi-Civita connections on N and M^m_s, respectively.

For vector fields X, Y tangent to N and ξ normal to N, the formulas of Gauss and Weingarten are given respectively by (cf. [1, 2]):

$$\nabla_X Y = \nabla'_X Y + h(X, Y), \tag{2.1}$$

$$\nabla_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

where h, A and D are the second fundamental form, the shape operator, and the normal connection of N in M^m_s.

The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle \,, \tag{2.3}$$

where $\langle\ ,\ \rangle$ is the inner product. The *mean curvature vector* is defined by

$$H = \left(\frac{1}{n}\right) \mathrm{trace}\, h. \tag{2.4}$$

The equations of Gauss, Codazzi and Ricci are given respectively by

$$R'(X,Y)Z = R(X,Y)Z + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.5}$$

$$(R(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{2.6}$$

$$g(R^D(X,Y)\xi,\eta) = g(R(X,Y)\xi,\eta) + g([A_\xi, A_\eta]X,Y) \tag{2.7}$$

for X,Y,Z tangent to N and ξ,η normal to N, where R' (respectively, R) is the curvature tensor of N (respectively, of M^m_s), $(R(X,Y)Z)^{\perp}$ is the normal component of $R(X,Y)Z$, and $\bar{\nabla}h$ and R^D are defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla'_X Y,Z) - h(Y,\nabla'_X Z), \tag{2.8}$$

$$R^D(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}. \tag{2.9}$$

3. Para-Kähler manifolds

Definition 3.1. An *almost para-Hermitian manifold* is a manifold M endowed with an almost product structure $P \neq \pm I$ and a pseudo-Riemannian metric g such that

$$P^2 = I, \quad \text{and} \quad g(Pv,Pw) = -g(v,w) \tag{3.1}$$

for vectors $v,w \in T_p(M)$, $p \in M$, where I is the identity map.

The dimension of an almost para-Hermitian manifold M is even and the metric is neutral.

Definition 3.2. An almost para-Hermitian manifold (M,g,P) is called *para-Kähler* if it satisfies $\nabla P = 0$ identically, where ∇ is the Levi-Civita connection of M.

The simplest example of para-Kähler manifolds is the pseudo-Euclidean $2n$-space $\mathbb{E}^{2n}_n$ endowed with the neutral metric:

$$g_0 = -\sum_{i=1}^{n} dx_i^2 + \sum_{j=1}^{n} dy_j^2 \tag{3.2}$$

with P being defined by

$$P\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad P\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j} \tag{3.3}$$

for $j = 1,\ldots,n$. We simply called $(\mathbb{E}^{2n}_n, g_0, P)$ the *para-Kähler n-plane.*

For a para-Kähler manifold M, (3.1) implies that

$$g(Pv,w) + g(v,Pw) = 0, \quad v,w \in T_pM, \; p \in M. \tag{3.4}$$

In particular, we obtain

$$g(v,Pv) = 0. \tag{3.5}$$

If $\{v,Pv\}$ spans a non-degenerate plane section, the sectional curvature

$$H^{P(v)} = K(v \wedge Pv)$$

of Span$\{v,Pv\}$ is called a *para-sectional curvature.*

A *para-Kähler space form*, by definition, is a para-Kähler manifold of constant para-sectional curvature. The Riemann curvature tensor of a para-Kähler space forms $M^{2n}_n(4c)$ of constant para-sectional curvature $4c$ satisfies

$$\begin{aligned} R(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle PY,Z\rangle PX \\ - \langle PX,Z\rangle PY + 2\langle X,PY\rangle PZ\}. \end{aligned} \tag{3.6}$$

The para-Kähler n-plane $(\mathbb{E}^{2n}_n, g_0, P)$ is the standard model of flat para-Kähler space form.

Definition 3.3. Let $z : I \to (\mathbb{E}^{2n}_n, g_0, P)$ be a unit speed curve in $(\mathbb{E}^{2n}_n, g_0, P)$. A normal vector field $F(t)$ along $z(t)$ is called a *parallel normal vector field* if $F'(t)$ is tangent to the curve z at each point, i.e., the covariant derivative of F along the curve has no normal component along the curve.

Put

$$S^{2n-1}_n = \{\mathbf{x} \in \mathbb{E}^{2n}_n : \langle \mathbf{x}, \mathbf{x}\rangle = 1\}. \tag{3.7}$$

Then S^{2n-1}_n is the unit pseudo hypersphere of the para-Kähler n-plane $(\mathbb{E}^{2n}_n, g_0, P)$. It follows from (3.5) that $\langle \mathbf{x}, P\mathbf{x}\rangle = 0$. Thus, for a unit speed curve $z : I \to S^{n-1}_n \subset \mathbb{E}^{2n}_n$, Pz is always tangent to S^{2n-1}_n.

Definition 3.4. A unit speed curve $z : I \to S^{n-1}_n \subset \mathbb{E}^{2n}_n$ is called *para-Legendre* if $\langle z'(t), Pz(t)\rangle = 0$ for each $t \in I$.

For a unit speed space-like para-Legendre curve $z : I \to S^{n-1}_n \subset \mathbb{E}^{2n}_n$, we have

$$\langle z, z\rangle = \langle z', z'\rangle = 1, \ \ \langle z, z'\rangle = \langle z, Pz\rangle = \langle z', Pz\rangle = 0. \tag{3.8}$$

Thus we may extend z, Pz, z', Pz' to an orthonormal frame

$$z, Pz, z', Pz', w_3, Pw_3, \ldots, w_n, Pw_n \tag{3.9}$$

along the curve. From (3.8) we find

$$\langle z'', Pz\rangle = \langle z'', z'\rangle = 0, \ \ \langle z'', z\rangle = -1. \tag{3.10}$$

Hence it follows from (3.8)-(3.10) that

$$z''(t) = -z(t) + \mu(t)Pz'(t) - \sum_{j=3}^{n} a_j(t)w_j(t) + \sum_{j=3}^{n} b_j(t)Pw_j(t) \tag{3.11}$$

for some functions $\mu, a_3, \ldots, a_n, b_3, \ldots, b_n$.

Definition 3.5. A unit speed space-like para-Legendre curve $z : I \to S^{n-1}_n \subset \mathbb{E}^{2n}_n$ is called *special para-Legendre* if (3.11) reduces to

$$z''(t) = -z(t) + \mu(t)Pz'(t) - \sum_{j=3}^{n} a_j(t)w_j(t) \tag{3.12}$$

for some parallel normal vector fields $w_3, \ldots, w_n$.

4. Lagrangian submanifolds of Para-Kähler manifolds

We recall the following results from [8, 9].

Lemma 4.1. *Let N be a Lagrangian submanifold of a para-Kähler manifold M^{2n}_n. Then we have*

(i) $P(\nabla'_X Y) = D_X(PY)$,

(ii) $A_{PX}Y = -P(h(X,Y))$,

(iii) $\langle h(X,Y), PZ\rangle = \langle h(Y,Z), PX\rangle = \langle h(Z,X), PY\rangle$,

(iv) $P(R'(X,Y)Z) = R^D(X,Y)PZ$

for X, Y, Z tangent to N.

The equations of Gauss and Codazzi for a Lagrangian submanifold N of a para-Kähler space form $M^{2n}_n(4c)$ are given respectively by

$$R'(X,Y;Z,W) = \left\langle A_{h(Y,Z)}X, W\right\rangle - \left\langle A_{h(X,Z)}Y, W\right\rangle + c\left(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\right), \tag{4.1}$$

$$(\bar\nabla_X h)(Y,Z) = (\bar\nabla_Y h)(X,Z) \tag{4.2}$$

for X, Y, Z, W tangent to N.

If we put $h = P\circ\sigma$ (equivalently $\sigma = P\circ h$), then (3.1) and Lemma 4.1(iii) imply that

$$\left\langle A_{h(Y,Z)}X, W\right\rangle = -\left\langle \sigma(\sigma(Y,Z),X), W\right\rangle.$$

Therefore equation (4.1) of Gauss can be rephrased as

$$R'(X,Y)Z = \sigma(\sigma(X,Z),Y) - \sigma(\sigma(Y,Z),X) + c\langle Y,Z\rangle X - c\langle X,Z\rangle Y.$$

It follows Lemma 4.1(i) that the equation of Ricci is nothing but the equation of Gauss for Lagrangian submanifolds of para-Kähler manifolds.

The fundamental existence and uniqueness theorems for Lagrangian submanifolds in $(\mathbb{E}^{2n}_n, g_0, P)$ are the following.

Existence Theorem. *Let N be a simply-connected Riemannian n-manifold. If σ is a TN-valued symmetric bilinear form on N such that*

(a) $g(\sigma(X,Y),Z)$ *is totally symmetric,*

(b) $(\nabla\sigma)(X,Y,Z)$ *is totally symmetric,*

(c) $R'(X,Y)Z = \sigma(\sigma(X,Z),Y) - \sigma(\sigma(Y,Z),X)$,

then there is a Lagrangian isometric immersion $L : N \to (\mathbb{E}^{2n}_n, g_0, P)$ whose second fundamental form is given by $h = P\circ\sigma$.

Uniqueness Theorem. *Let $L_1, L_2 : N \to (\mathbb{E}^{2n}_n, g_0, P)$ be two Lagrangian isometric immersions of a Riemannian n-manifold N with second fundamental forms h^1 and h^2, respectively. If*

$$g(h^1(X,Y), PL_{1\star}Z) = g(h^2(X,Y), PL_{2\star}Z)$$

for all vector fields X, Y, Z tangent to N, then there is an isometry Φ of $(\mathbb{E}^{2n}_n, g_0, P)$ such that $L_1 = \Phi\circ L_2$.

A pseudo-Riemannian submanifold N of a pseudo-Riemannian manifold is called *totally umbilical* if its second fundamental form satisfies

$$h(X,Y) = \langle X,Y\rangle H \tag{4.3}$$

for X,Y tangent to N.

The following result was proved in [9].

Proposition 4.1. *Every totally umbilical Lagrangian submanifold of a para-Kähler space form $M^{2n}_n(4c)$ with $n \geq 2$ is totally geodesic.*

Lagrangian H-umbilical submanifolds in Kähler manifolds were introduced in [3, 4]. Such submanifolds in complex space forms were classified in [3, 4, 5].

The following definition of Lagrangian H-umbilical submanifolds of para-Kähler manifolds was given in [9].

Definition 4.1. A Lagrangian submanifold N of a para-Kähler manifold is called *Lagrangian H-umbilical* if the second fundamental form satisfies

$$\begin{aligned} &h(e_1,e_1) = \lambda Pe_1,\ h(e_2,e_2) = \cdots = h(e_n,e_n) = \mu Pe_1, \\ &h(e_1,e_j) = \mu Pe_j,\ h(e_j,e_k) = 0,\ 2 \leq j \neq k \leq n, \end{aligned} \tag{4.4}$$

for some functions λ,μ with respect to some orthonormal local frame $\{e_1,\ldots,e_n\}$.

In view of Proposition 4.1, Lagrangian H-umbilical submanifolds are the simplest Lagrangian submanifolds next to totally geodesic ones.

The following classification theorem was obtained in [9].

Theorem 4.1. *Let $L : N \to (\mathbb{E}^{2n}_n, g_0, P)$ be a Lagrangian H-umbilical immersion of a Riemannian n-manifold N into the para-Kähler n-plane with $n \geq 3$. Then*

(i) *If N is of constant sectional curvature, then either N is flat or L is congruent to an open portion of*

$$\frac{1}{2b}\Big(2\cosh^2(bs)\cosh t, z\sinh(2bs)\sinh t, \sinh(2bs)\cosh t, 2z\cosh^2(bs)\sinh t\Big),\ b \neq 0,$$

where $z = (z_2,\ldots,z_n) \in \mathbb{E}^{n-1}$ satisfies $z_2^2 + z_3^2 + \cdots + z_n^2 = 1$.

(ii) *If N contains no open subset of constant sectional curvature, then L is locally congruent to one of the following three types of submanifolds:*

(ii.1) *a Lagrangian submanifold defined by*

$$\Bigg(a^2\sum_{j=2}^n x_j^2 + \frac{e^{2r}}{8} - \frac{e^{-2r}}{2r'^2} - \int^s \frac{2r'^2 + r''}{e^{2r}r'^3}ds, \frac{1-a^2e^{2r}}{2}x_2, \ldots, \frac{1-a^2e^{2r}}{2}x_n,$$
$$a^2\sum_{j=2}^n x_j^2 - \frac{e^{2r}}{8} - \frac{e^{-2r}}{2r'^2} - \int^s \frac{2r'^2 + r''}{e^{2r}r'^3}ds, \frac{1+a^2e^{2r}}{2}x_2, \ldots, \frac{1+a^2e^{2r}}{2}x_n\Bigg),$$

where $r = r(s)$ is a non-constant function and a is positive number;

(ii.2) *a Lagrangian submanifold defined by*

$$\frac{1}{2}\Bigg(\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi} + \frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)\sin t, \left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi} + \frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)z\cos t,$$
$$\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi} - \frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)\sin t, \left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi} - \frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)z\cos t\Bigg),\ \lambda = 2\mu + \frac{\mu}{\varphi},$$

where $z=(z_2,\dots,z_n)\in\mathbb{E}^{n-1}$ *satisfies* $z_2^2+z_3^2+\cdots+z_n^2=1$ *and* $\mu(s)$ *and* $\varphi(s)$ *are nonzero functions satisfies* $\mu^2\neq\varphi^2$ *and* $\varphi\varphi'-\mu\mu'=(\mu^2-\varphi^2)\varphi$;

(ii.3) *a Lagrangian submanifold defined by*

$$\frac{1}{2}\left(\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}+\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)\cosh t,\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}-\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)z\sinh t,\right.$$
$$\left.\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}-\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)\cosh t,\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}+\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)z\sinh t\right),\ \lambda=2\mu+\frac{\mu}{\varphi},$$

where $z=(z_2,\dots,z_n)\in\mathbb{E}^{n-1}$ *satisfies* $z_2^2+z_3^2+\cdots+z_n^2=1$ *and* $\mu(s)$ *and* $\varphi(s)$ *are nonzero functions satisfies* $\mu^2\neq\varphi^2$ *and* $\varphi\varphi'-\mu\mu'=(\mu^2-\varphi^2)\varphi$.

5. Classification of flat Lagrangian H-umbilical submanifolds

In view of Theorem 4.1, we classify in this section all flat Lagrangian H-umbilical submanifolds in the para-Kähler n-plane.

Theorem 5.1. *Let* $L:N\to(\mathbb{E}^{2n}_n,g_0,P), n\geq 2$, *be a Lagrangian H-umbilical immersion of a flat Riemannian n-manifold into the para-Kähler n-plane. Then locally L is congruent to one of the following two types of submanifolds:*

(a) *a Lagrangian submanifolds defined by*

$$L(t,u_2,\dots,u_n)=(\gamma_1(t),0,\dots,0,\gamma_{n+1}(t),u_2,\dots,u_n), \tag{5.1}$$

where $(\gamma_1(t),\gamma_{n+1}(t))$ *is a space-like curve in* $\mathbb{E}^2_1$;

(b) *a Lagrangian submanifold defined by*

$$L(t,u_2,\dots,u_n)=u_2z(t)+\sum_{j=3}^n u_jw_j(t)+\int_0^t b(t)z'(t)dt, \tag{5.2}$$

where $b:I\to\mathbf{R}$ *is a real-valued function defined on an open interval* $I\ni 0$ *and* $z:I\to S^{2n-1}_n\subset\mathbb{E}^{2n}_n$ *is a space-like unit speed special para-Legendre curve satisfying*

$$z''(t)=-z(t)+\varphi(t)Pz'(t)-\sum_{j=3}^n a_j(t)w_j(t) \tag{5.3}$$

for some nonzero function φ *and parallel normal vector fields* $w_3,\dots,w_n$ *along* z.

Conversely, (5.1) *and* (5.2) *define flat Lagrangian H-umbilical submanifolds of the para-Kähler n-plane* $(\mathbb{E}^{2n}_n,g_0,P)$.

Proof. Assume that $L:N\to(\mathbb{E}^{2n}_n,g,P)$ is a Lagrangian H-umbilical isometric immersion of a flat Riemannian n-manifold N into the para-Kähler n-plane without totally geodesic points. Since N is flat, the second fundamental form h satisfies

$$\begin{aligned}&h(e_1,e_1)=\lambda Pe_1,\\&h(e_1,e_j)=h(e_j,e_k)=0,\quad j,k=2,\dots,n,\end{aligned} \tag{5.4}$$

for some nowhere zero function λ with respect to some suitable orthonormal local frame $\{e_1,\dots,e_n\}$. Without loss of generality, we may assume that $\lambda>0$.

From (5.4) and Codazzi's equation, we find

$$e_j\ln\lambda=\omega_1^j(e_1),\quad \omega_1^j(e_k)=0,\quad 2\leq j,k\leq n. \tag{5.5}$$

Let $\mathcal{D}$ and $\mathcal{D}^{\perp}$ denote the distributions on N spanned by $\{e_1\}$ and $\{e_2, \ldots, e_n\}$, respectively. Then $\mathcal{D}$ is clearly integrable, since it is one-dimensional. Also, it follows from (5.4) and (5.5) that $\mathcal{D}^{\perp}$ is also integrable and the leaves of $\mathcal{D}^{\perp}$ are totally geodesic submanifolds of $\mathbb{E}^{2n}_n$. Since $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are integrable and they are perpendicular, there exist local coordinates $\{x_1, x_2, \ldots, x_n\}$ such that $\partial/\partial x_1$ spans $\mathcal{D}$ and $\{\partial/\partial x_2, \ldots, \partial/\partial x_n\}$ spans $\mathcal{D}^{\perp}$. Because $\mathcal{D}$ is one-dimensional, we may choose x_1 such that $\partial/\partial x_1 = \lambda^{-1} e_1$.

With respect to $\partial/\partial x_1, \ldots, \partial/\partial x_n$, (5.4) becomes

$$\begin{aligned} &h\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) = P\left(\frac{\partial}{\partial x_1}\right), \\ &h\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}\right) = h\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = 0, \;\; j, k = 2, \ldots, n. \end{aligned} \tag{5.6}$$

Let Q^{n-1} be an integral submanifold of $\mathcal{D}^{\perp}$. Then Q^{n-1} is a totally geodesic submanifold of $\mathbb{E}^{2n}_n$. Thus Q^{n-1} is an open portion of a Euclidean $(n-1)$-space $\mathbb{E}^{n-1}$. Hence N is an open portion of the twisted product manifold ${}_f I \times \mathbb{E}^{n-1}$ with twisted product metric [1, 16]

$$g = f^2 dx_1^2 + dx_2^2 + dx_3^2 + \cdots + dx_n^2, \tag{5.7}$$

where $f = \lambda^{-1}$ and I is an open interval on which λ is defined.

From (5.7) we know that the Levi-Civita connection of N satisfies

$$\begin{aligned} &\nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_1} = \frac{f_1}{f} \frac{\partial}{\partial x_1} - f \sum_{k=2}^{n} f_k \frac{\partial}{\partial x_k}, \\ &\nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_j} = \frac{f_j}{f} \frac{\partial}{\partial x_1}, \quad \nabla_{\partial/\partial x_j} \frac{\partial}{\partial x_k} = 0, \end{aligned} \tag{5.8}$$

for $2 \leq j, k \leq n$, where $f_i = \partial f/\partial x_i$, $i = 1, \ldots, n$. Using (5.8) we find

$$R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}\right) \frac{\partial}{\partial x_1} = f \sum_{k=2}^{n} f_{jk} \frac{\partial}{\partial x_k}, \quad j = 2, \ldots, n. \tag{5.9}$$

Since N is flat, (5.9) implies that $f_{jk} = 0$, $j, k = 2, \ldots, n$. Therefore, f is given by

$$f = \beta(x_1) + \sum_{j=2}^{n} \alpha_j(x_1) x_j, \tag{5.10}$$

for some functions $\beta, \alpha_2, \ldots, \alpha_n$. In view of (5.10), (5.8) reduces to

$$\begin{aligned} &\nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_1} = \frac{1}{f}\left(\beta'(x_1) + \sum_{j=2}^{n} \alpha_j'(x_1) x_j\right) \frac{\partial}{\partial x_1} - f \sum_{k=2}^{n} \alpha_k \frac{\partial}{\partial x_k}, \\ &\nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_j} = \frac{\alpha_j}{f} \frac{\partial}{\partial x_1}, \quad \nabla_{\partial/\partial x_j} \frac{\partial}{\partial x_k} = 0, \quad j, k = 2, \ldots, n. \end{aligned} \tag{5.11}$$

By combining (5.6), (5.11) and the formula of Gauss we obtain the following PDE system:

$$L_{x_1x_1} = \frac{1}{f}(\beta'(x_1) + \sum_{j=2}^{n} \alpha'_j(x_1)x_j)L_{x_1} - f\sum_{k=2}^{n} \alpha_k L_{x_k} + PL_{x_1}, \tag{5.12}$$

$$L_{x_1x_j} = \frac{\alpha_j}{f} L_{x_1}, \tag{5.13}$$

$$L_{x_jx_k} = 0, \quad j,k = 2,\dots,n. \tag{5.14}$$

Integrating (5.14) yields

$$L = \sum_{j=2}^{n} A_j(t)x_j + \gamma(t), \ \ t = x_1, \tag{5.15}$$

for some $\mathbb{E}^{2n}_n$-valued functions $A_2,\dots,A_n,\gamma$ of t. Thus

$$L_t = \sum_{j=2}^{n} A'_j(t)x_j + \gamma'(t), \tag{5.16}$$

$$L_{x_j} = A_j(t), \quad j = 2,\dots,n. \tag{5.17}$$

From (5.7) and (5.17), we know that $A_2,\dots,A_n$ are orthonormal tangent vector fields on N. By applying (5.13), (5.16) and (5.17), we obtain

$$\alpha_j(t)\gamma'(t) = \beta(t)A'_j(t), \tag{5.18}$$

$$\alpha_j(t)A'_k(t) = \alpha_k(t)A'_j(t), \quad j,k = 2,\dots,n. \tag{5.19}$$

Case (1): $\alpha_2 = \cdots = \alpha_n = 0$. Equation (5.7) and system (5.12)-(5.14) reduce to

$$g = \beta^2(t)dt^2 + dx_2^2 + \cdots + dx_n^2, \tag{5.20}$$

$$L_{tt} = (\ln\beta(t))'L_{x_1} + PL_t, \ L_{tx_j} = L_{x_jx_k} = 0, \quad j,k = 2,\dots,n. \tag{5.21}$$

Also, it follows from (5.18) that $A'_2(x_1) = \cdots = A'_n(x_1) = 0$, due to $\beta \neq 0$ by (5.10). Thus $A_2,\dots,A_n$ are constant vectors, say $c_2,\dots,c_n \in \mathbb{E}^{2n}_n$. Therefore (5.15) becomes

$$L(t,x_2,\dots,x_n) = \gamma(t) + \sum_{j=2}^{n} c_jx_j. \tag{5.22}$$

From (5.22) we find

$$L_t = \gamma'(t), \ \ L_{x_j} = c_j, \ \ j = 2,\dots,n. \tag{5.23}$$

Now, by applying (5.20) and (5.22), we conclude that $c_2,\dots,c_n$ are orthonormal space-like vectors and $\gamma(t)$ is a space-like curve in $\mathbb{E}^{2n}_n$ with $\beta(t)$ as its speed. Without loss of generality, we may put

$$\begin{aligned} &\gamma(t) = (\gamma_1(t),\dots,\gamma_{2n}(t)),\\ &c_2 = (0,\dots,0,\overbrace{1}^{(n+2)-th},0,\dots,0),\\ &\qquad\vdots\\ &c_n = (0,\dots,0,1). \end{aligned} \tag{5.24}$$

Since the velocity vector β' is perpendicular to $c_2,\dots,c_n$, we have

$$\gamma'_{n+2} = \cdots = \gamma'_{2n} = 0.$$

Thus, after applying a suitable translation on $\mathbb{E}^{2n}_n$, we may put

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_{n+1}(t), 0, \ldots, 0). \tag{5.25}$$

It follows from (5.23)-(5.25) and the Lagrangian condition that $\gamma_2' = \cdots = \gamma_n' = 0$. Consequently, after applying a suitable translation on $\mathbb{E}^{2n}_n$, we obtain

$$L(t, x_2, \ldots, x_n) = (\gamma_1(t), 0, \ldots, 0, \gamma_{n+1}(t), x_2, \ldots, x_n), \tag{5.26}$$

where $(\gamma_1(t), \gamma_{n+1}(t))$ is a space-like curve in $\mathbb{E}^2_1$ with speed $\beta(t)$. This gives flat Lagrangian submanifolds of type (a) of the theorem.

Case (2): *At least one of* $\alpha_2, \ldots, \alpha_n$ *is nonzero.* Without loss of generality, we may assume $\alpha_2 \neq 0$. We may reparameterize x_1 by $t = \int_0^{x_1} \alpha_2(x_1)dx_1$, then we obtain from (5.10) that

$$\hat{f} = b(t) + u_2 + \sum_{j=3}^{n} a_j(t)u_j, \tag{5.27}$$

where $u_j = x_j, j = 2, \ldots, n$, $b(t) = \beta(x_1(t))$ and $a_j(t) = \alpha_j(x_1(t))$. Without loss of generality, we may assume that $b(t)$ is defined on an open interval I containing 0. Hence (5.7) becomes

$$g = \hat{f}^2 dt^2 + du_2^2 + \cdots + du_n^2. \tag{5.28}$$

From (5.6) we derive that

$$\begin{aligned} &h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \varphi(t)P\left(\frac{\partial}{\partial t}\right), \\ &h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial u_j}\right) = h\left(\frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right) = 0, \quad j, k = 2, \ldots, n, \end{aligned} \tag{5.29}$$

where $\varphi(t) = 1/(\alpha_2(x_1(t))$. By applying (5.8), (5.28), (5.27), (5.29) and the formula of Gauss, we get

$$L_{tt} = \frac{1}{\hat{f}}(b'(t) + \sum_{j=3}^{n} a_j'(t)u_j)L_t - \hat{f}\sum_{k=2}^{n} a_k L_{u_k} + \varphi(t)PL_t, \tag{5.30}$$

$$L_{tu_j} = \frac{a_j}{\hat{f}}L_t, \tag{5.31}$$

$$L_{u_j u_k} = 0, \quad j, k = 2, \ldots, n, \tag{5.32}$$

with $a_2 = 1$. After solving (5.32), we find that

$$L = u_2 z(t) + \sum_{j=3}^{n} u_j w_j(t) + B(t) \tag{5.33}$$

for some $\mathbb{E}^{2n}_n$-valued functions $z(t), w_3(t), \ldots, w_n(t), B(t)$. Thus

$$L_t = u_2 z'(t) + \sum_{j=3}^{n} u_j w_j'(t) + B'(t), \tag{5.34}$$

$$L_{u_2} = z(t), \;\; L_{u_j} = w_j(t), \;\; j = 3, \ldots, n. \tag{5.35}$$

It follows from (5.28) and (5.34) that $z(t), w_3(t), \ldots, w_n(t)$ are space-like orthonormal tangent vector fields. Now, after applying (5.27), (5.31) and (5.33), we find that

$$B'(t) = b(t)z'(t), \;\; w_k'(t) = a_k(t)z'(t), \;\; k = 3, \ldots, n. \tag{5.36}$$

Thus, $w_3(t), \ldots, w_n(t)$ are parallel normal vector fields along $z(t)$.

By substituting (5.36) into (5.34) we find that

$$L_t = \hat{f}z'(t). \tag{5.37}$$

It follows from (5.27), (5.28) and (5.37) that $z'(t)$ is a unit vector field. Since L is Lagrangian, we derive from (5.35) and (5.37) that $\langle z', Pz \rangle = 0$. Thus, $z = z(t)$ is a unit speed space-like para-Legendre curve in S^{2n-1}_n. Moreover, from (5.35), (5.37) and the Lagrangian condition, we know that

$$z(t), Pz(t), z'(t), Pz'(t), w_3(t), Pw_3(t) \ldots, w_n(t), Pw_n(t)$$

form an orthonormal frame. Furthermore, by using (5.33) and (5.36) we conclude that

$$L(t, u_2, \ldots, u_n) = u_2 z(t) + \sum_{j=3}^{n} u_j w_j(t) + \int_0^t b(t)z'(t)dt. \tag{5.38}$$

Finally, it follows from (5.30), (5.36), and (5.38) that the unit speed space-like para-Legendre curve z satisfies (5.3). Consequently, the unit speed space-like para-Legendre curve z in $S^{2n-1}_n \subset \mathbb{E}^{2n}_n$ is special para-Legendre. Thus, we obtain flat Lagrangian submanifolds of type (b).

The converse can be verified by direct computation. □

It follows from Theorem 6.1 that there exists infinitely many flat Lagrangian submanifolds of type (b) in $(\mathbb{E}^{2n}_n, g_0, P)$ as described in Theorem 5.1.

6. Existence of special para-Legendre curves

Theorem 6.1. *Let $\varphi, a_2, a_3, \ldots, a_n$ $(n \geq 2)$ be real-valued functions defined on an open interval $I \ni 0$ with $a_2 = 1$ and φ nowhere zero. Then there exists a unit speed space-like special para-Legendre curve*

$$z : I \to S^{2n-1}_n \subset (\mathbb{E}^{2n}_n, g_0, P)$$

satisfying (5.3) *for some parallel orthonormal normal vector fields $w_3, \ldots, w_n$ along the curve z.*

Proof. Let $\varphi(t), a_3(t), \ldots, a_m(t)$ be $n-1$ functions of t defined on an open interval $I \ni 0$ with φ nowhere zero. Put

$$f = u_2 + \sum_{j=3}^{n} a_j(t)u_j. \tag{6.1}$$

Consider the twisted product manifold $N := {}_f I \times \mathbb{E}^{n-1}$ equipped with the twisted product metric

$$g = f^2 dt^2 + du_2^2 + \cdots + du_n^2. \tag{6.2}$$

Then N is a flat Riemannian n-manifold. Define a symmetric bilinear form σ on N by

$$\text{(6.3)}\qquad \begin{aligned} &\sigma\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=\varphi\frac{\partial}{\partial t},\\ &\sigma\left(\frac{\partial}{\partial t},\frac{\partial}{\partial u_j}\right)=\sigma\left(\frac{\partial}{\partial u_j},\frac{\partial}{\partial u_k}\right)=0,\ j,k=2,\dots,n. \end{aligned}$$

Then $\langle\sigma(X,Y),Z\rangle$ is totally symmetric in X,Y and Z.

From (6.1)-(6.3) it follows that $(\nabla\sigma)(X,Y,Z)$ is totally symmetric in X,Y and Z. Moreover σ and the Riemann curvature tensor R of N satisfy

$$R'(X,Y)Z=\sigma(\sigma(Y,Z),X)-\sigma(\sigma(X,Z),Y). \tag{6.4}$$

Thus, the Existence and Uniqueness Theorems imply that, up to rigid motions of $(\mathbb{E}^{2n}_n,g_0,P)$, there is a unique Lagrangian immersion $L:N\to(\mathbb{E}^{2n}_n,g_0,P)$ whose second fundamental form is given by $h=P\circ\sigma$.

It follows from (6.1)-(6.3) and $h=P\circ\sigma$ that L satisfies

$$L_{tt}=\frac{1}{f}\sum_{j=3}^{n}a_j'(t)u_jL_t-f\sum_{k=2}^{n}a_kL_{u_k}+\varphi PL_t, \tag{6.5}$$

$$L_{tu_j}=\frac{a_j}{f}L_t,\quad L_{u_ju_k}=0,\quad j,k=2,\dots,n, \tag{6.6}$$

where $a_2=1$. Solving (6.6) yields

$$L=\sum_{j=2}^{n}u_jw_j(t)+c_0, \tag{6.7}$$

$$L_t=fw_2'(t),\ L_{u_j}=w_j(t),\ \ j=2,\dots,n, \tag{6.8}$$

$$B'(t)=0,\ w_k'(t)=a_k(t)w_2'(t),\ \ k=3,\dots,n, \tag{6.9}$$

for some $\mathbb{E}^{2n}_n$-valued functions $w_2,\dots,w_n$ and constant vector c_0. From (6.2) and (6.8), it follows that $w_2'(t)$ is a unit vector field and $w_2(t),\dots,w_n(t)$ are orthonormal vector fields. Put $z(t)=w_2(t)$. Then $z:I\to S^{2n-1}_n\subset\mathbb{E}^{2n}_n$ is a unit speed curve defined on some open interval I.

Because L is Lagrangian, (6.8) and (6.9) imply that $z=z(t)$ is a unit speed space-like para-Legendre curve in $S^{2n-1}_n\subset(\mathbb{E}^{2n}_n,g_0,P)$ and

$$z(t),Pz(t),z'(t)Pz'(t),w_3(t),Pw_3(t)\dots,w_n(t),Pw_n(t)$$

form an orthonormal frame such that $w_3,\dots,w_n$ are parallel orthonormal normal vector fields along z. Finally, from (6.5) and (6.8), we conclude that z is a special para-Legendre curve in S^{2n-1}_n satisfying (5.3). □

7. Explicit examples of special para-Legendre curves

Now, we provide some simple explicit examples of unit speed space-like para-Legendre curves in $S^5_3\subset(\mathbb{E}^6_3,g_0,P)$ as follows:

Example 7.1. Let a,b be real numbers such that $a^2+b^2>1$ and $a^4<1$. Put

$$\gamma=1-a^2,\ \ \mu=\sqrt{a^2+b^2-1}.$$

Consider the following curve in $(\mathbb{E}^6_3, g_0, P)$:

$$z(t) = \Bigg(\frac{\sinh(bt)\sinh(\mu t)}{\mu}, \frac{\mu\{\sinh(bt)\cosh(\mu t) - a^2\} - b\cosh(bt)\sinh(\mu t)}{\mu\sqrt{1-a^4}},$$
$$\frac{a\mu\{\sinh(bt)\cosh(\mu t) - a^2\} - ab\cosh(bt)\sinh(\mu t)}{\mu\sqrt{1-a^4}},$$
$$\frac{\cosh(bt)\sinh(\mu t)}{\mu}, \frac{\mu\cosh(bt)\cosh(\mu t) - b\sinh(bt)\sinh(\mu t)}{\mu\sqrt{1-a^4}},$$
$$\frac{a\mu\cosh(bt)\cosh(\mu t) - ab\sinh(bt)\sinh(\mu t)2}{\mu\sqrt{1-a^4}} \Bigg).$$

Then $z = z(t)$ is a unit speed space-like special para-Legendre curve in $S^{2n-1}_n \subset (\mathbb{E}^{2n}_n, g_0, P)$ satisfying

$$z''(t) = 2bPz'(t) - z(t) - aw(t), \tag{7.1}$$

where

$$w(t) = \frac{1-a^2}{\sqrt{1-a^4}}(0, a, -1, 0, 0, 0) - az(t). \tag{7.2}$$

is the associated unit parallel normal vector field.

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