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ON $\mathcal{T}$-CURVATURE TENSOR IN K-CONTACT AND SASAKIAN MANIFOLDS

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Abstract. Some properties of quasi-$\mathcal{T}$-flat, ξ-$\mathcal{T}$-flat and φ-$\mathcal{T}$-flat K-contact and Sasakian manifolds are obtained. We give the necessary and sufficient condition for the K-contact manifold to be ξ-$\mathcal{T}$-flat under some algebraic condition. Among others, it is proved that a compact φ-$\mathcal{T}$-flat K-contact manifold with regular contact vector field, under an algebraic condition, is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature.

1. Introduction

Let M be a $(2n+1)$-dimensional almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). Then at each point $p \in M$ the tangent space T_pM can be decomposed into the direct sum $T_pM = \varphi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by ξ_p. Thus, the conformal curvature tensor $\mathcal{C}$ is a map $\mathcal{C} : T_pM \times T_pM \times T_pM \to \varphi(T_pM) \oplus \{\xi_p\}$, $p \in M$. We have the following well-known particular cases: (1) the projection of the image of $\mathcal{C}$ in $\varphi(TpM)$ is zero, (2) the projection of the image of $\mathcal{C}$ in $\{\xi_p\}$ is zero, and (3) the projection of the image of $\mathcal{C}|_{\varphi(TpM)\times\varphi(TpM)\times\varphi(TpM)}$ in $\varphi(TpM)$ is zero. An almost contact metric manifold satisfying the case (1), (2) and (3) is said to be conformally symmetric [18], ξ-conformally flat [17] and φ-conformally flat [3], respectively. The first case was studied by Zhen [18] by proving that a conformally symmetric K-contact manifold is locally isometric to the unit sphere. The second case was studied by Zhen et al. [17], they proved that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstein Sasakian manifold. The third case was studied by Cabrerizo et al. [3], they gave the some necessary conditions for a K-contact manifold to be φ-conformally flat. In 2008, Tripathi and Dwivedi [12]

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defined and studied quasi projectively flat, ξ-projectively flat and φ-projectively flat almost contact metric manifold. Necessary and sufficient conditions for a K-contact manifold to be quasi projectively flat and φ-projectively flat were obtained. Finally, they proved that a compact φ-projectively flat K-contact manifold with regular contact vector field is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature 4. In 2010, Dwivedi and Kim [4] defined some necessary and/or sufficient condition(s) for K-contact and/or Sasakian manifolds to be quasi conharmonically flat, ξ-conharmonically flat and φ-conharmonically flat. In last, they proved that a compact φ-conharmonically flat K-contact manifold with regular contact vector field is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature $3 - \dfrac{2}{2n-1}$.

Motivated by these studies, we study the $\mathcal{T}$-curvature tensor [13] in K-contact and Sasakian manifolds. Section 2 contains some preliminaries. In Section 3, we give the definition of $\mathcal{T}$-curvature tensor. In particular, $\mathcal{T}$-curvature tensor is reduced to be quasi-conformal curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, pseudo-projective curvature tensor, projective curvature tensor, $\mathcal{M}$-projective curvature tensor, $\mathcal{W}_i$-curvature tensors ($i = 0, \ldots, 9$), $\mathcal{W}_j^*$-curvature tensors ($j = 0, 1$). In Section 4, we give definitions of quasi $\mathcal{T}$-flat, ξ-$\mathcal{T}$-flat and φ-$\mathcal{T}$-flat almost contact metric manifolds analogous to those of conformal curvature tensor, projective curvature tensor and conharmonic curvature tensor. We obtain some properties of ξ-T-flat, quasi T-flat and φ-T-flat K-contact manifolds. We give a necessary and sufficient condition for a K-contact manifold to be ξ-T-flat under some algebraic condition. In last section, we establish that a φ-T-flat compact regular K-contact manifold with regular contact vector field, under an algebraic condition, is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature.

2. Preliminaries

Let $(M, \varphi, \xi, \eta, g)$ be an $(2n+1)$-dimensional almost contact metric manifold consisting of a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g. Let $\mathfrak{X}(M)$ be the Lie algebra of vector fields in M. Consider X, Y, Z, V, $W \in \mathfrak{X}(M)$ throughout the paper, unless otherwise specifically stated. Then

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

From (2.1) and (2.2) we have

$$g(X, \varphi Y) = -g(\varphi X, Y), \qquad g(X, \xi) = \eta(X). \tag{2.3}$$

An almost contact metric manifold is

1. a contact metric manifold if $g(X, \varphi Y) = d\eta(X, Y)$.
2. a K-contact manifold if $\nabla\xi = -\varphi$, where ∇ is Levi-Civita connection and
3. a Sasakian manifold if $(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$.

A K-contact manifold is always contact metric manifold but converse is true if the Lie derivative of φ in the characteristic direction ξ vanishes. A Sasakian manifold is a K-contact manifold but the converse is true if dimension is 3. A contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.4}$$

In a Sasakian manifold (M,φ,ξ,η,g), we easily get

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.5}$$

where R is the curvature tensor.

In a $(2n+1)$-dimensional almost contact metric manifold M, if $\{e_1,\ldots,e_{2n},\xi\}$ is a local orthonormal basis of vector fields in M, then $\{\varphi e_1,\ldots,\varphi e_{2n},\xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i,e_i) = \sum_{i=1}^{2n} g(\varphi e_i,\varphi e_i) = 2n, \tag{2.6}$$

$$\sum_{i=1}^{2n} g(e_i,Z)S(Y,e_i) = \sum_{i=1}^{2n} g(\varphi e_i,Z)S(Y,\varphi e_i) = S(Y,Z) - S(Y,\xi)\eta(Z), \tag{2.7}$$

$$\sum_{i=1}^{2n} g(e_i,\varphi Z)S(Y,e_i) = \sum_{i=1}^{2n} g(\varphi e_i,\varphi Z)S(Y,\varphi e_i) = S(Y,\varphi Z), \tag{2.8}$$

where S is the Ricci tensor. If M is a K-contact manifold then we have

$$S(X,\xi) = 2n\eta(X), \tag{2.9}$$

$$(\nabla_X\varphi)Y = -R(X,\xi)Y, \tag{2.10}$$

$$(\nabla_X\varphi)\varphi Y + \varphi(\nabla_X\varphi)Y = -g(\varphi X,Y)\xi - \eta(Y)\varphi X. \tag{2.11}$$

Then from (2.10) and (2.11), we get

$$\varphi R(X,\xi)Y + R(X,\xi)\varphi Y = g(\varphi X,Y)\xi - \eta(Y)\varphi X \tag{2.12}$$

and, in particular,

$$R(X,\xi)\xi = X - \eta(X)\xi, \tag{2.13}$$

$$S(\xi,\xi) = 2n. \tag{2.14}$$

From (2.14) we get

$$\sum_{i=1}^{2n} S(e_i,e_i) = \sum_{i=1}^{2n} S(\varphi e_i,\varphi e_i) = r - 2n, \tag{2.15}$$

where r is the scalar curvature. In a K-contact manifold we also get

$$R(\xi,Y,Z,\xi) = g(\varphi Y,\varphi Z). \tag{2.16}$$

Consequently,

$$\sum_{i=1}^{2n} R(e_i,Y,Z,e_i) = \sum_{i=1}^{2n} R(\varphi e_i,Y,Z,\varphi e_i) = S(Y,Z) - g(\varphi Y,\varphi Z). \tag{2.17}$$

Definition 2.1. An almost contact metric manifold M is said to be *η-Einstein manifold* [1, p. 105] if the Ricci tensor satisfies

$$S(X,Y) = b_1 g(X,Y) + b_2 \eta(X)\eta(Y), \tag{2.18}$$

where b_1,b_2 are smooth functions on the manifold and η is 1-form. In particular, if $b_2 = 0$, then M is an *Einstein manifold.*

Remark 2.1. In [17, Lemma 1.1], the authors proved that for a K-contact metric manifold

$$g((\nabla_\xi Q)X - (\nabla_X Q)\xi, \xi) = 3g(Q\varphi X, \xi).$$

Since $Q\xi = 2n\xi$, therefore

$$g((\nabla_\xi Q)X - (\nabla_X Q)\xi, \xi) = 0.$$

3. $\mathcal{T}$-curvature tensor

In a $(2n+1)$-dimensional Riemannian manifold M, the $\mathcal{T}$-curvature tensor [13] is given by

$$\begin{aligned} \mathcal{T}(X,Y)Z \;=\;& a_0\, R\,(X,Y)\,Z \\ &+ a_1\, S\,(Y,Z)\,X + a_2\, S\,(X,Z)\,Y + a_3\, S(X,Y)Z \\ &+ a_4\, g\,(Y,Z)\,QX + a_5\, g\,(X,Z)\,QY + a_6\, g(X,Y)QZ \\ &+ a_7\, r\,(g\,(Y,Z)\,X - g\,(X,Z)\,Y)\,, \end{aligned} \tag{3.1}$$

where R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively.

In particular, the $\mathcal{T}$-curvature tensor is reduced to be

(1) the *quasi-conformal curvature tensor* $\mathcal{C}_*$ [16] if

$$a_1 = -\,a_2 = a_4 = -\,a_5, \quad a_3 = a_6 = 0, \quad a_7 = -\frac{1}{2n+1}\left(\frac{a_0}{2n} + 2a_1\right),$$

(2) the *conformal curvature tensor* $\mathcal{C}$ [5, p. 90] if

$$a_0 = 1, \quad a_1 = -\,a_2 = a_4 = -\,a_5 = -\frac{1}{2n-1}, \quad a_3 = a_6 = 0, \quad a_7 = \frac{1}{2n(2n-1)},$$

(3) the *conharmonic curvature tensor* $\mathcal{L}$ [6] if

$$a_0 = 1, \quad a_1 = -\,a_2 = a_4 = -\,a_5 = -\frac{1}{2n-1}, \quad a_3 = a_6 = 0, \quad a_7 = 0,$$

(4) the *concircular curvature tensor* $\mathcal{V}$ ([14], [15, p. 87]) if

$$a_0 = 1, \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{2n(2n+1)},$$

(5) the *pseudo-projective curvature tensor* $\mathcal{P}_*$ [11] if

$$a_1 = -\,a_2, \quad a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{2n+1}\left(\frac{a_0}{2n} + a_1\right),$$

(6) the *projective curvature tensor* $\mathcal{P}$ [15, p. 84] if

$$a_0 = 1, \quad a_1 = -\,a_2 = -\frac{1}{2n}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(7) the $\mathcal{M}$-*projective curvature tensor* [9] if

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{4n}, \quad a_3 = a_6 = a_7 = 0,$$

(8) the $\mathcal{W}_0$-*curvature tensor* [9, Eq. (1.4)] if

$$a_0 = 1, \quad a_1 = -a_5 = -\frac{1}{2n}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

(9) the $\mathcal{W}_0^*$-*curvature tensor* [9, Eq. (2.1)] if

$$a_0 = 1, \quad a_1 = -a_5 = \frac{1}{2n}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

(10) the $\mathcal{W}_1$-*curvature tensor* [9] if

$$a_0 = 1, \quad a_1 = -a_2 = \frac{1}{2n}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(11) the $\mathcal{W}_1^*$-*curvature tensor* [9] if

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{2n}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(12) the $\mathcal{W}_2$-*curvature tensor* [8] if

$$a_0 = 1, \quad a_4 = -a_5 = -\frac{1}{2n}, \quad a_1 = a_2 = a_3 = a_6 = a_7 = 0,$$

(13) the $\mathcal{W}_3$-*curvature tensor* [9] if

$$a_0 = 1, \quad a_2 = -a_4 = -\frac{1}{2n}, \quad a_1 = a_3 = a_5 = a_6 = a_7 = 0,$$

(14) the $\mathcal{W}_4$-*curvature tensor* [9] if

$$a_0 = 1, \quad a_5 = -a_6 = \frac{1}{2n}, \quad a_1 = a_2 = a_3 = a_4 = a_7 = 0,$$

(15) the $\mathcal{W}_5$-*curvature tensor* [10] if

$$a_0 = 1, \quad a_2 = -a_5 = -\frac{1}{2n}, \quad a_1 = a_3 = a_4 = a_6 = a_7 = 0,$$

(16) the $\mathcal{W}_6$-*curvature tensor* [10] if

$$a_0 = 1, \quad a_1 = -a_6 = -\frac{1}{2n}, \quad a_2 = a_3 = a_4 = a_5 = a_7 = 0,$$

(17) the $\mathcal{W}_7$-*curvature tensor* [10] if

$$a_0 = 1, \quad a_1 = -a_4 = -\frac{1}{2n}, \quad a_2 = a_3 = a_5 = a_6 = a_7 = 0,$$

(18) the $\mathcal{W}_8$-*curvature tensor* [10] if

$$a_0 = 1, \quad a_1 = -a_3 = -\frac{1}{2n}, \quad a_2 = a_4 = a_5 = a_6 = a_7 = 0,$$

(19) the $\mathcal{W}_9$-*curvature tensor* [10] if

$$a_0 = 1, \quad a_3 = -a_4 = \frac{1}{2n}, \quad a_1 = a_2 = a_5 = a_6 = a_7 = 0.$$

Denoting

$$\mathcal{T}\,(X,Y,Z,V) = g(\mathcal{T}\,(X,Y)\,Z,V),$$

we write the curvature tensor $\mathcal{T}$ in its $(0,4)$ form as follows.

$$\begin{aligned}\mathcal{T}\,(X,Y,Z,V) \;=\;& a_0\,R\,(X,Y,Z,V) + a_1\,S\,(Y,Z)\,g\,(X,V)\\ &+ a_2\,S\,(X,Z)\,g\,(Y,V) + a_3\,S(X,Y)g(Z,V)\\ &+ a_4\,g\,(Y,Z)\,S\,(X,V) + a_5\,g\,(X,Z)\,S\,(Y,V)\\ &+ a_6\,S(Z,V)g(X,Y)\\ &+ a_7\,r\,(g\,(Y,Z)\,g\,(X,V) - g\,(X,Z)\,g\,(Y,V))\,.\end{aligned} \tag{3.2}$$

Remark 3.1. In [12], the projective curvature tensor is defined by

$$\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{2n}(g(Y,Z)QX - g(X,Z)QY),$$

which is $\mathcal{W}_2$-curvature tensor. In [9], the projective curvature tensor is defined by

$$W(X,Y)Z = R(X,Y)Z + \frac{1}{2n}(g(X,Z)QY - S(Y,Z)X),$$

which we call $\mathcal{W}_0$-curvature tensor.

4. Some structure theorems

Analogous to the considerations of conformal curvature tensor, conharmonic curvature tensor and projective curvature tensor, we give the following:

Definition 4.1. An almost contact metric manifold M is said to be

(1) quasi $\mathcal{T}$-flat if

$$g(\mathcal{T}(X,Y)Z,\varphi W) = 0, \tag{4.1}$$

(2) ξ-$\mathcal{T}$-flat if

$$\mathcal{T}(X,Y)\xi = 0, \tag{4.2}$$

(3) φ-$\mathcal{T}$-flat if

$$g(\mathcal{T}(\varphi X,\varphi Y)\varphi Z,\varphi W) = 0. \tag{4.3}$$

We begin with the following Theorem

Theorem 4.1. *Let M be a $(2n+1)$-dimensional K-contact manifold satisfying*

$$g(\mathcal{T}(\varphi X,Y)Z,\varphi W) = 0, \tag{4.4}$$

1. *If*

$$a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0.$$

Then

$$S = A_0 g + A_1 \eta\otimes\eta, \tag{4.5}$$

where

$$A_0 = \frac{a_0 - a_4(r-2n) - (2n-1)a_7 r}{a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6} \tag{4.6}$$

and

$$A_1 = \frac{2n(a_2 + a_3 + a_5 + a_6) - (a_0 + a_7\,r)}{a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6}. \tag{4.7}$$

Therefore M is an η-Einstein manifold. In particular, M becomes an Einstein manifold provided

$$2n(a_2 + a_3 + a_5 + a_6) - (a_0 + a_7 r) = 0.$$

Consequently, we have the following:

K-contact manifold	***Einstein/ η-Einstein***	$S =$
$g(\mathcal{C}_*(\varphi X, Y)Z, \varphi W) = 0$	η-Einstein	$\frac{1}{a_0 + (2n-2)a_1}\left(\left(1 + \frac{(2n-1)r}{2n(2n+1)}\right)a_0 + \left(2n + \frac{(2n-3)r}{2n+1}\right)a_1\right)g + \frac{1}{a_0 + 2na_1}\left(\left(-1 + \frac{r}{2n(2n+1)}\right)a_0 + \left(-4n + \frac{2r}{2n+1}\right)a_1\right)\eta \otimes \eta$
$g(\mathcal{C}(\varphi X, Y)Z, \varphi W) = 0$	η-Einstein	$\left(\frac{r}{2n} - 1\right)g + \left(2n + 1 - \frac{r}{2n}\right)\eta \otimes \eta$
$g(\mathcal{L}(\varphi X, Y)Z, \varphi W) = 0$	η-Einstein	$(r-1)g + (2n+1)\eta \otimes \eta$
$g(\mathcal{V}(\varphi X, Y)Z, \varphi W) = 0$	Einstein	$2ng$
$g(\mathcal{P}_*(\varphi X, Y)Z, \varphi W) = 0$	Einstein	$\frac{2na_0 + 2n(2n-1)a_1}{a_0 + (2n-1)a_1}g$
$g(\mathcal{P}(\varphi X, Y)Z, \varphi W) = 0$	Einstein	$2ng$
$g(\mathcal{M}(\varphi X, Y)Z, \varphi W) = 0$	Einstein	$2ng$
$g(\mathcal{W}_0(\varphi X, Y)Z, \varphi W) = 0$	Einstein	$2ng$
$g(\mathcal{W}_0^*(\varphi X, Y)Z, \varphi W) = 0$	η-Einstein	$\frac{2n}{4n-1}g - \frac{4n}{4n-1}\eta \otimes \eta$
$g(\mathcal{W}_1(\varphi X, Y)Z, \varphi W) = 0$	η-Einstein	$\frac{2n}{4n-1}g - \frac{4n}{4n-1}\eta \otimes \eta$
$g(\mathcal{W}_1^*(\varphi X, Y)Z, \varphi W) = 0$	Einstein	$2ng$
$g(\mathcal{W}_2(\varphi X, Y)Z, \varphi W) = 0$ [12]	Einstein	$2ng$
$g(\mathcal{W}_3(\varphi X, Y)Z, \varphi W) = 0$	η-Einstein	$\frac{2n(2n+1)}{2n-1}g - \frac{4n}{2n-1}\eta \otimes \eta$
$g(\mathcal{W}_4(\varphi X, Y)Z, \varphi W) = 0$	η-Einstein	$g - \eta \otimes \eta$
$g(\mathcal{W}_5(\varphi X, Y)Z, \varphi W) = 0$	η-Einstein	$g - \eta \otimes \eta$
$g(\mathcal{W}_6(\varphi X, Y)Z, \varphi W) = 0$	Einstein	$2ng$
$g(\mathcal{W}_8(\varphi X, Y)Z, \varphi W) = 0$	Einstein	$2ng$
$g(\mathcal{W}_9(\varphi X, Y)Z, \varphi W) = 0$	Einstein	$2ng$

2. *If $a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 = 0$ and $4n^2a_7 + (2n+1)a_4 \neq 0$ then*

$$r = \frac{2n\,(a_0 + a_2 + a_3 + (2n+1)a_4 + a_5 + a_6)}{4n^2a_7 + (2n+1)a_4}. \tag{4.8}$$

Consequently, for a K-contact manifold which satisfies $g(\mathcal{W}_7(\varphi X, Y)Z, \varphi W) = 0$, we have

$$r = \frac{2n(4n+1)}{2n+1}.$$

3. *If $a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 = 0$ and $4n^2a_7 + (2n+1)a_4 = 0$, then*

$$a_0 + a_2 + a_3 + (2n+1)a_4 + a_5 + a_6 = 0. \tag{4.9}$$

Proof. Let M be a $(2n+1)$-dimensional K-contact manifold, then from (3.2) we have

$$\begin{aligned}\mathcal{T}(\varphi X,Y,Z,\varphi W) &= a_0\,R(\varphi X,Y,Z,\varphi W)\\ &+ a_1\,S\,(Y,Z)\,g(\varphi X,\varphi W)+a_2\,S\,(\varphi X,Z)\,g(Y,\varphi W)\\ &+ a_3\,S(\varphi X,Y)g(Z,\varphi W)+a_4\,g\,(Y,Z)\,S(\varphi X,\varphi W)\\ &+ a_5\,g\,(\varphi X,Z)\,S(Y,\varphi W)+a_6\,g(\varphi X,Y)S(Z,\varphi W)\\ &+ a_7\,r\,(g\,(Y,Z)\,g(\varphi X,\varphi W)-g\,(\varphi X,Z)\,g(Y,\varphi W))\,.\end{aligned} \tag{4.10}$$

If $\{e_1,\dots,e_{2n},\xi\}$ is a local orthonormal basis of vector fields in M, then from (4.10) we get

$$\begin{aligned}\sum_{i=1}^{2n}\mathcal{T}(\varphi e_i,Y,Z,\varphi e_i) &= a_0\sum_{i=1}^{2n}R(\varphi e_i,Y,Z,\varphi e_i)\\ &+\sum_{i=1}^{2n}(\ a_1\,S\,(Y,Z)\,g(\varphi e_i,\varphi e_i)+a_2\,S\,(\varphi e_i,Z)\,g(Y,\varphi e_i)\\ &+a_3\,S(\varphi e_i,Y)g(Z,\varphi e_i)+a_4\,g\,(Y,Z)\,S(\varphi e_i,\varphi e_i)\\ &+a_5\,g\,(\varphi e_i,Z)\,S(Y,\varphi e_i)+a_6\,g(\varphi e_i,Y)S(Z,\varphi e_i)\\ &+a_7\,r\,(g\,(Y,Z)\,g(\varphi e_i,\varphi e_i)-g\,(\varphi e_i,Z)\,g(Y,\varphi e_i))).\end{aligned} \tag{4.11}$$

Using (2.6), (2.7), (2.15) and (2.17) in (4.11), we get

$$\begin{aligned}&\sum_{i=1}^{2n}\mathcal{T}(\varphi e_i,Y,Z,\varphi e_i)\\ =\ &(a_0+2na_1+a_2+a_3+a_5+a_6)S(Y,Z)\\ &-(a_2+a_6)S\,(Z,\xi)\,\eta(Y)-\ (a_3+a_5)S(Y,\xi)\eta(Z)\\ &+(a_4(r-2n)+2n\,a_7\,r)g\,(Y,Z)+\ (-a_0-a_7\,r)g(\varphi Y,\varphi Z).\end{aligned} \tag{4.12}$$

If M satisfies (4.4) then from (4.12) we get

$$\begin{aligned}&(a_0+2na_1+a_2+a_3+a_5+a_6)S(Y,Z)\\ =\ &(a_2+a_6)S\,(Z,\xi)\,\eta(Y)+\ (a_3+a_5)S(Y,\xi)\eta(Z)\\ &+(a_0+a_7\,r)g(\varphi Y,\varphi Z)-\ (a_4(r-2n)+2n\,a_7\,r)g\,(Y,Z)\,.\end{aligned} \tag{4.13}$$

Case 1. If $a_0+2na_1+a_2+a_3+a_5+a_6\neq 0$, then by using (2.2) and (2.9) in (4.13), we get (4.5).
Case 2. If $a_0+2na_1+a_2+a_3+a_5+a_6=0$ and $4n^2a_7+(2n+1)a_4\neq 0$, we get from (4.13)

$$\begin{aligned}0\ =\ &(a_0-(2n-1)a_7r-(r-2n)a_4)\,g(Y,Z)\\ &-\ ((a_0+a_7\,r)-2n(a_2+a_3+a_5+a_6))\,\eta(Y)\eta(Z).\end{aligned}$$

Contracting the above equation, we get

$$\left(4n^2a_7+(2n+1)a_4\right)r=2n\,(a_0+a_2+a_3+(2n+1)a_4+a_5+a_6)\,. \tag{4.14}$$

From (4.14), we have (4.8).
Case 3. If $a_0+2na_1+a_2+a_3+a_5+a_6=0$ and $4n^2a_7+(2n+1)a_4=0$, then from (4.14), we get (4.9). This proves the result. □

Remark 4.1. The conclusions of Theorem 4.1 remain true if the condition of the equation (4.4) is replaced by the condition of being quasi $\mathcal{T}$-flat.

Remark 4.2. In [4, Lemma 3.5], it is proved that a $(2n+1)$-dimensional quasi conharmonically flat Sasakian manifold is η-Einstein. But here we prove this result for K-contact manifold, which is weaker condition than that of Sasakian manifold.

Next, we have the following.

Theorem 4.2. *Let M be a $(2n+1)$-dimensional is ξ-$\mathcal{T}$-flat K-contact manifold.*

1. *If $a_4 \neq 0$, then*

$$S = A_2\, g + A_3 \eta \otimes \eta, \tag{4.15}$$

where

$$A_2 = -\frac{a_0 + 2na_1 + a_7 r}{a_4} \tag{4.16}$$

and

$$A_3 = \frac{a_0 - 2na_2 - 2na_3 - 2na_5 - 2na_6 + a_7 r}{a_4}. \tag{4.17}$$

Therefore it is an η-Einstein manifold. In particular, M becomes an Einstein manifold provided

$$a_0 - 2na_2 - 2na_3 - 2na_5 - 2na_6 + a_7 r = 0.$$

Consequently, we have the following:

***K*-contact manifold**	***Einstein/ η-Einstein***	**$S =$**
ξ-quasi-conformally flat	*η-Einstein*	$-\left(\frac{a_0}{a_1} + 2n - \frac{r}{2n+1}\left(\frac{a_0}{2na_1} + 2\right)\right) g + \left(\frac{a_0}{a_1} + 4n - \frac{r}{2n+1}\left(\frac{a_0}{2na_1} + 2\right)\right) \eta \otimes \eta$
ξ-conformally flat	*η-Einstein*	$\left(\frac{r}{2n} - 1\right) g + \left(2n + 1 - \frac{r}{2n}\right) \eta \otimes \eta$
ξ-conharmonically flat	*η-Einstein*	$-g + (2n+1)\eta \otimes \eta$
ξ-$\mathcal{M}$-projectively flat	*Einstein*	$2ng$
ξ-$\mathcal{W}_2$-flat [12]	*Einstein*	$2ng$
ξ-$\mathcal{W}_3$-flat	*η-Einstein*	$-2ng + 4n\eta \otimes \eta$
ξ-$\mathcal{W}_7$-flat	*η-Einstein*	$2n\eta \otimes \eta$
ξ-$\mathcal{W}_9$-flat	*Einstein*	$2ng$

2. *If $a_4 = 0$ and $a_7 \neq 0$, then*

$$r = -\frac{a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6}{a_7}. \tag{4.18}$$

Consequently, if M is ξ-concircularly flat or ξ-pseudo-projectively flat then $r = 2n(2n+1)$.

3. *If $a_4 = 0$ and $a_7 = 0$, then*

$$a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6 = 0. \tag{4.19}$$

In particular, for projective curvature tensor, $\mathcal{W}_0$-curvature tensor, $\mathcal{W}_1^*$-curvature tensor, $\mathcal{W}_6$-curvature tensor and $\mathcal{W}_8$-curvature tensor, the equation (4.19) is true.

Proof. By (3.2) and (2.3), we get

$$\begin{aligned} \mathcal{T}(X,Y,\xi,W) &= a_0\, R(X,Y,\xi,W) \\ &\quad + a_1\, S\,(Y,\xi)\, g(X,W) + a_2 S\,(X,\xi)\, g(Y,W) \\ &\quad + a_3\, S(X,Y)\eta(W) + a_4\eta(Y)S(X,W) \\ &\quad + a_5\eta(X)S(Y,W) + a_6\, g(X,Y)S(\xi,W) \\ &\quad + a_7\, r\,(\eta(Y)g(X,W) - \eta(X)g(Y,W))\,. \end{aligned} \tag{4.20}$$

Using $Y=\xi$ in (4.20), we get

$$\begin{aligned} \mathcal{T}(X,\xi,\xi,W) &= a_0\, R(X,\xi,\xi,W) \\ &\quad + a_1\, S\,(\xi,\xi)\, g(X,W) + a_2\, S\,(X,\xi)\, g(\xi,W) \\ &\quad + a_3\, S(X,\xi)\eta(W) + a_4\,\eta(\xi)S(X,W) \\ &\quad + a_5\eta(X)S(\xi,W) + a_6\, g(X,\xi)S(\xi,W) \\ &\quad + a_7\, r\,(\eta(\xi)g(X,W) - \eta(X)g(\xi,W))\,. \end{aligned} \tag{4.21}$$

Case 1. If $a_4 \neq 0$, then using (2.9), (2.13) and (2.14) and the fact that M is ξ-$\mathcal{T}$-flat in (4.21), we get (4.15).
Case 2. If $a_4 = 0$ and $a_7 \neq 0$, then by using (2.9), (2.13) and (2.14) and the fact that M is ξ-$\mathcal{T}$-flat in (4.21), we get

$$(a_0 + 2na_1 + a_7 r)\, g(Y,Z) = ((a_0 + a_7\, r) - 2n(a_2 + a_3 + a_5 + a_6))\,\eta(Y)\eta(Z),$$

Contracting the above equation, we get

$$a_7 r = -\,(a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6)\,, \tag{4.22}$$

If $a_7 \neq 0$, we get (4.18).
Case 3. If $a_4 = 0$ and $a_7 = 0$, from (4.22) we get (4.19). This proves the result. □

Remark 4.3. For $\mathcal{W}_0^*$-curvature tensor, $\mathcal{W}_1$-curvature tensor, $\mathcal{W}_4$-curvature tensor and $\mathcal{W}_5$-curvature tensor (4.19) do not hold.

Remark 4.4. In [4], it is proved that if M is a $(2n{+}1)$-dimensional ξ-conharmonically flat K-contact manifold, then the Ricci tensor satisfies

$$\begin{aligned} S(X,Y) &= -g(X,Y) + S(X,\xi)\eta(Y) + S(Y,\xi)\eta(X) \\ &\quad -(2n-1)\eta(X)\eta(Y). \end{aligned}$$

But from Theorem 4.2, we see that any ξ-conharmonically flat K-contact manifold is η-Einstein because in this case $S = -g + (2n+1)\eta\otimes\eta$.

Theorem 4.3. *Let M be a $(2n+1)$-dimensional K-contact manifold such that $a_4 \neq 0$ and $a_4 + 2na_7 \neq 0$. Then M is ξ-$\mathcal{T}$-flat if and only if Ricci tensor satisfies the equations* (4.15),

$$r = -\frac{2n\,(a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6)}{a_4 + 2na_7} = B_0,\ \text{(say)}, \tag{4.23}$$

and

$$\begin{aligned} -\,a_0 R(X,Y)\xi &= (2na_1 + a_4 A_2 + a_7 B_0)\eta(Y)X \\ &\quad + (2na_2 + a_5 A_2 - a_7 B_0)\eta(X)Y \\ &\quad + (2na_6 + a_3 A_2)g(X,Y)\xi \\ &\quad + (a_3 + a_4 + a_5)A_3\eta(X)\eta(Y)\xi. \end{aligned} \tag{4.24}$$

Proof. From Theorem 4.2, we get (4.15). Then contracting (4.15), we have (4.23). Using (4.15) and (4.23) in (4.20), we get (4.24).

Conversely if (4.15), (4.23) and (4.24) are satisfied, then the K-contact manifold is ξ-$\mathcal{T}$-flat. □

In [17], it is proved that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstein Sasakian manifold. In fact, we have the following

Corollary 4.1. *Let M be a $(2n+1)$-dimensional K-contact manifold. Then the following statements are true:*

(a) *For $\mathcal{T} \in \{\mathcal{C}_*, \mathcal{C}, \mathcal{L}\}$, M is ξ-$\mathcal{T}$-flat if and only if it is η-Einstein and Sasakian.*

(b) *For $\mathcal{T} \in \{\mathcal{M}, \mathcal{W}_2\}$, M is ξ-$\mathcal{T}$-flat if and only if it is Einstein and Sasakian.*

Remark 4.5. M is ξ-conformally flat, then from (4.23), the scalar curvature r is in indeterminate form.

Theorem 4.4. *If a $(2n+1)$-dimensional K-contact manifold M is φ-$\mathcal{T}$-flat such that*

$$a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0,$$

then

$$\begin{aligned} & a_0 R(\varphi X, \varphi Y, \varphi Z, \varphi W) \\ = & -(A_4 a_1 + A_4 a_4 + r a_7) g(\varphi Y, \varphi Z) g(\varphi X, \varphi W) \\ & -(A_4 a_2 + A_4 a_5 - r a_7) g(\varphi X, \varphi Z) g(\varphi Y, \varphi W) \\ & -(A_4 a_3 + A_4 a_6) g(\varphi X, \varphi Y) g(\varphi Z, \varphi W). \end{aligned} \tag{4.25}$$

Proof. Let M be a $(2n+1)$-dimensional K-contact manifold. From (3.2) we have

$$\begin{aligned} \mathcal{T}(\varphi X, \varphi Y, \varphi Z, \varphi W) \ = \ & a_0 R(\varphi X, \varphi Y, \varphi Z, \varphi W) \\ & + a_1 S(\varphi Y, \varphi Z) g(\varphi X, \varphi W) + a_2 S(\varphi X, \varphi Z) g(\varphi Y, \varphi W) \\ & + a_3 S(\varphi X, \varphi Y) g(\varphi Z, \varphi W) + a_4 g(\varphi Y, \varphi Z) S(\varphi X, \varphi W) \\ & + a_5 g(\varphi X, \varphi Z) S(\varphi Y, \varphi W) + a_6 g(\varphi X, \varphi Y) S(\varphi Z, \varphi W) \\ & + a_7 r \left(g(\varphi Y, \varphi Z) g(\varphi X, \varphi W) - g(\varphi X, \varphi Z) g(\varphi Y, \varphi W) \right). \end{aligned} \tag{4.26}$$

For an orthonormal basis of vector fields $\{e_1, \ldots, e_{2n}, \xi\}$ in M, from (4.26) it follows that

$$\begin{aligned} \mathcal{T}(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i) \ = \ & a_0 R(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i) \\ & + a_1 S(\varphi Y, \varphi Z) g(\varphi e_i, \varphi e_i) + a_2 S(\varphi e_i, \varphi Z) g(\varphi Y, \varphi e_i) \\ & + a_3 S(\varphi e_i, \varphi Y) g(\varphi Z, \varphi e_i) + a_4 g(\varphi Y, \varphi Z) S(\varphi e_i, \varphi e_i) \\ & + a_5 g(\varphi e_i, \varphi Z) S(\varphi Y, \varphi e_i) + a_6 g(\varphi e_i, \varphi Y) S(\varphi Z, \varphi e_i) \\ & + a_7 r \left(g(\varphi Y, \varphi Z) g(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z) g(\varphi Y, \varphi e_i) \right). \end{aligned} \tag{4.27}$$

Using (2.6), (2.8), (2.17) and (4.3) in (4.27), we get

$$\begin{aligned} 0 \ = \ & (a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6) S(\varphi Y, \varphi Z) \\ & + (-a_0 + (r - 2n) a_4 + (2n-1) a_7 r) g(\varphi Y, \varphi Z). \end{aligned} \tag{4.28}$$

(4.28) can be rewritten as

$$S(\varphi Y, \varphi Z) = A_4\, g(\varphi Y, \varphi Z), \tag{4.29}$$

where

$$A_4 = \frac{(a_0 - (r-2n)a_4 - (2n-1)ra_7)}{(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)}$$

and $a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0$. Using (4.3) and (4.29) in (4.26), we get (4.25). □

Theorem 4.5. *Let M be a $(2n+1)$-dimensional Sasakian manifold such that $a_0 \neq 0$ and*

$$a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0.$$

If M is φ-$\mathcal{T}$-flat, then

$$\begin{aligned}
&- a_0 R(X,Y,Z,W) \\
&= (A_4a_1 + A_4a_4 + a_7r)g(Y,Z)g(X,W) \\
&+ (A_4a_2 + A_4a_5 - a_7r)g(X,Z)g(Y,W) \\
&+ (A_4a_3 + A_4a_6)g(X,Y)g(Z,W) \\
&- (A_4a_3 + A_4a_6)g(X,Y)\eta(Z)\eta(W) \\
&- (A_4a_3 + A_4a_6)g(Z,W)\eta(X)\eta(Y) \\
&- (A_4a_1 + A_4a_4 + a_7r + a_0)\eta(Y)\eta(Z)g(X,W) \\
&- (A_4a_1 + A_4a_4 + a_7r + a_0)\eta(X)\eta(W)g(Y,Z) \\
&- (A_4a_2 + A_4a_5 - a_7r - a_0)\eta(Y)\eta(W)g(X,Z) \\
&- (A_4a_2 + A_4a_5 - a_7r - a_0)\eta(X)\eta(Z)g(Y,W) \\
&+ \left(\sum_{i=1}^{6} a_i\right) A_4\, \eta(X)\eta(Y)\eta(Z)\eta(W).
\end{aligned} \tag{4.30}$$

If M satisfies (4.30)*, then*

$$\begin{aligned}
&T(X,Y,Z,\varphi W) \\
&= (a_1 + a_4)(A_5 - A_4)g(Y,Z)g(X,\varphi W) \\
&+ (a_2 + a_5)(A_5 - A_4)g(X,Z)g(Y,\varphi W) \\
&+ (a_3 + a_6)(A_5 - A_4)g(X,Y)g(Z,\varphi W) \\
&+ ((a_3 + a_6)A_4 + ((2n-1) - A_5)a_3)g(Z,\varphi W)\eta(X)\eta(Y) \\
&+ ((a_1 + a_4)A_4 + a_7r + a_0 \\
&\qquad +((2n-1) - A_5)a_1)g(X,\varphi W)\eta(Y)\eta(Z) \\
&+ ((a_2 + a_5)A_4 - a_7r - a_0 \\
&\qquad +((2n-1) - A_5)a_2)g(Y,\varphi W)\eta(X)\eta(Z).
\end{aligned} \tag{4.31}$$

Proof. Let M be a $(2n+1)$-dimensional Sasakian manifold. Now, assume that M is φ-$\mathcal{T}$-flat. In a Sasakian manifold, in view of (2.1), (2.4) and (2.5) we can easily verify that

$$\begin{aligned}
R(\varphi^2X,\varphi^2Y,\varphi^2Z,\varphi^2W) &= R(X,Y,Z,W) \\
&- g(Y,Z)\eta(X)\eta(W) + g(X,Z)\eta(Y)\eta(W) \\
&+ g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z).
\end{aligned} \tag{4.32}$$

Replacing X, Y, Z, W by $\varphi X, \varphi Y, \varphi Z, \varphi W$ respectively in (4.25), we get

$$\begin{aligned} & -a_0 R(\varphi^2 X, \varphi^2 Y, \varphi^2 Z, \varphi^2 W) \\ = \; & (a_1 A_4 + a_4 A_4 + a_7 r) g(\varphi Y, \varphi Z) g(\varphi X, \varphi W) \\ & + (a_2 A_4 + a_5 A_4 - a_7 r) g(\varphi X, \varphi Z) g(\varphi Y, \varphi W) \\ & + (a_3 A_4 + a_6 A_4) g(\varphi X, \varphi Y) g(\varphi Z, \varphi W). \end{aligned} \tag{4.33}$$

From (4.32) and (4.33), we get (4.30). For an orthonormal basis of vector fields $\{e_1, \ldots, e_{2n}, \xi\}$ in M, from (4.30) it follows that

$$S(Y, Z) = A_5\, g(\varphi Y, \varphi Z) + (2n-1)\eta(Y)\eta(Z), \tag{4.34}$$

where

$$A_5 = -\frac{(2n-1)(a_1 A_4 + a_4 A_4 + a_7 r) + A_4(a_2 + a_3 + a_5 + a_6) - a_7 r - a_0}{a_0}.$$

Using (4.34) and (4.30) in (3.2), we get (4.31). □

5. Compact regular K-contact manifold

A $(2n+1)$-dimensional K-contact manifold M is said to be regular if for each point $p \in M$ there is a cubical coordinate neighborhood U of p such that the integral curves of ξ in U pass through U only once. Moreover, the orbits of ξ are closed curves if M is compact. Let B denotes the space of orbits of ξ. Then there is the natural projection $\pi : M \to B$ and B is a $2n$-dimensional differentiable manifold such that π is a differentiable map. In 1958, Boothby and Wang [2] proved that if M is a $(2n+1)$-dimensional compact regular contact manifold, then M is a principal S^1-bundle over B, where S^1 is a 1-dimensional compact Lie group which is isomorphic to the 1-parameter group of global transformations generated by ξ. Now, we prove the following result:

Theorem 5.1. *Let M be a $(2n+1)$-dimensional φ-$\mathcal{T}$-flat compact regular K-contact manifold such that $a_0 \neq 0$ and*

$$a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0.$$

Then M is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature c given by

$$c = -\frac{a_1 A_4 + a_4 A_4 + a_7 r - 3a_0}{a_0}. \tag{5.1}$$

Consequently, we have the following:

***K*-contact manifold**	***c***
φ-*quasi-conformally flat*	$-\dfrac{2a_1}{(a_0+(2n-2)a_1)}\left(\left(1+\dfrac{2n-1}{2n(2n+1)}r\right) + \left((2n-r)+\dfrac{2(2n-1)}{2n+1}r\right)\dfrac{a_1}{a_0}\right) + \dfrac{r}{2n+1}\left(\dfrac{1}{2n}+\dfrac{2a_1}{a_0}\right)+3$
φ-*conformally flat* [3]	$\dfrac{r-4n}{2n(2n-1)}+3$
φ-*conharmonically flat* [4]	$3-\dfrac{2}{2n-1}$
φ-*concircularly flat*	3
φ-*pseudo-projectively flat*	$-\dfrac{a_1}{(a_0+(2n-1)a_1)}\left(\left(1+\dfrac{r}{2n}\right)+\dfrac{2n-1}{2n+1}\dfrac{a_1}{a_0}r\right) + \dfrac{r}{2n+1}\left(\dfrac{1}{2n}+\dfrac{a_1}{a_0}\right)+3$
φ-*projectively flat*	4
φ-$\mathcal{M}$-*projective flat*	4
φ-$\mathcal{W}_0$-*flat*	4
φ-$\mathcal{W}_0^*$-*flat*	$\dfrac{4(3n-1)}{4n-1}$
φ-$\mathcal{W}_1$-*flat*	$\dfrac{4(3n-1)}{4n-1}$
φ-$\mathcal{W}_1^*$-*flat*	4
φ-$\mathcal{W}_2$-*flat* [12]	4
φ-$\mathcal{W}_3$-*flat*	$\dfrac{4(3n-1)}{4n-1}$
φ-$\mathcal{W}_4$-*flat*	3
φ-$\mathcal{W}_5$-*flat*	3
φ-$\mathcal{W}_6$-*flat*	4
φ-$\mathcal{W}_8$-*flat*	4
φ-$\mathcal{W}_9$-*flat*	4

Proof. Let M be a compact regular K-contact manifold. Since in a K-contact manifold ξ is a Killing vector field, the metric g is invariant under the action of the group S^1. Hence a metric $\tilde{g}$ and a $(1,1)$ tensor field J on B can be defined by

$$\tilde{g}(X,Y) = g(X^*,Y^*), \tag{5.2}$$

$$JX = \pi_*\varphi X^* \tag{5.3}$$

for any vector fields $X, Y \in TB$, where $*$ denotes the horizontal lift with respect to η. It is well-known that $(J,\tilde{g})$ is an almost Kaehler structure on B [7]. Let $\tilde{R}$ denote the Riemann curvature tensor on B. Then we have [3]

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R(X^*,Y^*,Z^*,W^*) + 2g(X^*,\varphi Y^*)g(\varphi Z^*,W^*)\\ &\quad - g(Z^*,\varphi X^*)g(\varphi Y^*,W^*) + g(Z^*,\varphi Y^*)g(\varphi X^*,W^*)\end{aligned}$$

for all $X, Y, Z, W \in TB$. So from (5.3), we obtain [3]

$$
\begin{aligned}
(5.4) \qquad & \tilde{R}(JX, JY, JZ, JW) \\
= \; & R(\varphi X^*, \varphi Y^*, \varphi Z^*, \varphi W^*) + 2g(X^*, \varphi Y^*)g(\varphi Z^*, W^*) \\
& - g(Z^*, \varphi X^*)g(\varphi Y^*, W^*) + g(Z^*, \varphi Y^*)g(\varphi X^*, W^*).
\end{aligned}
$$

Moreover, if M is φ-$\mathcal{T}$-flat then from Theorem 4.4 and (5.4) we have

$$
\begin{aligned}
(5.5) \qquad & - a_0\tilde{R}(JX, JY, JZ, JW) \\
= \; & (a_1A_4 + a_4A_4 + ra_7)g(\varphi Y^*, \varphi Z^*)g(\varphi X^*, \varphi W^*) \\
& + (a_2A_4 + a_5A_4 - ra_7)g(\varphi X^*, \varphi Z^*)g(\varphi Y^*, \varphi W^*) \\
& + (a_3A_4 + a_6A_4)g(\varphi X^*, \varphi Y^*)g(\varphi Z^*, \varphi W^*) \\
& - 2a_0g(X^*, \varphi Y^*)g(\varphi Z^*, W^*) \\
& + a_0g(Z^*, \varphi X^*)g(\varphi Y^*, W^*) \\
& - a_0g(Z^*, \varphi Y^*)g(\varphi X^*, W^*).
\end{aligned}
$$

In (5.5) replacing X and W by JX and JW respectively, we have

$$
\begin{aligned}
-a_0\tilde{R}(X, JY, JZ, W) \;=\; & (a_1A_4 + a_4A_4 + ra_7)g(Y^*, Z^*)g(X^*, W^*) \\
& + (a_2A_4 + a_5A_4 - ra_7)g(X^*, \varphi Z^*)g(\varphi Y^*, W^*) \\
& + (a_3A_4 + a_6A_4)g(X^*, \varphi Y^*)g(\varphi Z^*, W^*) \\
& - 2a_0g(X^*, Y^*)g(Z^*, W^*) \\
& - a_0g(Z^*, X^*)g(Y^*, W^*) \\
& - a_0g(Z^*, \varphi Y^*)g(\varphi X^*, W^*),
\end{aligned}
$$

which for a unit vector field $X \in TB$ gives

$$
\tilde{R}(X, JX, JX, X) = -\frac{a_1A_4 + a_4A_4 + ra_7 - 3a_0}{a_0}.
$$

Thus the base manifold B is of constant holomorphic sectional curvature given by (5.1). □

References

[1] Blair, D. E., Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, 203 (Boston, MA: Birkhauser Boston, Inc.) (2002).
[2] Boothby, W. M. and Wang, H. C., On contact manifolds, Ann. Math., 68 (1958), 721–734.
[3] Cabrerizo, J. L., Fernández, L. M., Fernández, M. and Zhen, G., The structure of a class of K-contact manifolds, Acta Math. Hungar., 82 (1999), no. 4, 331–340.
[4] Dwivedi, M. K. and Kim, J.-S., On conharmonic curvature tensor in K-contact and Sasakian manifolds, Bull. Malays. Math. Sci. Soc. 34(1) (2011), 171-180.
[5] Eisenhart, L. P., Riemannian Geometry, Princeton University Press, 1949.
[6] Ishii, Y., On conharmonic transformations, Tensor (N.S.), 7 (1957), 73–80.
[7] Ogiue, K., On fibrings of almost contact manifolds, Kodai Math. Sem. Rep., 17 (1965) 53–62.
[8] Pokhariyal G. P. and Mishra, R. S., Curvature tensors and their relativistic significance, Yokohama Math. J., 18 (1970), 105–108.
[9] Pokhariyal, G. P. and Mishra, R. S., Curvature tensors and their relativistic significance II, Yokohama Math. J., 19 (1971), no. 2, 97–103.
[10] Pokhariyal, G. P., Relativistic significance of curvature tensors, Internat. J. Math. Math. Sci., 5 (1982), no. 1, 133–139.
[11] Prasad, B., A pseudo projective curvature tensor on a Riemannian manifold, Bull. Calcutta Math. Soc., 94 (2002), no. 3, 163–166.
[12] Tripathi, M. M. and Dwivedi, M. K., The structure of some classes of K-contact manifolds, Proc. Indian Acad. Sci. (Math. Sci.), 118 (2008), no. 3, 371–379.

[13] Tripathi, M. M. and Gupta, P., $\mathcal{T}$-curvature tensor on a semi-Riemannian manifold, J. Adv. Math. Stud. 4 (2011), No. 1, 117-129.

[14] Yano, K., Concircular Geometry I. Concircular transformations, Math. Institute, Tokyo Imperial Univ. Proc., 16 (1940), 195–200.

[15] Yano, K. and Bochner, S., Curvature and Betti numbers, Annals of Mathematics Studies 32, Princeton University Press, 1953.

[16] Yano, K. and Sawaki, S., Riemannian manifolds admitting a conformal transformation group, J. Diff. Geom., 2 (1968), 161–184.

[17] Zhen, G., Cabrerizo, J. L., Fernández, L. M. and Fernández, M., On ξ-conformally flat contact metric manifolds, Indian J. Pure Appl. Math., 28 (1997), 725–734.

[18] Zhen, G., On conformal symmetric K-contact manifolds, Chinese Quart. J. Math., 7 (1992), 5–10.

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