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CANONICAL EQUATIONS FOR NONSINGULAR QUADRICS AND HERMITIAN VARIETIES OF WITT INDEX AT LEAST $\frac{n-1}{2}$ OF $\mathrm{PG}(n,\mathbb{K})$, n ODD

BART DE BRUYN

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Abstract. Let V be a vector space of even dimension $n+1 \geq 4$ over a field $\mathbb{K}$, and let $\mathcal{F}$ be a nonsingular quadric or Hermitian variety of Witt index at least $\frac{n-1}{2}$ of $\mathrm{PG}(V)$. $\mathcal{F}$ is described by a certain equation, which depends on the chosen reference system. We give a survey of the methods which allow to determine a canonical equation for $\mathcal{F}$ without performing the actual coordinate transformations, and give complete proofs for these methods. We give necessary and sufficient conditions for two nonsingular quadrics or Hermitian varieties of Witt index at least $\frac{n-1}{2}$ to be projectively equivalent (respectively, to be equivalent under an automorphism of $\mathrm{PG}(V)$). Many of the things we will discuss here are only implicit in the literature.

1. Introduction

Throughout this manuscript, $n \geq 3$ denotes a given strictly positive odd integer, $\mathbb{K}$ denotes a given field and V denotes a given $(n+1)$-dimensional vector space over $\mathbb{K}$.

1.1. **The case of quadrics.** Let X_{ij}, $i,j \in \{0,\ldots,n\}$, be $(n+1)^2$ indeterminates. The $(n+1)^2$-tuple $(X_{00}, X_{01}, \ldots, X_{0n}, X_{10}, X_{11}, \ldots, X_{1n}, \ldots, X_{n0}, X_{n1}, \ldots, X_{nn})$ will also be denoted by $(X_{ij} \,|\, 0 \leq i,j \leq n)$ and the $\frac{(n+1)(n+2)}{2}$-tuple $(X_{00}, X_{01}, \ldots, X_{0n}, X_{11}, X_{12}, \ldots, X_{1n}, \ldots, X_{n-1,n-1}, X_{n-1,n}, X_{n,n})$ will also be denoted by $(X_{ij} \mid 0 \leq i \leq j \leq n)$. The X_{ij}'s will serve as indeterminates for the polynomial rings $\mathbb{Z}[X_{ij} \mid 0 \leq i,j \leq n]$ and $\mathbb{Z}[X_{ij} \mid 0 \leq i \leq j \leq n]$. Define

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the following polynomials of $\mathbb{Z}[X_{ij} \mid 0 \leq i \leq j \leq n]$:

$$h[(X_{ij} \mid 0 \leq i \leq j \leq n)] := \begin{vmatrix} 2 \cdot X_{00} & X_{01} & X_{02} & \cdots & X_{0n} \\ X_{01} & 2 \cdot X_{11} & X_{12} & \cdots & X_{1n} \\ X_{02} & X_{12} & 2 \cdot X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{0n} & X_{1n} & X_{2n} & \cdots & 2 \cdot X_{nn} \end{vmatrix},$$

$$g[(X_{ij} \mid 0 \leq i \leq j \leq n)] := \begin{vmatrix} 0 & X_{01} & X_{02} & \cdots & X_{0n} \\ -X_{01} & 0 & X_{12} & \cdots & X_{1n} \\ -X_{02} & -X_{12} & 0 & \cdots & X_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_{0n} & -X_{1n} & -X_{2n} & \cdots & 0 \end{vmatrix},$$

$$f_1[(X_{ij} \mid 0 \leq i \leq j \leq n)] := g[(X_{ij} \mid 0 \leq i \leq j \leq n)] + (-1)^{\frac{n-1}{2}} \cdot h[(X_{ij} \mid 0 \leq i \leq j \leq n)].$$

Since the matrix M_g whose determinant defines g is a skew-symmetric matrix, there exists a polynomial $Pf[(X_{ij} \mid 0 \leq i \leq j \leq n)] \in \mathbb{Z}[X_{ij} \mid 0 \leq i \leq j \leq n]$ such that $g[(X_{ij} \mid 0 \leq i \leq j \leq n)] = \Big(Pf[(X_{ij} \mid 0 \leq i \leq j \leq n)]\Big)^2$, see e.g. Jacobson [4, Theorem 6.4] or Lang [5, XV §9]. The polynomial $Pf[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ is called the *Pfaffian* of M_g. For an explicit description of this Pfaffian in terms of the indeterminates X_{ij}, see e.g. Marcus [6, p. 280].

We will prove the following proposition in Section 2:

Proposition 1.1. *All the coefficients of the polynomial $f_1 \in \mathbb{Z}[X_{ij} \mid 0 \leq i \leq j \leq n]$ are multiples of* 4.

Now, define the following polynomial of $\mathbb{Z}[X_{ij} \mid 0 \leq i \leq j \leq n]$:

$$f[(X_{ij} \mid 0 \leq i \leq j \leq n)] := \frac{f_1[(X_{ij} \mid 0 \leq i \leq j \leq n)]}{4}.$$

Let $\phi_{\mathbb{K}} : \mathbb{Z} \to \mathbb{K}$ denote the unique group homomorphism from $(\mathbb{Z}, +)$ to $(\mathbb{K}, +)$ which maps the element $1 \in \mathbb{Z}$ to the identity element for the multiplication of $\mathbb{K}$. For every polynomial $p \in \mathbb{Z}[X_{ij} \mid 0 \leq i \leq j \leq n]$, let $p_{\mathbb{K}}$ denote the unique element of $\mathbb{K}[X_{ij} \mid 0 \leq i \leq j \leq n]$ obtained from p by replacing each coefficient of p by its image under the map $\phi_{\mathbb{K}}$.

Now, let $B = (\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$ be an ordered basis of V. For all $i, j \in \{0, \ldots, n\}$ satisfying $i \leq j$, choose an element $a_{ij} \in \mathbb{K}$ and consider the quadric Q of $\mathrm{PG}(V)$ whose equation with respect to B is given by

$$F(X_0, \ldots, X_n) := \sum_{0 \leq i \leq j \leq n} a_{ij} X_i X_j = 0. \tag{1.1}$$

A point $p \in Q$ is called a *singular point* of Q if $pr \subseteq Q$ for every point $r \in Q \setminus \{p\}$. The quadric Q is called *singular* if it has a singular point; otherwise, it is called *nonsingular*. If Q is nonsingular, then its Witt index is at most $\frac{n+1}{2}$. If the quadric Q is singular, then the linear system $\frac{\partial F}{\partial X_0} = \frac{\partial F}{\partial X_1} = \cdots = \frac{\partial F}{\partial X_n} = 0$ has a nonzero solution for $(X_0, \dots, X_n)$, i.e., the matrix

$$A := \begin{bmatrix} 2\cdot a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\ a_{01} & 2\cdot a_{11} & a_{12} & \cdots & a_{1n} \\ a_{02} & a_{12} & 2\cdot a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0n} & a_{1n} & a_{2n} & \cdots & 2\cdot a_{nn} \end{bmatrix}. \tag{1.2}$$

is singular. If $\mathrm{char}(\mathbb{K}) \neq 2$, then also the converse is true: if A is singular, then Q is singular. A similar conclusion cannot be drawn in the case $\mathrm{char}(\mathbb{K}) = 2$.

The quadric Q of $\mathrm{PG}(V)$ is called a *hyperbolic quadric* if it is a nonsingular quadric of Witt index $\frac{n+1}{2}$. If Q is a nonsingular quadric of Witt index $\frac{n-1}{2}$, then Q is called an *elliptic quadric* when its equation (1) defines a nonsingular quadric of Witt index $\frac{n+1}{2}$ over a suitable quadratic Galois extension of $\mathbb{K}$; otherwise it is called a *pseudo-elliptic quadric*. We will see in Section 3.3 that this definition is independent from the chosen ordered basis B and the quadratic equation which determines Q with respect to such a basis B.

Suppose the quadric Q is nonsingular and that in some way we know that the Witt index of Q is at least $\frac{n-1}{2}$. The following questions can then be raised.

- (Q1) Is there some nice criterion (stated in terms of the coefficients a_{ij}) which allows us to decide whether Q is a hyperbolic quadric, an elliptic quadric or a pseudo-elliptic quadric?
- (Q2) Is there a canonical equation to which Q can be reduced by means of linear coordinate transformations?
- (Q3) Is there a direct way to decide which canonical equation this is without explicitly performing the coordinate transformations?

The following theorem provides complete answers to the questions (Q1), (Q2) and (Q3).

Theorem 1.1. *Let Q be a nonsingular quadric of Witt index at least $\frac{n-1}{2}$ of $\mathrm{PG}(V)$ whose equation with respect to a given ordered basis of V is given by $F(X_0, X_1, \dots, X_n) = \sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j = 0$. Let A be the matrix as defined in equation* (1.2) *above.*

(1) *If $\mathrm{char}(\mathbb{K}) \neq 2$, then $\det(A) \neq 0$ and Q is either a hyperbolic or elliptic quadric. If $\eta := h_{\mathbb{K}}(a_{ij} \mid 0 \leq i \leq j \leq n) = \det(A)$, then Q is projectively equivalent to the quadric with equation $X_0^2 - (-1)^{\frac{n+1}{2}}\eta \cdot X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$. The quadric Q is a hyperbolic quadric if and only if $(-1)^{\frac{n+1}{2}}\eta$ is a square of $\mathbb{K}$. In this case, Q is projectively equivalent to the quadric with equation $X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n = 0$.*

(2) *Suppose $\mathrm{char}(\mathbb{K}) = 2$ and $\det(A) \neq 0$. Then Q is either a hyperbolic or elliptic quadric. Moreover, if $\eta_1 := f_{\mathbb{K}}(a_{ij} \mid 0 \leq i \leq j \leq n)$ and $\eta_2 = g_{\mathbb{K}}(a_{ij} \mid 0 \leq i \leq j \leq n)$, then $\eta_2 \neq 0$ and Q is projectively equivalent to the quadric with equation $X_0^2 + X_0X_1 + \frac{\eta_1}{\eta_2}X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$. The quadric Q is a hyperbolic quadric if and only if there exists a $\lambda \in \mathbb{K}$ such that $\frac{\eta_1}{\eta_2} = \lambda^2 + \lambda$. In this case, Q is projectively equivalent to the quadric with equation $X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n = 0$.*

(3) *Suppose* char($\mathbb{K}$) $= 2$ *and* $\det(A) = 0$. *Then Q is a pseudo-elliptic quadric. Moreover,* $\mathrm{rank}(A) = n-1$ *and Q is projectively equivalent to the quadric with equation* $X_0^2+\eta X_1^2+X_2X_3+\cdots+X_{n-1}X_n = 0$ *where* $\eta := F(u_0,\ldots,u_n)\cdot F(v_0,\ldots,v_n)$ *with* $[u_0\cdots u_n]^T$ *and* $[v_0\cdots v_n]^T$ *two linearly independent* $(n\times 1)$*-matrices over* $\mathbb{K}$ *belonging to the kernel of the matrix A. The element η is a non-square of* $\mathbb{K}$.

The following theorem, in combination with Theorem 1.1, gives necessary and sufficient conditions for two nonsingular quadrics of Witt index at least $\frac{n-1}{2}$ to be projectively equivalent (respectively, to be equivalent under an automorphism of $\mathrm{PG}(V)$). Notice that by Theorem 1.1(1)+(2), any two hyperbolic quadrics of $\mathrm{PG}(V)$ are projectively equivalent.

Theorem 1.2. (1) *Suppose* char($\mathbb{K}$) $\neq 2$ *and* $\mu_1,\mu_2 \in \mathbb{K}\setminus\{0\}$. *Then the quadrics Q_1 and Q_2 with respective equations* $X_0^2+\mu_1X_1^2+X_2X_3+\cdots+X_{n-1}X_n = 0$ *and* $X_0^2+\mu_2X_1^2+X_2X_3+\cdots+X_{n-1}X_n = 0$ *(with respect to some given reference system) are projectively equivalent if and only if there exists a* $\lambda\in\mathbb{K}\setminus\{0\}$ *such that* $\mu_2 = \mu_1\cdot\lambda^2$. *The quadrics Q_1 and Q_2 are equivalent under an automorphism of* $\mathrm{PG}(V)$ *if and only if there exists an automorphism θ of* $\mathbb{K}$ *and a* $\lambda\in\mathbb{K}\setminus\{0\}$ *such that* $\mu_2 = \mu_1^\theta\cdot\lambda^2$.

(2) *Suppose* char($\mathbb{K}$) $= 2$ *and* $\mu_1,\mu_2\in\mathbb{K}$. *Then the quadrics Q_1 and Q_2 with respective equations* $X_0^2+X_0X_1+\mu_1X_1^2+X_2X_3+\cdots+X_{n-1}X_n = 0$ *and* $X_0^2+X_0X_1+\mu_2X_1^2+X_2X_3+\cdots+X_{n-1}X_n = 0$ *are projectively equivalent if and only if there exists a* $\lambda\in\mathbb{K}$ *such that* $\mu_2 = \lambda^2+\lambda+\mu_1$. *The quadrics Q_1 and Q_2 are equivalent under an automorphism of* $\mathrm{PG}(V)$ *if and only if there exists an automorphism θ of* $\mathbb{K}$ *and a* $\lambda\in\mathbb{K}$ *such that* $\mu_2 = \lambda^2+\lambda+\mu_1^\theta$.

(3) *Suppose* char($\mathbb{K}$) $= 2$ *and* μ_1,μ_2 *non-squares of* $\mathbb{K}$. *Then the quadrics Q_1 and Q_2 with respective equations* $X_0^2+\mu_1X_1^2+X_2X_3+\cdots+X_{n-1}X_n = 0$ *and* $X_0^2+\mu_2X_1^2+X_2X_3+\cdots+X_{n-1}X_n = 0$ *are projectively equivalent if and only if there exist* $\lambda_1,\lambda_2\in\mathbb{K}$ *with* $\lambda_2\neq 0$ *such that* $\mu_2 = \lambda_1^2+\lambda_2^2\cdot\mu_1$. *The quadrics Q_1 and Q_2 are equivalent under an automorphism of* $\mathrm{PG}(V)$ *if and only if there exists an automorphism θ of* $\mathbb{K}$ *and* $\lambda_1,\lambda_2\in\mathbb{K}$ *with* $\lambda_2\neq 0$ *such that* $\mu_2 = \lambda_1^2+\lambda_2^2\cdot\mu_1^\theta$.

One of the motivations for writing this survey paper was the need to deal with questions (Q1), (Q2) and (Q3) in some specific cases where the characteristic of the field is equal to 2 (for the purpose to study isometric embeddings between some classes of near polygons).

Theorem 1.1(1) was proved in the finite case by Hirschfeld [2, Section 3] and Hirschfeld & Thas [3, Section 22.2]. Theorem 1.1(2) is inspired by the German paper of Witt [9] which discusses an invariant of a quadratic form discovered by Arf [1]. (For a discussion of this invariant, see also Scharlau [7, §9.4].) For the finite case, Theorem 1.1(2) is also mentioned without proof in Hirschfeld [2, Section 3] and Hirschfeld & Thas [3, Theorem 22.2.1]. (Notice here that, using the notations and conventions of [3], the number $\alpha := \frac{|B|-(-1)^{\frac{n+1}{2}}|A|}{4|B|}$ is not necessarily fixed under a projectivity, but the set $\{\alpha+t+t^2 \mid t\in\mathbb{K}\}$ certainly is).

1.2. **The case of Hermitian varieties.** Let ψ be an involutory automorphism of $\mathbb{K}$ and let $\mathbb{K}_\psi\subset\mathbb{K}$ denote the fix field of ψ. Then $[\mathbb{K}:\mathbb{K}_\psi] = 2$ and $\mathbb{K}/\mathbb{K}_\psi$ is a Galois extension. Let $B = (\bar{e}_0,\bar{e}_1,\ldots,\bar{e}_n)$ be an ordered basis of V, and let a_{ij},

$i, j \in \{0, \ldots, n\}$, be elements of $\mathbb{K}$ such that $a_{ij}^{\psi} = a_{ji}$. So, $A^{\psi} = A^T$, where A is the $(n+1) \times (n+1)$-matrix $(a_{ij})_{0 \leq i,j \leq n}$. Let $\mathcal{H}$ be the ψ-Hermitian variety of $\mathrm{PG}(V)$ whose equation with respect to B is given by

$$\sum_{0 \leq i,j \leq n} a_{ij} X_i X_j^{\psi} = 0,$$

i.e.

$$[X_0 \cdots X_n] \cdot A \cdot [Y_0 \cdots Y_n]^T = 0,$$

where we take the convention to identify each (1×1)-matrix with its unique entry.

A point $p \in \mathcal{H}$ is called a *singular point* of $\mathcal{H}$ if $pr \subseteq \mathcal{H}$ for every point $r \in \mathcal{H} \setminus \{p\}$. The ψ-Hermitian variety $\mathcal{H}$ is called *singular* if it contains a singular point; otherwise it is called *nonsingular*. $\mathcal{H}$ is singular if and only if $\det(A) = 0$. If $\mathcal{H}$ is nonsingular, then its Witt index is at most $\frac{n+1}{2}$.

We will now state results for Hermitian varieties which are similar to the ones for quadrics mentioned in Theorems 1.1 and 1.2.

Theorem 1.3. *If $\mathcal{H}$ is a nonsingular ψ-Hermitian variety of Witt index at least $\frac{n-1}{2}$, then $\mathcal{H}$ is projectively equivalent to the ψ-Hermitian variety with equation $X_0^{\psi+1} - (-1)^{\frac{n+1}{2}} \det(A) \cdot X_1^{\psi+1} + X_2 X_3^{\psi} + X_3 X_2^{\psi} + \cdots + X_{n-1} X_n^{\psi} + X_n X_{n-1}^{\psi} = 0$. $\mathcal{H}$ has Witt index $\frac{n+1}{2}$ if and only if there exists a $\lambda \in \mathbb{K} \setminus \{0\}$ such that $\lambda^{\psi+1} = (-1)^{\frac{n+1}{2}} \cdot \det(A)$. In this case, $\mathcal{H}$ is projectively equivalent to the ψ-Hermitian variety with equation $X_0 X_1^{\psi} + X_1 X_0^{\psi} + X_2 X_3^{\psi} + X_3 X_2^{\psi} + \cdots + X_{n-1} X_n^{\psi} + X_n X_{n-1}^{\psi} = 0$.*

Notice that since $A^T = A^{\psi}$, we have $\det(A) \in K_{\psi} \setminus \{0\}$ if $\mathcal{H}$ is nonsingular.

Theorem 1.4. *Let ψ_1 and ψ_2 be two involutory automorphisms of $\mathbb{K}$ and let $\mu_i \in \mathbb{K}_{\psi_i} \setminus \{0\}$, $i \in \{1, 2\}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hermitian varieties of $\mathrm{PG}(V)$ whose equations with respect to a given ordered basis of V are given by respectively $X_0^{\psi_1+1} + \mu_1 \cdot X_1^{\psi_1+1} + X_2 X_3^{\psi_1} + X_3 X_2^{\psi_1} + \cdots + X_{n-1} X_n^{\psi_1} + X_n X_{n-1}^{\psi_1} = 0$ and $X_0^{\psi_2+1} + \mu_2 \cdot X_1^{\psi_2+1} + X_2 X_3^{\psi_2} + X_3 X_2^{\psi_2} + \cdots + X_{n-1} X_n^{\psi_2} + X_n X_{n-1}^{\psi_2} = 0$. Then $\mathcal{H}_1$ and $\mathcal{H}_2$ are projectively equivalent if and only if $\psi_1 = \psi_2$ and there exists a $\lambda \in \mathbb{K} \setminus \{0\}$ such that $\mu_2 = \lambda^{\psi_1+1} \cdot \mu_1$. The Hermitian varieties $\mathcal{H}_1$ and $\mathcal{H}_2$ are equivalent under an automorphism of $\mathrm{PG}(V)$ if and only if there exists an automorphism θ of $\mathbb{K}$ and a $\lambda \in \mathbb{K} \setminus \{0\}$ such that $\psi_2 = \theta^{-1} \psi_1 \theta$ and $\mu_2 = \lambda^{\psi_2+1} \mu_1^{\theta}$.*

2. Proof of Proposition 1.1

In this section, we will prove Proposition 1.1.

Let $M = (m_{ij})_{0 \leq i,j \leq n}$ be one of the two matrices whose determinant defines either $h[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ or $g[(X_{ij} \mid 0 \leq i \leq j \leq n)]$. For every permutation σ of $\{0, \ldots, n\}$, $sgn(\sigma) \cdot m_{0,\sigma(0)} \cdot m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}$ is of the form $\epsilon X_{i_0 j_0} X_{i_1 j_1} \cdots X_{i_n j_n}$ where $\epsilon \in \mathbb{Z}$ and $i_0, j_0, \ldots, i_n, j_n$ is a sequence of $2n+2$ natural numbers satisfying: (i) $i_k \leq j_k$ for every $k \in \{0, \ldots, n\}$; (ii) each number of the set $\{0, \ldots, n\}$ occurs precisely two times in the sequence. So, in order to prove Proposition 1.1, it suffices to prove the following lemma.

Lemma 2.1. *Let $i_0, j_0, \ldots, i_n, j_n$ be a sequence of $2n+2$ natural numbers satisfying (i) $i_k \leq j_k$ for every $k \in \{0, \ldots, n\}$, (ii) each number of the set $\{0, \ldots, n\}$ occurs precisely two times in the sequence. Then the coefficient of $X_{i_0 j_0} X_{i_1 j_1} \cdots X_{i_n j_n}$ in $f_1[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ is a multiple of 4.*

We divide the proof of Lemma 2.1 into three parts according to the number of $k \in \{0,\ldots,n\}$ satisfying $i_k = j_k$.

(I) Let $i_0, j_0, \ldots, i_n, j_n$ be a sequence of $2n+2$ natural numbers satisfying the following properties: (i) $i_k \leq j_k$ for every $k \in \{0,\ldots,n\}$; (ii) there are at least two $k \in \{0,\ldots,n\}$ for which $i_k = j_k$; (iii) each number of the set $\{0,\ldots,n\}$ occurs precisely two times in the sequence.

Lemma 2.2. (1) *The coefficient of* $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ *in* $h[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ *is a multiple of* 4.

(2) *The coefficient of* $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ *in* $g[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ *is equal to* 0.

Proof. Let $k, l \in \{0,\ldots,n\}$ such that $k \neq l$, $i_k = j_k$ and $i_l = j_l$.

(1) If we expand the determinant which defines h into $(n+1)!$ terms, then we see that the monomial $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ can only occur in terms which involve the product $(2 \cdot X_{i_ki_k}) \cdot (2 \cdot X_{i_li_l})$. So, the coefficient of this monomial must indeed be a multiple of 4.

(2) This follows from the fact that the indeterminates $X_{i_ki_k}$ and $X_{i_li_l}$ do not occur in the matrix whose determinant defines g. □

Corollary 2.1. *The coefficient of* $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ *in* $f_1[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ *is a multiple of* 4.

(II) Let $i_0, j_0, \ldots, i_n, j_n$ be a sequence of $2n+2$ natural numbers satisfying the following properties: (i) $i_k \leq j_k$ for every $k \in \{0,\ldots,n\}$; (ii) there exists precisely one $k \in \{0,\ldots,n\}$ for which $i_k = j_k$; (iii) each number of the set $\{0,\ldots,n\}$ occurs precisely two times in the sequence.

Lemma 2.3. (1) *The coefficient of* $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ *in* $h[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ *is a multiple of* 4.

(2) *The coefficient of* $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ *in* $g[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ *is equal to* 0.

Proof. Let k^* denote the unique element of $\{0,\ldots,n\}$ for which $i_{k^*} = j_{k^*}$.

(1) Let M_h denote the symmetric matrix whose determinant defines h and let $M = (m_{ij})_{1 \leq i \leq j \leq n}$ denote the symmetric matrix obtained from M_h by deleting the (k^*+1)-th row and the (k^*+1)-th column. The coefficient of $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ in $h[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ is equal to two times the coefficient of $X_{i_0j_0}\cdots \widehat{X_{i_{k^*}j_{k^*}}} \cdots X_{i_nj_n}$ in $\det(M)$, i.e. equal to $2 \cdot \Sigma_\sigma sgn(\sigma)$ where the summation ranges over all permutations σ of $\{1,\ldots,n\}$ such that $m_{1,\sigma(1)} \cdot m_{2,\sigma(2)} \cdot \cdots \cdot m_{n,\sigma(n)} = X_{i_0j_0}\cdots \widehat{X_{i_{k^*}j_{k^*}}} \cdots X_{i_nj_n}$ $(*)$. Now, if σ is a permutation of $\{1,\ldots,n\}$ satisfying $(*)$, then $\sigma \neq \sigma^{-1}$ (since n is odd and $i_l \neq j_l$ for every $l \in \{0,\ldots,n\} \setminus \{k^*\}$) and also σ^{-1} satisfies $(*)$ (since M is symmetric). So, the number of permutations σ of $\{1,\ldots,n\}$ satisfying $(*)$ is a multiple of 2. It follows that the coefficient $2 \cdot \Sigma_\sigma sgn(\sigma)$ of $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ in h is a multiple of 4.

(2) This follows from the fact that the indeterminate $X_{i_{k^*}i_{k^*}}$ does not occur in the matrix whose determinant defines g. □

Corollary 2.2. *The coefficient of* $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ *in* $f_1[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ *is a multiple of* 4.

(III) Let $i_0, j_0, \dots, i_n, j_n$ be a sequence of $2n+2$ natural numbers satisfying the following properties: (i) $i_k < j_k$ for every $k \in \{0, \dots, n\}$; (ii) each number of the set $\{0, \dots, n\}$ occurs precisely two times in the sequence.

For every permutation σ of $\{0, \dots, n\}$ without fixpoints, let N_σ denote the number of $k \in \{0, \dots, n\}$ for which $\sigma(k) < k$. Let ψ denote the ring homomorphism from $\mathbb{Z}[X_{ij} \mid 0 \le i, j \le n]$ to $\mathbb{Z}[X_{ij} \mid 0 \le i \le j \le n]$ which maps X_{ij} to X_{ij} if $i \le j$ and X_{ij} to X_{ji} if $j < i$. Consider the following polynomials in $\mathbb{Z}[X_{ij} \mid 0 \le i, j \le n]$:

$$\tilde{h}[(X_{ij} \mid 0 \le i, j \le n)] := \begin{vmatrix} 2 \cdot X_{00} & X_{01} & X_{02} & \cdots & X_{0n} \\ X_{10} & 2 \cdot X_{11} & X_{12} & \cdots & X_{1n} \\ X_{20} & X_{21} & 2 \cdot X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{n0} & X_{n1} & X_{n2} & \cdots & 2 \cdot X_{nn} \end{vmatrix},$$

$$\tilde{g}[(X_{ij} \mid 0 \le i, j \le n)] := \begin{vmatrix} 0 & X_{01} & X_{02} & \cdots & X_{0n} \\ -X_{10} & 0 & X_{12} & \cdots & X_{1n} \\ -X_{20} & -X_{21} & 0 & \cdots & X_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_{n0} & -X_{n1} & -X_{2n} & \cdots & 0 \end{vmatrix}.$$

Since $h = \psi(\tilde{h})$ and $g = \psi(\tilde{g})$, we have:

Lemma 2.4. (1) *The coefficient of* $X_{i_0 j_0} X_{i_1 j_1} \cdots X_{i_n j_n}$ *in* $h[(X_{ij} \mid 0 \le i \le j \le n)]$ *is equal to* $\sum sgn(\sigma)$, *where the summation ranges over all permutations* σ *of* $\{0, \dots, n\}$ *satisfying* $\psi(X_{0,\sigma(0)} \cdots X_{n,\sigma(n)}) = X_{i_0 j_0} X_{i_1 j_1} \cdots X_{i_n j_n}$.

(2) *The coefficient of* $X_{i_0 j_0} X_{i_1 j_1} \cdots X_{i_n j_n}$ *in* $g[(X_{ij} \mid 0 \le i \le j \le n)]$ *is equal to* $\sum sgn(\sigma) \cdot (-1)^{N_\sigma}$, *where the summation ranges over all permutations* σ *of* $\{0, \dots, n\}$ *satisfying* $\psi(X_{0,\sigma(0)} \cdots X_{n,\sigma(n)}) = X_{i_0 j_0} X_{i_1 j_1} \cdots X_{i_n j_n}$.

Let Γ be the undirected graph with vertex set $\{0, \dots, n\}$ where for each $k \in \{0, \dots, n\}$ an edge is drawn between the vertices i_k and j_k. Notice that any two distinct vertices of Γ are connected by either 0, 1 or 2 edges. The graph Γ has valency 2 and hence is a disjoint union of cycles. We define the following numbers:

- n_1 = the number of cycles of length 2;
- n_2 = the number of cycles whose length is even and at least 4;
- n_3 = the number of cycles of odd length.

The following lemma immediately follows from the fact that the total number $n+1$ of vertices of Γ is even.

Lemma 2.5. (1) n_3 *is even.*

(2) *If* $n_2 = n_3 = 0$, *then* $n_1 = \frac{n+1}{2}$.

Definitions. (1) Let Γ' be a directed graph with vertex set $\{0, \dots, n\}$. We say that Γ' is *compatible with* Γ if the following conditions are satisfied: (i) the inner and outer degrees of all vertices of Γ' are equal to 1; (ii) the undirected graph obtained from Γ' by replacing each directed edge by an undirected edge is equal to Γ. We denote by $\mathcal{D}(\Gamma)$ the set of all directed graphs on the vertex set $\{0, \dots, n\}$ which are compatible with Γ.

(2) Let σ be a permutation of $\{0, \dots, n\}$ without fixpoints. Then Γ'_σ denotes the directed graph with vertex set $\{0, \dots, n\}$ where for each $k \in \{0, \dots, n\}$ there is an

edge from k to $\sigma(k)$. The graph Γ_σ is obtained from Γ'_σ by replacing each directed edge by an undirected edge. The graph Γ'_σ has no loops and the inner and outer degrees of all its vertices are equal to 1. Conversely, if Γ' is a directed graph on the vertex set $\{0,\ldots,n\}$ without loops such that the inner and outer degrees of all its vertices are equal to 1, then $\Gamma' = \Gamma'_\sigma$ for a unique permutation σ of $\{0,\ldots,n\}$. In particular, this latter remark holds for any directed graph on the vertex set $\{0,\ldots,n\}$ which is compatible with Γ. We denote by Σ the set of all permutations σ of $\{0,\ldots,n\}$ without fixpoints for which $\Gamma'_\sigma \in \mathcal{D}(\Gamma)$. Clearly, $|\Sigma| = |\mathcal{D}(\Gamma)|$ and $\sigma^{-1} \in \Sigma$ if $\sigma \in \Sigma$.

(3) Let $C_1, C_2, \ldots, C_{n_2}$ denote the n_2 cycles of Γ whose length is even and at least 4. Let $C_{n_2+1}, \ldots, C_{n_2+n_3}$ denote the n_3 cycles of Γ whose length is odd. Each of these cycles can be turned in precisely two ways into a directed cycle (i.e. into a connected directed graph in which the indegrees and outdegrees of all vertices are equal to 1). We label one of these directed cycles by the symbol "+" and the other by the symbol "-". There exists a bijective correspondence μ between the set $\{+,-\}^{n_2+n_3}$ and the set $\mathcal{D}(\Gamma)$. If $\bar{c} = (c_1, c_2, \ldots, c_{n_2+n_3}) \in \{+,-\}^{n_2+n_3}$, then $\mu(\bar{c})$ is the unique element of $\mathcal{D}(\Gamma)$ such that for each $i \in \{1,\ldots,n_2+n_3\}$, the directed cycle of $\mu(\bar{c})$ induced on C_i corresponds with the label "c_i".

Lemma 2.6. (1) $|\Sigma| = |\mathcal{D}(\Gamma)| = 2^{n_2+n_3}$.

(2) *For each* $\sigma \in \Sigma$, $sgn(\sigma) = (-1)^{n_1+n_2}$.

(3) *Let* σ *be a permutation of* $\{0,\ldots,n\}$ *without fixpoints. Then* $\sigma \in \Sigma$ *if and only if* ψ *maps the elements* $X_{0,\sigma(0)}X_{1,\sigma(1)}\cdots X_{n,\sigma(n)}$ *of* $\mathbb{Z}[X_{ij} \mid 0 \le i,j \le n]$ *to the element* $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ *of* $\mathbb{Z}[X_{ij} \mid 0 \le i \le j \le n]$.

Proof. (1) This follows from the fact that the map μ defines a bijection between the set $\{+,-\}^{n_2+n_3}$ and the set $\mathcal{D}(\Gamma)$.

(2) If $\sigma \in \Sigma$, then since Γ'_σ is compatible with Γ, Γ'_σ consists of n_1 cycles of length 2, n_2 cycles whose length is even and at least 4 and n_3 cycles of odd length. Hence, $sgn(\sigma) = (-1)^{n_1+n_2}$.

(3) We have $\sigma \in \Sigma$ if and only if $\Gamma_\sigma = \Gamma$, i.e. if and only if the multisets $\{\{0,\sigma(0)\},\{1,\sigma(1)\},\ldots,\{n,\sigma(n)\}\}$ and $\{\{i_0,j_0\},\{i_1,j_1\},\ldots,\{i_n,j_n\}\}$ are equal. Since $i_k < j_k$ for every $k \in \{0,\ldots,n\}$, this precisely happens when ψ maps $X_{0,\sigma(0)}X_{1,\sigma(1)}\cdots X_{n,\sigma(n)}$ to $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$. □

Combining Lemma 2.6 with Lemma 2.4(1), we find

Corollary 2.3. *The coefficient of* $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ *in* $h[(X_{ij} \mid 0 \le i \le j \le n)]$ *is equal to* $(-1)^{n_1+n_2} \cdot 2^{n_2+n_3}$.

Now, for every $\sigma \in \Sigma$ and every $i \in \{1,\ldots,n_2+n_3\}$, let $N_\sigma^{(i)}$ denote the number of $k \in \{0,\ldots,n\}$ for which: (*i*) $\sigma(k) < k$, (*ii*) $\{k,\sigma(k)\}$ is an edge of the cycle C_i. The following is obvious.

Lemma 2.7. (1) *For every* $\sigma \in \Sigma$, $N_\sigma = n_1 + N_\sigma^{(1)} + N_\sigma^{(2)} + \cdots + N_\sigma^{(n_2+n_3)}$.

(2) *If* $\sigma_1, \sigma_2 \in \Sigma$ *and* $i \in \{1,\ldots,n_2+n_3\}$ *such that* $\mu^{-1}(\Gamma'_{\sigma_1})$ *and* $\mu^{-1}(\Gamma'_{\sigma_2})$ *have the same* i*-th components, then* $N_{\sigma_1}^{(i)} = N_{\sigma_2}^{(i)}$.

(3) *If* $\sigma_1, \sigma_2 \in \Sigma$ *and* $i \in \{1,\ldots,n_2+n_3\}$ *such that* $\mu^{-1}(\Gamma'_{\sigma_1})$ *and* $\mu^{-1}(\Gamma'_{\sigma_2})$ *have distinct* i*-th components, then* $N_{\sigma_1}^{(i)} + N_{\sigma_2}^{(i)}$ *is the number of vertices of the cycle* C_i.

By Lemma 2.7(2)+(3), we have

Corollary 2.4. (1) *If $\sigma_1, \sigma_2 \in \Sigma$ and $i \in \{1, \ldots, n_2\}$, then $(-1)^{N_{\sigma_1}^{(i)}} = (-1)^{N_{\sigma_2}^{(i)}}$.*

(2) *If $\sigma_1, \sigma_2 \in \Sigma$ and $i \in \{n_2+1, \ldots, n_2+n_3\}$ such that $\mu^{-1}(\Gamma'_{\sigma_1})$ and $\mu^{-1}(\Gamma'_{\sigma_2})$ have the same i-th components, then $(-1)^{N_{\sigma_1}^{(i)}} = (-1)^{N_{\sigma_2}^{(i)}}$.*

(3) *If $\sigma_1, \sigma_2 \in \Sigma$ and $i \in \{n_2+1, \ldots, n_2+n_3\}$ such that $\mu^{-1}(\Gamma'_{\sigma_1})$ and $\mu^{-1}(\Gamma'_{\sigma_2})$ have distinct i-th components, then $(-1)^{N_{\sigma_1}^{(i)}} = -(-1)^{N_{\sigma_2}^{(i)}}$.*

Now, let σ^* be a fixed element of Σ and put $M_1 := N_{\sigma^*}^{(1)} + N_{\sigma^*}^{(2)} + \cdots + N_{\sigma^*}^{(n_2)}$ and $M_2 := N_{\sigma^*}^{(n_2+1)} + N_{\sigma^*}^{(n_2+2)} + \cdots + N_{\sigma^*}^{(n_2+n_3)}$.

Lemma 2.8. *The coefficient of $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ in $g[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ is equal to $(-1)^{n_2+M_1+M_2} \cdot 2^{n_2} \cdot 0^{n_3}$, where $0^{n_3} = 1$ if $n_3 = 0$.*

Proof. By Lemma 2.4(2), Lemma 2.6(2)+(3), Lemma 2.7(1) and Corollary 2.4, the coefficient of $X_{i_0j_0}X_{i_1j_1}\cdots X_{i_nj_n}$ in $g[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ is equal to

$$\begin{aligned}
& \sum_{\sigma\in\Sigma}(-1)^{N_\sigma} \cdot sgn(\sigma) \\
= \quad & (-1)^{n_1+n_2} \cdot \sum_{\sigma\in\Sigma}(-1)^{N_\sigma} \\
= \quad & (-1)^{n_1+n_2} \cdot (-1)^{n_1+M_1+M_2} \cdot \underbrace{(1+1)\cdots(1+1)}_{n_2 \text{ times}} \cdot \underbrace{(1+(-1))\cdots(1+(-1))}_{n_3 \text{ times}} \\
= \quad & (-1)^{n_2+M_1+M_2} \cdot 2^{n_2} \cdot 0^{n_3}.
\end{aligned}$$

□

Lemma 2.9. *The coefficient N of $X_{i_0j_0}\cdots X_{i_nj_n}$ in $f_1[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ is a multiple of 4.*

Proof. By Corollary 2.3 and Lemma 2.8, we have $N = (-1)^{n_1+n_2+\frac{n-1}{2}} \cdot 2^{n_2+n_3} + (-1)^{n_2+M_1+M_2} \cdot 2^{n_2} \cdot 0^{n_3}$.

Suppose $n_3 \neq 0$. Then $n_3 \geq 2$ by Lemma 2.5(1). Hence $N = 4 \cdot (-1)^{n_1+n_2+\frac{n-1}{2}} \cdot 2^{n_2+n_3-2}$ is a multiple of 4.

Suppose $n_3 = 0$ and $n_2 \geq 1$. Then $M_2 = 0$ and the number $N = 4 \cdot 2^{n_2-1} \cdot \frac{(-1)^{n_1+n_2+\frac{n-1}{2}}+(-1)^{n_2+M_1}}{2}$ is a multiple of 4.

Suppose $n_2 = n_3 = 0$. Then $n_1 = \frac{n+1}{2}$ (recall Lemma 2.5(2)) and $M_1 = M_2 = 0$. Hence, $N = (-1)^{\frac{n+1}{2}+\frac{n-1}{2}} + 1 = 0$. □

3. Proofs of Theorems 1.1 and 1.2

3.1. **Coordinate transformations.** Let $B = (\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$ be an ordered basis of V. For all $i, j \in \{0, \ldots, n\}$ satisfying $i \leq j$, choose an element $a_{ij} \in \mathbb{K}$ and consider the quadric Q of $\mathrm{PG}(V)$ whose equation with respect to B is given by $\sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j = 0$.

Now, consider the coordinate transformation with associate nonsingular matrix $M = (m_{ij})_{0\leq i,j\leq n}$: $X_i = \sum_{k=0}^{n} m_{ik}Y_k$, $i \in \{0, \ldots, n\}$. The equation of Q in the new coordinates is equal to $\sum_{0\leq k\leq l\leq n} a'_{kl}Y_kY_l = 0$, where $a'_{kk} = \sum_{0\leq i\leq j\leq n} a_{ij} \cdot$

$(m_{ik}m_{jk})$ and $a'_{kl} = \sum_{0\leq i\leq j\leq n} a_{ij} \cdot (m_{ik}m_{jl} + m_{il}m_{jk})$ for all $k, l \in \{0,\ldots,n\}$ satisfying $k \leq l$. In the sequel, we will say that the quadratic polynomial $\sum_{0\leq i\leq j\leq n} a'_{ij} X_iX_j$ is *related to* $\sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j$ *via a linear coordinate transformation.*

The two latter equations above imply that $A' = M^T \cdot A \cdot M$, where A' and A are the matrices obtained from the coefficients a'_{ij} and a_{ij} as described in equation (1.2) of Section 1.1. It follows that $h_{\mathbb{K}}[(a'_{ij} \mid 0 \leq i \leq j \leq n)] = [\det(M)]^2 \cdot h_{\mathbb{K}}[(a_{ij} \mid 0 \leq i \leq j \leq n)]$. So, we have established the following.

Lemma 3.1. *If the quadratic polynomials* $\sum a_{ij}X_iX_j$ *and* $\sum a'_{ij}X_iX_j$ *are related via a linear coordinate transformation, then*

(1) *there exists a* $\lambda \in \mathbb{K}\setminus\{0\}$ *such that* $h_{\mathbb{K}}[(a'_{ij} \mid 0 \leq i \leq j \leq n)] = \lambda^2 \cdot h_{\mathbb{K}}[(a_{ij} \mid 0 \leq i \leq j \leq n)]$;

(2) *the matrices* A *and* A' *have the same rank.*

3.2. The quadratic equations describing certain quadrics.

Lemma 3.2. *Let* $B = (\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$ *be an ordered basis of* V *and suppose* Q *is a quadric of* $\mathrm{PG}(V)$ *whose equation with respect to* B *is given by* $X_0^2 + \alpha X_0X_1 + \beta X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$. *If the equation* $\sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j = 0$ *describes* Q *with respect to the same ordered basis* B, *then* $\sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j = \lambda \cdot (X_0^2 + \alpha X_0X_1 + \beta X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n)$ *for some* $\lambda \in \mathbb{K} \setminus \{0\}$.

Proof. • For all $i \in \{2,\ldots,n\}$, we have $a_{ii} = 0$ since $\langle \bar{e}_i \rangle \in Q$.

• For all $i, j \in \{2,\ldots,n\}$ satisfying $i < j$ and $(i,j) \notin \{(2,3),(4,5),\ldots,(n-1,n)\}$, we have $a_{ij} = 0$ since $\langle \bar{e}_i + \bar{e}_j \rangle \in Q$.

• For every $i \in \{2,\ldots,n\}$ and every $\lambda \in \mathbb{K}$, we have $a_{00} + a_{0i}\lambda \neq 0$ since $\langle \bar{e}_0 + \lambda\bar{e}_i \rangle \notin Q$. Hence, $a_{00} \neq 0$ and $a_{0i} = 0$ for all $i \in \{2,\ldots,n\}$.

• Suppose $\beta \neq 0$. For every $i \in \{2,\ldots,n\}$ and every $\lambda \in \mathbb{K}$, we have $a_{11} + a_{1i}\lambda \neq 0$ since $\langle \bar{e}_1 + \lambda\bar{e}_i \rangle \notin Q$. Hence, $a_{11} \neq 0$ and $a_{1i} = 0$ for all $i \in \{2,\ldots,n\}$.

• Suppose $\beta = 0$. Then $a_{11} = 0$ since $\langle \bar{e}_1 \rangle \in Q$. For every $i \in \{2,\ldots,n\}$, we have $a_{1i} = 0$ since $\langle \bar{e}_1 + \bar{e}_i \rangle \in Q$.

• For every $i \in \{1,\ldots,\frac{n-1}{2}\}$, we have $a_{00} = a_{2i,2i+1}$ since $\langle \bar{e}_0 + \bar{e}_{2i} - \bar{e}_{2i+1} \rangle \in Q$.

• For every $i \in \{1,\ldots,\frac{n-1}{2}\}$, we have $a_{11} = \beta \cdot a_{2i,2i+1}$ since $\langle \bar{e}_1 + \bar{e}_{2i} - \beta\bar{e}_{2i+1} \rangle \in Q$.

• For every $i \in \{1,\ldots,\frac{n-1}{2}\}$, we have $a_{00} + a_{01} + a_{11} - (1+\alpha+\beta)a_{2i,2i+1} = 0$ since $\langle \bar{e}_0 + \bar{e}_1 + \bar{e}_{2i} - (1+\alpha+\beta)\bar{e}_{2i+1} \rangle \in Q$. Hence, $a_{01} = \alpha \cdot a_{2i,2i+1}$ for all $i \in \{1,\ldots,\frac{n-1}{2}\}$.

By the above discussion, we have that $\sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j = a_{00}(X_0^2 + \alpha X_0X_1 + \beta X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n)$ where $a_{00} \neq 0$. □

Remark. The conclusion of the above lemma would not necessarily be true if n would be allowed to be equal to 1.

3.3. Canonical quadratic equations for hyperbolic and (pseudo-)elliptic quadrics.

Lemma 3.3. *Let* Q *be a nonsingular quadric of Witt index at least* $\frac{n-1}{2}$ *of* $\mathrm{PG}(V)$.

(1) *If* $\mathrm{char}(\mathbb{K}) \neq 2$, *then there exists an* $\alpha \in \mathbb{K} \setminus \{0\}$ *and a reference system with respect to which* Q *is described by the equation* $X_0^2 - \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$. *The quadric* Q *has Witt index* $\frac{n+1}{2}$ *if and only if* α *is a square of*

$\mathbb{K}$. *In this case there exists a reference system with respect to which Q has equation $X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n = 0$.*

(2) *If* $\mathrm{char}(\mathbb{K}) = 2$, *then there exists a reference system with respect to which Q has equation $X_0^2 + \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ with α a non-square of $\mathbb{K}$ or equation $X_0^2 + X_0X_1 + \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ with $\alpha \in \mathbb{K}$. In the former case, Q always has Witt index $\frac{n-1}{2}$. In the latter case, Q has Witt index $\frac{n+1}{2}$ if and only if there exists a $\lambda \in \mathbb{K}$ such that $\lambda^2 + \lambda = \alpha$. In this case there exists a reference system with respect to which Q has equation $X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n = 0$.*

Proof. Suppose Q is described by the quadratic form q of V. Let b denote the symmetric bilinear form of V associated with q, i.e. $b(\bar{x}, \bar{y}) = q(\bar{x} + \bar{y}) - q(\bar{x}) - q(\bar{y})$ for all $\bar{x}, \bar{y} \in V$. Let m' be the Witt index of Q and put $m := \frac{n-1}{2}$. The following properties of Q hold more generally for any non-degenerate polar space of rank m', see Tits [8, Chapter 7]:

(P1) There exist two disjoint $(m'-1)$-dimensional subspaces of Q.

(P2) Let W_1 and W_2 be two disjoint $(m'-1)$-dimensional subspaces of Q, and consider the map Π_{W_1,W_2} which associates with every subspace U_1 of W_1 the subspace U_2 of W_2 consisting of all points of W_2 which are collinear on Q with every point of U_1. Then Π_{W_1,W_2} defines an isomorphism between the projective space W_1 and the dual of the projective space W_2.

Now, let π_1 and π_2 be two m'-dimensional subspaces of V satisfying (i) $\pi_1 \cap \pi_2 = \{\bar{o}\}$ and (ii) $q(\bar{x}) = 0$ for all $\bar{x} \in \pi_1 \cup \pi_2$. Let $\bar{e}_1, \ldots, \bar{e}_{m'}$ be a basis of π_1. For every $i \in \{1, \ldots, m'\}$, there exists by Property (P2) a unique vector $\bar{f}_i$ of π_2 satisfying $b(\bar{e}_i, \bar{f}_i) = 1$ and $b(\bar{e}_j, \bar{f}_i) = 0$ for all $j \in \{1, \ldots, m'\} \setminus \{i\}$. By (P2), it also follows that $\{\bar{f}_1, \cdots, \bar{f}_{m'}\}$ is a basis of π_2.

If $m' = m + 1 = \frac{n+1}{2}$, then $\{\bar{e}_1, \ldots, \bar{e}_{m'}, \bar{f}_1, \ldots, \bar{f}_{m'}\}$ is a basis of V and $q(X_0\bar{e}_1 + X_1\bar{f}_1 + X_2\bar{e}_2 + X_3\bar{f}_2 + \cdots + X_{n-1}\bar{e}_{m'} + X_n\bar{f}_{m'}) = X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n$.

Consider now the most general case $m' \in \{m, m+1\}$. Let W denote the subspace of V consisting of all vectors which are b-orthogonal with all vectors of $U := \langle \bar{e}_1, \ldots, \bar{e}_m, \bar{f}_1, \ldots, \bar{f}_m \rangle$. Since U has co-dimension 2 in V, W has dimension at least 2.

Suppose $\bar{x}$ is a vector of $W \cap U$. Then there exist $k_1, \ldots, k_m, l_1, \ldots, l_m \in \mathbb{K}$ such that $\bar{x} = k_1\bar{e}_1 + \cdots + k_m\bar{e}_m + l_1\bar{f}_1 + \cdots + l_m\bar{f}_m$. We have $l_i = b(\bar{x}, \bar{e}_i) = 0$ and $k_i = b(\bar{x}, \bar{f}_i) = 0$ for all $i \in \{1, \ldots, m\}$. Hence, $\bar{x} = \bar{o}$. So, $W \cap U = \{\bar{o}\}$. Since $\dim(U) = n - 1$ and $\dim(W) \geq 2$, we have $\dim(W) = 2$ and $V = U \oplus W$. We have $q(X_2\bar{e}_1 + X_3\bar{f}_1 + \cdots + X_{n-1}\bar{e}_m + X_n\bar{f}_m) = X_2X_3 + \cdots + X_{n-1}X_n$ and $q(\bar{x}_1 + \bar{x}_2) = q(\bar{x}_1) + q(\bar{x}_2)$ for any vectors $\bar{x}_1 \in W$ and $\bar{x}_2 \in U$. If $\mathrm{char}(\mathbb{K}) \neq 2$, then it is easily verified that there exists a basis $\{\bar{e}'_{m+1}, \bar{f}'_{m+1}\}$ of W such that $f(X_0, X_1) := q(X_0\bar{e}'_{m+1} + X_1\bar{f}'_{m+1}) = \beta(X_0^2 - \alpha X_1^2)$ for some $\alpha, \beta \in \mathbb{K} \setminus \{0\}$. If $\mathrm{char}(\mathbb{K}) = 2$, then it is easily verified that there exists a basis $\{\bar{e}'_{m+1}, \bar{f}'_{m+1}\}$ of W such that $f(X_0, X_1) := q(X_0\bar{e}'_{m+1} + X_1\bar{f}'_{m+1})$ is equal to $\beta(X_0^2 + \alpha X_1^2)$ for some $\beta \in \mathbb{K} \setminus \{0\}$ and some non-square α of $\mathbb{K}$ or equal to $\beta(X_0^2 + X_0X_1 + \alpha X_1^2)$ for some $\alpha, \beta \in \mathbb{K}$ with $\beta \neq 0$. (Notice that in all the above cases $f(X_0, X_1) = 0$ determines a nonsingular quadric of $\mathrm{PG}(W)$ since Q itself is nonsingular.) With respect to the reference system $(\bar{e}'_{m+1}, \bar{f}'_{m+1}, \bar{e}_1, \beta \cdot \bar{f}_1, \ldots, \bar{e}_m, \beta \cdot \bar{f}_m)$, Q has equation $\frac{f(X_0,X_1)}{\beta} + X_2X_3 + \cdots + X_{n-1}X_n = 0$. Now, $\langle \bar{e}_1, \ldots, \bar{e}_m \rangle$ is a subspace of Q. So, Q has Witt index $\frac{n+1}{2}$ if and only if there exists an $\frac{n-1}{2}$-dimensional subspace of Q

containing $\langle \bar{e}_1, \ldots, \bar{e}_m \rangle$. This precisely happens when the equation $f(X_0, X_1) = 0$ has a nonzero solution for (X_0, X_1). The conclusions of the lemma follow. □

By Lemmas 3.2 and 3.3, we have

Corollary 3.1. *Let Q be a nonsingular quadric of Witt index $m \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$ of $\mathrm{PG}(V)$ and let B be an ordered basis of V. Then the quadratic polynomial in the quadratic equation which describes Q with respect to B is uniquely determined up to a nonzero factor of $\mathbb{K}$.*

From Corollary 3.1, it should now be clear that the definition of the notion "elliptic quadric" and "pseudo-elliptic quadric" as given in Section 1.1 is independent from the quadratic equation which represents Q. These notions are invariant under linear coordinate transformations and also the fact that one could multiply the quadratic polynomial in a given quadratic equation with a nonzero constant does not have any influence.

Lemma 3.4. (1) *Suppose* $\mathrm{char}(\mathbb{K}) \neq 2$, $\alpha \in \mathbb{K} \setminus \{0\}$ *and that B is an ordered basis of V. Let Q be the nonsingular quadric of* $\mathrm{PG}(V)$ *whose equation with respect to B is given by $X_0^2 - \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$. If α is a square of $\mathbb{K}$, then Q is a hyperbolic quadric; otherwise, Q is an elliptic quadric.*

(2) *Suppose* $\mathrm{char}(\mathbb{K}) = 2$, $\alpha \in \mathbb{K}$ *and that B is an ordered basis of V. Let Q be the nonsingular quadric of* $\mathrm{PG}(V)$ *whose equation with respect to B is given by $X_0^2 + X_0X_1 + \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$. If there exists a $\lambda \in \mathbb{K}$ such that $\alpha = \lambda^2 + \lambda$, then Q is a hyperbolic quadric; otherwise Q is an elliptic quadric.*

(3) *Suppose* $\mathrm{char}(\mathbb{K}) = 2$, *$\alpha$ a non-square of $\mathbb{K}$ and that $B = (\bar{e}_0, \ldots, \bar{e}_n)$ an ordered basis of V. Then the nonsingular quadric of* $\mathrm{PG}(V)$ *whose equation with respect to B is given by $X_0^2 + \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ is a pseudo-elliptic quadric.*

Proof. (1) If α is a square of $\mathbb{K}$, then Q is a hyperbolic quadric since Q has Witt index $\frac{n+1}{2}$. Suppose now that α is a non-square of $\mathbb{K}$. Let $\sqrt{\alpha}$ denote one of the two square roots of α in a given algebraic closure of $\mathbb{K}$. Then $\mathbb{K}(\sqrt{\alpha})$ is a quadratic Galois extension of $\mathbb{K}$ over which the equation $X_0^2 - \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ defines a nonsingular quadric of Witt index $\frac{n+1}{2}$. So, Q is an elliptic quadric.

(2) If there exists a $\lambda \in \mathbb{K}$ such that $\alpha = \lambda^2 + \lambda$, then Q is a hyperbolic quadric since Q has Witt index $\frac{n+1}{2}$. Suppose now that there exists no $\lambda \in \mathbb{K}$ for which $\alpha = \lambda^2 + \lambda$. Let $\widetilde{\alpha}$ be one of the two roots of the polynomial $X^2 + X + \alpha \in \mathbb{K}[X]$ in a given algebraic closure of $\mathbb{K}$. (Then the other root is $1 + \widetilde{\alpha} \neq \widetilde{\alpha}$.) Then $\mathbb{K}(\widetilde{\alpha})$ is a quadratic Galois extension of $\mathbb{K}$ over which the equation $X_0^2 + X_0X_1 + \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ defines a nonsingular quadric of Witt index $\frac{n+1}{2}$. So, Q is an elliptic quadric.

(3) We prove that Q is a pseudo-elliptic quadric. Suppose to the contrary that Q is an elliptic quadric. Then there exists a quadratic Galois extension $\mathbb{K}'$ of $\mathbb{K}$ over which $X_0^2 + \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ defines a nonsingular quadric Q' of Witt index $\frac{n+1}{2}$. Let $\overline{\mathbb{K}'}$ denote an algebraic closure of $\mathbb{K}'$ and let $\sqrt{\alpha}$ denote the unique square root of α in $\overline{\mathbb{K}'}$. The subspace $\langle \bar{e}_2, \bar{e}_4, \ldots, \bar{e}_{n-1} \rangle$ of Q' must be contained in a subspace of maximal dimension $\frac{n-1}{2}$. This subspace necessarily coincides with $\langle \sqrt{\alpha} \cdot \bar{e}_0 + \bar{e}_1, \bar{e}_2, \bar{e}_4, \ldots, \bar{e}_{n-1} \rangle$. This implies that $\mathbb{K}' = \mathbb{K}(\sqrt{\alpha})$. But

this is in contradiction with the fact that $\mathbb{K}(\sqrt{\alpha})$ is not a Galois extension of $\mathbb{K}$. So, Q must be a pseudo-elliptic quadric. □

Lemma 3.5. *Let Q be a nonsingular quadric of Witt index $m \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$ of $\mathrm{PG}(V)$ and suppose $\sum_{0 \leq i \leq j \leq n} a_{ij} X_i X_j = 0$ is the equation of Q with respect to a given ordered basis B of V. Let A be the matrix as defined in equation* (1.2) *of Section 1.1. Then Q is a pseudo-elliptic quadric if and only if* $\det(A) = 0$. *Moreover, if Q is a pseudo-elliptic quadric, then the rank of A is equal to $n-1$.*

Proof. By Lemma 3.1 and Corollary 3.1, we may without loss of generality suppose that B is an ordered basis of V with respect to which Q has an "easy" equation.

If Q is hyperbolic, then we may suppose that $\sum a_{ij} X_i X_j = X_0 X_1 + \cdots + X_{n-1} X_n$. In this case, $\det(A) = (-1)^{\frac{n+1}{2}} \neq 0$.

If Q is elliptic, then we may suppose that $\sum a_{ij} X_i X_j = X_0^2 - \alpha X_1^2 + X_2 X_3 + \cdots + X_{n-1} X_n$ if $\mathrm{char}(\mathbb{K}) \neq 2$ and $\sum a_{ij} X_i X_j = X_0^2 + X_0 X_1 + \beta X_1^2 + X_2 X_3 + \cdots + X_{n-1} X_n$ if $\mathrm{char}(\mathbb{K}) = 2$. Here, α is a non-square of $\mathbb{K}$ if $\mathrm{char}(\mathbb{K}) \neq 2$ and $X^2 + X + \beta \in \mathbb{K}[X]$ is irreducible if $\mathrm{char}(\mathbb{K}) = 2$. If $\mathrm{char}(\mathbb{K}) \neq 2$, we have $\det(A) = (-1)^{\frac{n+1}{2}} \cdot 4\alpha \neq 0$. If $\mathrm{char}(\mathbb{K}) = 2$, we have $\det(A) = 1 \neq 0$.

If Q is pseudo-elliptic, then we may suppose that $\sum a_{ij} X_i X_j = X_0^2 + \alpha X_1^2 + X_2 X_3 + \cdots + X_{n-1} X_n$, where α is a non-square of $\mathbb{K}$. Clearly, A is a singular matrix of rank $n-1$. □

3.4. **Proofs of Theorem 1.1(1) and Theorem 1.2(1).** Throughout this subsection, we suppose that $\mathrm{char}(\mathbb{K}) \neq 2$.

Let Q be a nonsingular quadric of Witt index at least $\frac{n-1}{2}$ of $\mathrm{PG}(V)$ whose equation with respect to a given ordered basis of V is given by $\sum a_{ij} X_i X_j = 0$. By Lemma 3.3, we know that there exist quadratic polynomials $\sum b_{ij} X_i X_j$ and $\sum c_{ij} X_i X_j$ such that the following holds:

- $\sum b_{ij} X_i X_j$ is related to $\sum a_{ij} X_i X_j$ via a linear coordinate transformation;
- $\sum c_{ij} X_i X_j = \lambda_3 \cdot \Big(\sum b_{ij} X_i X_j \Big) = X_0^2 - \alpha X_1^2 + X_2 X_3 + \cdots + X_{n-1} X_n$ for some $\lambda_3, \alpha \in \mathbb{K} \setminus \{0\}$.

Now, put $\lambda_2 := \lambda_3^{\frac{n+1}{2}}$. By Lemma 3.1, we have $h_{\mathbb{K}}[(b_{ij} \mid 0 \leq i \leq j \leq n)] = \lambda_1^2 \cdot h_{\mathbb{K}}[(a_{ij} \mid 0 \leq i \leq j \leq n)]$ for some $\lambda_1 \in \mathbb{K} \setminus \{0\}$. So, we have $(-1)^{\frac{n+1}{2}} 4\alpha = h_{\mathbb{K}}[(c_{ij} \mid 0 \leq i \leq j \leq n)] = \lambda_2^2 \cdot h_{\mathbb{K}}[(b_{ij} \mid 0 \leq i \leq j \leq n)] = \lambda_1^2 \lambda_2^2 \cdot h_{\mathbb{K}}[(a_{ij} \mid 0 \leq i \leq j \leq n)]$. Putting $\eta := h_{\mathbb{K}}[(a_{ij} \mid 0 \leq i \leq j \leq n)] = \det(A)$, we find $\alpha = (-1)^{\frac{n+1}{2}} (\frac{\lambda_1 \lambda_2}{2})^2 \eta$. Hence, $\eta \neq 0$ and Q has equation $X_0^2 - (-1)^{\frac{n+1}{2}} \eta \cdot (\frac{\lambda_1 \lambda_2}{2} X_1)^2 + X_2 X_3 + \cdots + X_{n-1} X_n = 0$ with respect to a suitable reference system. This completes the proof of Theorem 1.1(1).

Now, suppose $\mu_1, \mu_2 \in \mathbb{K} \setminus \{0\}$ and consider the quadrics Q_1 and Q_2 with respective equations $X_0^2 + \mu_1 X_1^2 + X_2 X_3 + \cdots + X_{n-1} X_n = 0$ and $X_0^2 + \mu_2 X_1^2 + X_2 X_3 + \cdots + X_{n-1} X_n = 0$ with respect to a given ordered basis $B = (\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$ of V.

If $\mu_2 = \mu_1 \cdot \lambda^2$ for some $\lambda \in \mathbb{K} \setminus \{0\}$, then Q_1 and Q_2 are projectively equivalent since $X_0^2 + \mu_2 X_1^2 + X_2 X_3 + \cdots + X_{n-1} X_n = X_0^2 + \mu_1 (\lambda X_1)^2 + X_2 X_3 + \cdots + X_{n-1} X_n$.

Conversely, suppose that Q_1 and Q_2 are projectively equivalent. Then by Corollary 3.1, there exists a quadratic polynomial $\sum_{0 \leq i \leq j \leq n} a_{ij} X_i X_j$ such that the following holds:

(i) $X_0^2+\mu_1 X_1^2+X_2X_3+\cdots+X_{n-1}X_n$ and $\sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j$ are related via a linear coordinate transformation. By Lemma 3.1, $h_{\mathbb{K}}[(a_{ij} \mid 0\leq i\leq j\leq n)] = \lambda_1^2\cdot\mu_1\cdot 4(-1)^{\frac{n-1}{2}}$ for some $\lambda_1\in\mathbb{K}\setminus\{0\}$.

(ii) $\sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j = \lambda_3\cdot(X_0^2+\mu_2X_1^2+X_2X_3+\cdots+X_{n-1}X_n)$ for some $\lambda_3\in\mathbb{K}\setminus\{0\}$. Then $h_{\mathbb{K}}[(a_{ij} \mid 0\leq i\leq j\leq n)] = \left(\lambda_3^{\frac{n+1}{2}}\right)^2\cdot\mu_2\cdot 4(-1)^{\frac{n-1}{2}}$.

By (i) and (ii), it follows that $\mu_2=\lambda^2\cdot\mu_1$ for some $\lambda\in\mathbb{K}\setminus\{0\}$.

For every automorphism θ of $\mathbb{K}$, let Q_1^θ denote the quadric whose equation with respect to B is given by $X_0^2+\mu_1^\theta\cdot X_1^2+X_2X_3+\cdots+X_{n-1}X_n=0$. Notice that a point $\langle\sum_{i=0}^n X_i\bar{e}_i\rangle$ of $\mathrm{PG}(V)$ belongs to Q_1 if and only if $\langle\sum_{i=0}^n X_i^\theta\bar{e}_i\rangle$ belongs to Q_1^θ. The quadrics Q_1 and Q_2 are equivalent under an automorphism of $\mathrm{PG}(V)$ if and only if Q_2 and Q_1^θ are projectively equivalent for some automorphism θ of $\mathbb{K}$, i.e. if and only if there exists an automorphism θ of $\mathbb{K}$ and a $\lambda\in\mathbb{K}\setminus\{0\}$ such that $\mu_2=\mu_1^\theta\cdot\lambda^2$. This completes the proof of Theorem 1.2(1).

3.5. **A decomposition of the polynomials** $f_{\mathbb{K}}[(X_{ij} \mid 0\leq i\leq j\leq n)]$ **and** $g_{\mathbb{K}}[(X_{ij} \mid 0\leq i\leq j\leq n)]$**.** Let $I_{\mathbb{Z}}$, respectively $I_{\mathbb{K}}$, denote the ideal of $\mathbb{Z}[X_{ij} \mid 0\leq i\leq j\leq n]$, respectively $\mathbb{K}[X_{ij} \mid 0\leq i\leq j\leq n]$, generated by the set $\{X_{ij} \mid (i,j)\notin\{(0,0),(0,1),(1,1),(2,3),(4,5),\ldots,(n-1,n)\}\}$.

Every polynomial $p\in\mathbb{Z}[X_{ij} \mid 0\leq i\leq j\leq n]$ can be written in a unique way as $p'+p''$ where $p'\in\mathbb{Z}[X_{00},X_{01},X_{11},X_{23},X_{45},\ldots,X_{n-1,n}]$ and $p''\in I_{\mathbb{Z}}$. We have

$$h'[(X_{ij} \mid 0\leq i\leq j\leq n)] = (-1)^{\frac{n-1}{2}}\cdot(4\cdot X_{00}X_{11}-X_{01}^2)X_{23}^2\cdots X_{n-1,n}^2,$$
$$g'[(X_{ij} \mid 0\leq i\leq j\leq n)] = X_{01}^2X_{23}^2\cdots X_{n-1,n}^2.$$

Hence,

$$f_1'[(X_{ij} \mid 0\leq i\leq j\leq n)] = 4X_{00}X_{11}X_{23}^2\cdots X_{n-1,n}^2$$

and

$$f'[(X_{ij} \mid 0\leq i\leq j\leq n)] = X_{00}X_{11}X_{23}^2\cdots X_{n-1,n}^2.$$

This allows us to conclude the following:

Lemma 3.6. *We have* $f_{\mathbb{K}}[(X_{ij} \mid 0\leq i\leq j\leq n)] = X_{00}X_{11}X_{23}^2\cdots X_{n-1,n}^2 + f_{\mathbb{K}}''[(X_{ij} \mid 0\leq i\leq j\leq n)]$ *and* $g_{\mathbb{K}}[(X_{ij} \mid 0\leq i\leq j\leq n)] = X_{01}^2X_{23}^2\cdots X_{n-1,n}^2 + g_{\mathbb{K}}''[(X_{ij} \mid 0\leq i\leq j\leq n)]$*, where* $X_{00}X_{11}X_{23}^2\cdots X_{n-1,n}^2$ *and* $X_{01}^2X_{23}^2\cdots X_{n-1,n}^2$ *are regarded as polynomials of* $\mathbb{K}[X_{ij} \mid 0\leq i\leq j\leq n]$ *and* $f_{\mathbb{K}}''[(X_{ij} \mid 0\leq i\leq j\leq n)], g_{\mathbb{K}}''[(X_{ij} \mid 0\leq i\leq j\leq n)]\in I_{\mathbb{K}}$.

Corollary 3.2. *If* $\sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j = X_0^2+X_0X_1+\alpha X_1^2+X_2X_3+\cdots+X_{n-1}X_n$ *for some* $\alpha\in\mathbb{K}$*, then* $f_{\mathbb{K}}[(a_{ij} \mid 0\leq i\leq j\leq n)]=\alpha$ *and* $g_{\mathbb{K}}[(a_{ij} \mid 0\leq i\leq j\leq n)]=1$.

3.6. **The polynomials** $R_{\mathbb{K}}^M[(X_{ij} \mid 0\leq i\leq j\leq n)]$**.** Let $M=(m_{ij})_{0\leq i\leq j\leq n}$ be an $(n+1)\times(n+1)$-matrix with entries in $\mathbb{Z}$ such that $\det(M)\in\{1,-1\}$. For all $k,l\in\{0,\ldots,n\}$ with $k\leq l$, put

$$Y_{kk} := \sum_{0\leq i\leq j\leq n}(m_{ik}m_{jk})X_{ij},$$
$$Y_{kl} := \sum_{0\leq i\leq j\leq n}(m_{ik}m_{jl}+m_{il}m_{jk})X_{ij}.$$

From the discussion given in Section 3.1, we immediately have that

$$h[(Y_{ij} \mid 0 \leq i \leq j \leq n)] \quad = \quad h[(X_{ij} \mid 0 \leq i \leq j \leq n)].$$

Now, let $R_1^M[(X_{ij} \mid 0 \leq i \leq j \leq n)] \in \mathbb{Z}[X_{ij} \mid 0 \leq i \leq j \leq n]$ such that

$$Pf[(Y_{ij} \mid 0 \leq i \leq j \leq n)] \quad = \quad Pf[(X_{ij} \mid 0 \leq i \leq j \leq n)] + R_1^M[(X_{ij} \mid 0 \leq i \leq j \leq n)].$$

Lemma 3.7. *There exists a polynomial* $R^M[(X_{ij} \mid 0 \leq i \leq j \leq n)] \in \mathbb{Z}[X_{ij} \mid 0 \leq i \leq j \leq n]$ *such that* $R_1^M[(X_{ij} \mid 0 \leq i \leq j \leq n)] = 2 \cdot R^M[(X_{ij} \mid 0 \leq i \leq j \leq n)]$. *Moreover,* $f[(Y_{ij} \mid 0 \leq i \leq j \leq n)] = f[(X_{ij} \mid 0 \leq i \leq j \leq n)] + Pf[(X_{ij} \mid 0 \leq i \leq j \leq n)] \cdot R^M[(X_{ij} \mid 0 \leq i \leq j \leq n)] + \Big(R^M[(X_{ij} \mid 0 \leq i \leq j \leq n)]\Big)^2$.

Proof. By Proposition 1.1, all coefficients of the polynomial $\Big(Pf[(Y_{ij} \mid 0 \leq i \leq j \leq n)]\Big)^2 + (-1)^{\frac{n-1}{2}} \cdot h[(Y_{ij} \mid 0 \leq i \leq j \leq n)] = \Big(Pf[(X_{ij} \mid 0 \leq i \leq j \leq n)]\Big)^2 + (-1)^{\frac{n-1}{2}} \cdot h[(X_{ij} \mid 0 \leq i \leq j \leq n)] + 2 \cdot Pf[(X_{ij} \mid 0 \leq i \leq j \leq n)] \cdot R_1^M[(X_{ij} \mid 0 \leq i \leq j \leq n)] + \Big(R_1^M[(X_{ij} \mid 0 \leq i \leq j \leq n)]\Big)^2$ are multiples op 4. Applying Proposition 1.1 once more, we see that all coefficients of $\Big(R_1^M[(X_{ij} \mid 0 \leq i \leq j \leq n)]\Big)^2$ must be multiples of 2. This is only possible when all coefficients of $R_1^M[(X_{ij} \mid 0 \leq i \leq j \leq n)]$ are multiples of 2. The claims of the lemma now readily follow. $\square$

Corollary 3.3. *Put* $Z_{kk} := \sum_{0 \leq i \leq j \leq n} \phi_{\mathbb{K}}(m_{ik} m_{jk}) X_{ij}$ *and* $Z_{kl} := \sum_{0 \leq i \leq j \leq n} \phi_{\mathbb{K}}(m_{ik} m_{jl} + m_{il} m_{jk}) X_{ij}$ *for all* $k, l \in \{0, \ldots, n\}$ *satisfying* $k \leq l$. *Then* $f_{\mathbb{K}}[(Z_{ij} \mid 0 \leq i \leq j \leq n)] = f_{\mathbb{K}}[(X_{ij} \mid 0 \leq i \leq j \leq n)] + Pf_{\mathbb{K}}[(X_{ij} \mid 0 \leq i \leq j \leq n)] \cdot R_{\mathbb{K}}^M[(X_{ij} \mid 0 \leq i \leq j \leq n)] + \Big(R_{\mathbb{K}}^M[(X_{ij} \mid 0 \leq i \leq j \leq n)]\Big)^2$ *and* $g_{\mathbb{K}}[(Z_{ij} \mid 0 \leq i \leq j \leq n)] = g_{\mathbb{K}}[(X_{ij} \mid 0 \leq i \leq j \leq n)] + 4 \cdot Pf_{\mathbb{K}}[(X_{ij} \mid 0 \leq i \leq j \leq n)] \cdot R_{\mathbb{K}}^M[(X_{ij} \mid 0 \leq i \leq j \leq n)] + 4 \cdot \Big(R_{\mathbb{K}}^M[(X_{ij} \mid 0 \leq i \leq j \leq n)]\Big)^2$.

3.7. **Proofs of Theorem 1.1(2) and Theorem 1.2(2).** Throughout this subsection, $\mathbb{K}$ is a field of characteristic 2.

Notice that if Q is a hyperbolic or elliptic quadric whose equation with respect to a given ordered basis is given by $\sum_{0 \leq i \leq j \leq n} a_{ij} X_i X_j = 0$, then $g_{\mathbb{K}}[(a_{ij} \mid 0 \leq i \leq j \leq n)] = h_{\mathbb{K}}[(a_{ij} \mid 0 \leq i \leq j \leq n)] = \det(A) \neq 0$, where A is the matrix as defined in equation (1.2) of Section 1.1.

Proposition 3.1. *Suppose* $\mathrm{char}(\mathbb{K}) = 2$ *and that* Q *is a hyperbolic or elliptic quadric of* $\mathrm{PG}(V)$. *Let* $\sum_{0 \leq i \leq j \leq n} a_{ij} X_i X_j = 0$ *and* $\sum_{0 \leq i \leq j \leq n} a'_{ij} X_i X_j = 0$ *be equations of* Q *with respect to two ordered bases of* V. *Put* $\eta_1 = f_{\mathbb{K}}[(a_{ij} \mid 0 \leq i \leq j \leq n)]$, $\eta_2 = g_{\mathbb{K}}[(a_{ij} \mid 0 \leq i \leq j \leq n)]$, $\eta'_1 = f_{\mathbb{K}}[(a'_{ij} \mid 0 \leq i \leq j \leq n)]$ *and* $\eta'_2 = g_{\mathbb{K}}[(a'_{ij} \mid 0 \leq i \leq j \leq n)]$. *Then there exists a* $\lambda \in \mathbb{K}$ *such that* $\frac{\eta'_1}{\eta'_2} = \frac{\eta_1}{\eta_2} + \lambda^2 + \lambda$.

Proof. (1) Suppose that $\sum a_{ij} X_i X_j$ and $\sum a'_{ij} X_i X_j$ are related via one of the following coordinate transformations:

(i) $X_0 = Y_i$, $X_i = Y_0$, $X_j = Y_j$ ($j \in \{1, \ldots, n\} \setminus \{i\}$) for a certain $i \in \{1, \ldots, n\}$;

(ii) $X_0 = Y_0 + Y_1$, $X_j = Y_j$ for all $j \in \{1,\ldots,n\}$.
Let $M = (m_{ij})_{0\leq i\leq j\leq n}$ be an $(n+1)\times(n+1)$-matrix over $\mathbb{Z}$ with all entries equal to 0 or 1 such that $X_i = \sum_{k=0}^{n} \phi_{\mathbb{K}}(m_{ik})Y_k$ for every $i \in \{0,\ldots,n\}$. Then $\det(M) \in \{1,-1\}$ and

$$\begin{aligned} a'_{kk} &= \sum_{0\leq i\leq j\leq n} \phi_{\mathbb{K}}(m_{ik}m_{jk})\cdot a_{ij}, \\ a'_{kl} &= \sum_{0\leq i\leq j\leq n} \phi_{\mathbb{K}}(m_{ik}m_{jl}+m_{il}m_{jk})\cdot a_{ij} \end{aligned}$$

for all $k,l \in \{0,\ldots,n\}$ satisfying $k \leq l$. By Corollary 3.3, we have

$$\eta'_1 = \eta_1 + Pf_{\mathbb{K}}[(a_{ij}\mid 0\leq i\leq j\leq n)]\cdot R^M_{\mathbb{K}}[(a_{ij}\mid 0\leq i\leq j\leq n)] + \Big(R^M_{\mathbb{K}}[(a_{ij}\mid 0\leq i\leq j\leq n)]\Big)^2,$$

$$\eta'_2 = \eta_2 = \Big(Pf_{\mathbb{K}}[(a_{ij}\mid 0\leq i\leq j\leq n)]\Big)^2.$$

Hence,

$$\frac{\eta'_1}{\eta'_2} = \frac{\eta_1}{\eta_2} + \lambda^2 + \lambda,$$

where

$$\lambda = \frac{R^M_{\mathbb{K}}[(a_{ij}\mid 0\leq i\leq j\leq n)]}{Pf_{\mathbb{K}}[(a_{ij}\mid 0\leq i\leq j\leq n)]}.$$

(2) Suppose that $\sum a_{ij}X_iX_j$ and $\sum a'_{ij}X_iX_j$ are related via the following coordinate transformation ($\mu \neq 0$): $X_0 = \mu Y_0$, $X_j = Y_j$ for all $j \in \{1,\ldots,n\}$. Then $a'_{00} = \mu^2 a_{00}$, $a'_{0i} = \mu\cdot a_{0i}$ for every $i \in \{1,\ldots,n\}$ and $a'_{ij} = a_{ij}$ for all $i,j \in \{1,\ldots,n\}$ with $i \leq j$. We have $\eta'_1 = \mu^2\eta_1$ and $\eta'_2 = \mu^2\eta_2$. Hence, $\frac{\eta'_1}{\eta'_2} = \frac{\eta_1}{\eta_2}$.

(3) Notice that any linear coordinate transformation is a composition of coordinate transformations as mentioned in (1) and (2). So, if $\sum a_{ij}X_iX_j$ and $\sum a'_{ij}X_iX_j$ are related via a linear coordinate transformation, then by (1), (2) and the fact that the set $\{\lambda+\lambda^2 \mid \lambda \in \mathbb{K}\}$ is closed under addition, it follows that there exists a $\lambda \in \mathbb{K}$ such that $\frac{\eta'_1}{\eta'_2} = \frac{\eta_1}{\eta_2} + \lambda^2 + \lambda$.

(4) Suppose $\sum a'_{ij}X_iX_j = \mu\cdot\Big(\sum a_{ij}X_iX_j\Big)$ for some $\mu \in \mathbb{K}\setminus\{0\}$. Then $\eta'_1 = \mu^{n+1}\cdot\eta_1$ and $\eta'_2 = \mu^{n+1}\cdot\eta_2$. Hence, $\frac{\eta'_1}{\eta'_2} = \frac{\eta_1}{\eta_2}$.

The proposition now follows from (3), (4) and Corollary 3.1. □

Suppose Q is a hyperbolic or elliptic quadric of $\mathrm{PG}(V)$ whose equation with respect to a given reference system is equal to $\sum_{0\leq i\leq j\leq n} a_{ij}X_iX_j = 0$. By Lemma 3.3, there exists an $\alpha \in \mathbb{K}$ and a reference system with respect to which Q has equation $\sum_{0\leq i\leq j\leq n} a'_{ij}X_iX_j = X_0^2 + X_0X_1 + \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$. Put $\eta_1 = f_{\mathbb{K}}[(a_{ij}\mid 0\leq i\leq j\leq n)]$, $\eta_2 = g_{\mathbb{K}}[(a_{ij}\mid 0\leq i\leq j\leq n)]$, $\eta'_1 = f_{\mathbb{K}}[(a'_{ij}\mid 0\leq i\leq j\leq n)]$ and $\eta'_2 = g_{\mathbb{K}}[(a'_{ij}\mid 0\leq i\leq j\leq n)]$. By Corollary 3.2, we have $\eta'_1 = \alpha$ and $\eta'_2 = 1$. So, by Proposition 3.1, we have $\alpha = \frac{\eta_1}{\eta_2} + \lambda^2 + \lambda$ for some $\lambda \in \mathbb{K}$. Since $\sum_{0\leq i\leq j\leq n} a'_{ij}X_iX_j = (X_0+\lambda X_1)^2 + (X_0+\lambda X_1)X_1 + \frac{\eta_1}{\eta_2}X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n$, there exists a reference system with respect to which Q has equation $X_0^2 + X_0X_1 + \frac{\eta_1}{\eta_2}X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$. This completes the proof of Theorem 1.1(2).

Now, let $\mu_1, \mu_2 \in \mathbb{K}$ and consider the quadrics Q_1 and Q_2 with respective equations $X_0^2 + X_0X_1 + \mu_1X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ and $X_0^2 + X_0X_1 + \mu_2X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ with respect to a given ordered basis $B = (\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$ of V. If $\mu_2 = \mu_1 + \lambda^2 + \lambda$ for some $\lambda \in \mathbb{K}$, then Q_1 and Q_2 are projectively equivalent since $X_0^2 + X_0X_1 + \mu_2X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = (X_0 + \lambda X_1)^2 + X_1(X_0 + \lambda X_1) + \mu_1X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n$. Conversely, if Q_1 and Q_2 are projectively equivalent, then by Corollary 3.2 and Proposition 3.1, $\mu_2 = \mu_1 + \lambda^2 + \lambda$ for a certain $\lambda \in \mathbb{K}$.

Now, for every automorphism θ of $\mathbb{K}$, let Q_1^θ denote the quadric with equation $X_0^2 + X_0X_1 + \mu_1^\theta X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ with respect to B. Notice that a point $\langle \sum_{i=0}^n X_i\bar{e}_i \rangle$ of $\mathrm{PG}(V)$ belongs to Q_1 if and only if $\langle \sum_{i=0}^n X_i^\theta \bar{e}_i \rangle$ belongs to Q_1^θ. The quadrics Q_1 and Q_2 are equivalent under an automorphism of $\mathrm{PG}(V)$ if and only if Q_2 and Q_1^θ are projectively equivalent for some automorphism θ of $\mathbb{K}$, i.e. if and only if there exists an automorphism θ of $\mathbb{K}$ and a $\lambda \in \mathbb{K}$ such that $\mu_2 = \mu_1^\theta + \lambda^2 + \lambda$. This completes the proof of Theorem 1.2(2).

3.8. **Proofs of Theorem 1.1(3) and Theorem 1.2(3).** Throughout this paragraph, we suppose that $\mathrm{char}(\mathbb{K}) = 2$.

Let Q be a pseudo-elliptic quadric of $\mathrm{PG}(V)$ whose equation with respect to a given ordered basis $(\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$ of V is given by $\sum_{0 \le i \le j \le n} a_{ij}X_iX_j = 0$. Put $q(\sum_{i=0}^n X_i\bar{e}_i) = \sum_{0 \le i \le j \le n} a_{ij}X_iX_j$. Then q is a quadratic form of V. If b is the symmetric bilinear form of V associated to q, then $b(\sum_{i=0}^n X_i\bar{e}_i, \sum_{i=0}^n Y_i\bar{e}_i) = [X_0 \cdots X_n] \cdot A \cdot [Y_0 \cdots Y_n]^T$, where A is the matrix as defined in equation (1.2) of Section 1.1. The radical R of b consists of all vectors $\sum_{i=0}^n X_i\bar{e}_i$ of V such that $[X_0 \cdots X_n]^T$ belongs to the kernel of A.

Now, by Lemma 3.3, we can choose an ordered basis $(\bar{e}_0', \bar{e}_1', \ldots, \bar{e}_n')$ of V such that $q(\sum_{i=0}^n X_i\bar{e}_i') = \sum_{0 \le i \le j \le n} a_{ij}'X_iX_j = \lambda \cdot (X_0^2 + \alpha X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n)$ for some $\lambda \in \mathbb{K} \setminus \{0\}$ and some non-square α of $\mathbb{K}$. One calculates that $R = \langle \bar{e}_0', \bar{e}_1' \rangle$. Now, let $\delta_0\bar{e}_0' + \delta_1\bar{e}_1'$ and $\delta_2\bar{e}_0' + \delta_3\bar{e}_1'$ be two linearly independent vectors of R. So, $\delta_0\delta_3 + \delta_1\delta_2 \neq 0$. We have $\eta := q(\delta_0\bar{e}_0' + \delta_1\bar{e}_1') \cdot q(\delta_2\bar{e}_0' + \delta_3\bar{e}_1') = \lambda^2 \cdot (\delta_0^2 + \alpha\delta_1^2) \cdot (\delta_2^2 + \alpha\delta_3^2) = [\lambda(\delta_0\delta_2 + \alpha\delta_1\delta_3)]^2 + [\lambda(\delta_0\delta_3 + \delta_1\delta_2)]^2 \cdot \alpha = \gamma_1^2 + \gamma_2^2 \cdot \alpha$, where $\gamma_1 = \lambda(\delta_0\delta_2 + \alpha\delta_1\delta_3)$ and $\gamma_2 = \lambda(\delta_0\delta_3 + \delta_1\delta_2) \neq 0$. Since $X_0^2 + \eta X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = (X_0 + \gamma_1X_1)^2 + \alpha(\gamma_2X_1)^2 + X_2X_3 + \cdots + X_{n-1}X_n$, the quadric Q is projectively equivalent to the quadric with equation $X_0^2 + \eta X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$. This proves Theorem 1.1(3).

Let μ_1 and μ_2 be two non-squares of $\mathbb{K}$. Let Q_1 and Q_2 be the quadrics with respective equations $X_0^2 + \mu_1X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ and $X_0^2 + \mu_2X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ with respect to a given ordered basis $B = (\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$. If $\mu_2 = \lambda_1^2 + \lambda_2^2 \cdot \mu_1$ for some $\lambda_1, \lambda_2 \in \mathbb{K}$ with $\lambda_2 \neq 0$, then Q_1 and Q_2 are projectively equivalent since $(X_0 + \lambda_1X_1)^2 + \mu_1(\lambda_2X_1)^2 + X_2X_3 + \cdots + X_{n-1}X_n = X_0^2 + \mu_2X_1^2 + X_2X_3 + \cdots + X_{n-1}X_n$.

Conversely, suppose that Q_1 and Q_2 are projectively equivalent. Then by Corollary 3.1 there exists a quadratic form q of V, a $\lambda \in \mathbb{K} \setminus \{0\}$ and two ordered bases $(\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$ and $(\bar{e}_0', \bar{e}_1', \ldots, \bar{e}_n')$ of V such that $q(\sum_{i=0}^n X_i\bar{e}_i) = X_0^2 + \mu_1X_1^2 +$

$X_2X_3+\cdots+X_{n-1}X_n$ and $q(\sum_{i=0}^n X_i\bar{e}'_i)=\lambda(X_0^2+\mu_2X_1^2+X_2X_3+\cdots+X_{n-1}X_n)$. The radical of the bilinear form associated to q is equal to $\langle\bar{e}_0,\bar{e}_1\rangle=\langle\bar{e}'_0,\bar{e}'_1\rangle$. Put $\bar{e}'_0=\delta_0\bar{e}_0+\delta_1\bar{e}_1$ and $\bar{e}'_1=\delta_2\bar{e}_0+\delta_3\bar{e}_1$. Here, $\delta_0\delta_3+\delta_1\delta_2\neq 0$. We have $\mu_1=q(\bar{e}_0)\cdot q(\bar{e}_1)$ and $\lambda^2\mu_2=q(\bar{e}'_0)\cdot q(\bar{e}'_1)=q(\delta_0\bar{e}_0+\delta_1\bar{e}_1)\cdot q(\delta_2\bar{e}_0+\delta_3\bar{e}_1)=(\delta_0^2+\mu_1\delta_1^2)\cdot(\delta_2^2+\mu_1\delta_3^2)=(\delta_0\delta_2+\mu_1\delta_1\delta_3)^2+(\delta_0\delta_3+\delta_1\delta_2)^2\mu_1$. Hence, $\mu_2=\left(\frac{\delta_0\delta_2+\mu_1\delta_1\delta_3}{\lambda}\right)^2+\left(\frac{\delta_0\delta_3+\delta_1\delta_2}{\lambda}\right)^2\mu_1$, where $\frac{\delta_0\delta_3+\delta_1\delta_2}{\lambda}\neq 0$.

For every automorphism θ of $\mathbb{K}$, let Q_1^θ denote the quadric whose equation with respect to B is given by $X_0^2+\mu_1^\theta X_1^2+X_2X_3+\cdots+X_{n-1}X_n=0$. Notice that a point $\langle\sum_{i=0}^n X_i\bar{e}_i\rangle$ of $\mathrm{PG}(V)$ belongs to Q_1 if and only if $\langle\sum_{i=0}^n X_i^\theta\bar{e}_i\rangle$ belongs to Q_1^θ. The quadrics Q_1 and Q_2 are equivalent under an automorphism of $\mathrm{PG}(V)$ if and only if Q_2 and Q_1^θ are projectively equivalent for some automorphism θ of $\mathbb{K}$, i.e. if and only if there exists an automorphism θ of $\mathbb{K}$ and $\lambda_1,\lambda_2\in\mathbb{K}$ with $\lambda_2\neq 0$ such that $\mu_2=\lambda_1^2+\lambda_2^2\cdot\mu_1$. This completes the proof of Theorem 1.2(3).

4. Proofs of Theorems 1.3 and 1.4

4.1. **Coordinate transformations.** Let $B=(\bar{e}_0,\bar{e}_1,\ldots,\bar{e}_n)$ be an ordered basis of V, let ψ be an involutary automorphism of $\mathbb{K}$ and let a_{ij}, $i,j\in\{0,\ldots,n\}$, be elements of $\mathbb{K}$ satisfying $a_{ji}=a_{ij}^\psi$. Let $\mathcal{H}$ be the ψ-Hermitian variety of $\mathrm{PG}(V)$ whose equation with respect to B is given by

$$[X_0\cdots X_n]\cdot A\cdot[X_0^\psi\cdots X_n^\psi]^T=\sum_{0\leq i,j\leq n}a_{ij}X_iX_j^\psi=0.$$

Now, consider the coordinate transformation with associated nonsingular matrix $M=(m_{ij})_{0\leq i,j\leq n}$: $[X_0\cdots X_n]^T=M\cdot[Y_0\cdots Y_n]^T$. The equation of $\mathcal{H}$ in the new coordinates is equal to $[Y_0\cdots Y_n]\cdot A'\cdot[Y_0^\psi\cdots Y_n^\psi]^T=\sum_{0\leq i,j\leq n}a'_{ij}X_iX_j^\psi$, where $A'=(a'_{ij})_{0\leq i,j\leq n}=M^TAM^\psi$. In the sequel, we will say that $\sum_{0\leq i,j\leq n}a'_{ij}X_iX_j^\psi$ and $\sum_{0\leq i,j\leq n}a_{ij}X_iX_j^\psi$ are *related via a linear coordinate transformation.*

Since $A'=M^TAM^\psi$ we have that $\det(A')=\det(M^T)\cdot\det(A)\cdot\det(M^\psi)=[\det(M)]^{\psi+1}\cdot\det(A)$. So, we have established the following.

Lemma 4.1. *Let ψ_1 and ψ_2 be two involutary automorphisms of $\mathbb{K}$. If $\sum a_{ij}X_iX_j^{\psi_1}$ and $\sum a'_{ij}X_iX_j^{\psi_2}$ are related via a linear coordinate transformation, then $\psi_1=\psi_2$ and there exists a $\lambda\in\mathbb{K}\setminus\{0\}$ such that $\det(A')=\det(A)\cdot\lambda^{\psi_1+1}$.*

4.2. **The equations representing certain Hermitian varieties.**

Lemma 4.2. *Let $B=(\bar{e}_0,\bar{e}_1,\ldots,\bar{e}_n)$ be an ordered basis of V, let ψ_1 and ψ_2 be two involutary automorphisms of $\mathbb{K}$, let $\alpha\in\mathbb{K}\setminus\{0\}$ such that $\alpha^{\psi_1}=\alpha$ and let a_{ij}, $i,j\in\{0,\ldots,n\}$, be elements of $\mathbb{K}$ satisfying $a_{ji}=a_{ij}^{\psi_2}$. Suppose $\mathcal{H}$ is the ψ_1-Hermitian variety of $\mathrm{PG}(V)$ whose equation with respect to B is given by $X_0^{\psi_1+1}+\alpha X_1^{\psi_1+1}+X_2X_3^{\psi_1}+X_3X_2^{\psi_1}+\cdots+X_{n-1}X_n^{\psi_1}+X_nX_{n-1}^{\psi_1}=0$. If the equation $\sum_{0\leq i,j\leq n}a_{ij}X_iX_j^{\psi_2}$ describes $\mathcal{H}$ with respect to the same ordered basis B, then $\psi_1=\psi_2$ and $\sum_{0\leq i,j\leq n}a_{ij}X_iX_j^{\psi_2}=\lambda\cdot(X_0^{\psi_1+1}+\alpha X_1^{\psi_1+1}+X_2X_3^{\psi_1}+X_3X_2^{\psi_1}+\cdots+X_{n-1}X_n^{\psi_1}+X_nX_{n-1}^{\psi_1})$ for some $\lambda\in\mathbb{K}\setminus\{0\}$ satisfying $\lambda^{\psi_1}=\lambda$.*

Proof. Notice first that $\mathbb{K}/\mathbb{K}_{\psi_i}$ ($i\in\{1,2\}$) is a Galois extension and ψ_i is the unique nontrivial element of $Gal(\mathbb{K}/\mathbb{K}_{\psi_i})$. So, if $\mathbb{K}_{\psi_1}=\mathbb{K}_{\psi_2}$, then $\psi_1=\psi_2$.

- For all $i\in\{2,\ldots,n\}$, we have $a_{ii}=0$ since $\langle\bar{e}_i\rangle\in\mathcal{H}$.

• Let $i,j \in \{2,\dots,n\}$ such that $i \neq j$ and $\{i,j\} \notin \{\{2,3\},\{4,5\},\dots,\{n-1,n\}\}$. For every $\lambda \in \mathbb{K}$, we have $\lambda a_{ij} + \lambda^{\psi_2} a_{ji} = \lambda a_{ij} + (\lambda a_{ij})^{\psi_2} = 0$ since $\langle \lambda \bar{e}_i + \bar{e}_j \rangle \in \mathcal{H}$. This implies that $a_{ij} = 0$.

• We prove that $\psi_1 = \psi_2$ and $a_{2i,2i+1} = a_{2i+1,2i} \neq 0$ for every $i \in \{1,\dots,\frac{n-1}{2}\}$. Let $k \in \mathbb{K} \setminus \{0\}$ such that $k + k^{\psi_1} = 0$ (e.g., $k = l - l^{\psi_1}$ where $l \in \mathbb{K} \setminus \mathbb{K}_{\psi_1}$). Then since $\langle k\bar{e}_{2i} + \bar{e}_{2i+1} \rangle \in \mathcal{H}$, we have $k a_{2i,2i+1} + k^{\psi_2} a_{2i+1,2i} = 0$. Now, a point $\langle k\lambda \bar{e}_{2i} + \bar{e}_{2i+1} \rangle$, $\lambda \in \mathbb{K}$, belongs to $\mathcal{H}$ if and only if $k\lambda + (k\lambda)^{\psi_1} = k(\lambda - \lambda^{\psi_1}) = 0$, i.e. if and only if $\lambda \in \mathbb{K}_{\psi_1}$. On the other hand, $\langle k\lambda \bar{e}_{2i} + \bar{e}_{2i+1} \rangle \in \mathcal{H}$ if and only if $k\lambda a_{2i,2i+1} + (k\lambda)^{\psi_2} a_{2i+1,2i} = k a_{2i,2i+1}(\lambda - \lambda^{\psi_2}) = 0$. Hence, $a_{2i,2i+1} \neq 0$ and $\mathbb{K}_{\psi_1} = \mathbb{K}_{\psi_2}$. This implies that $\psi_1 = \psi_2$. Since $0 = k a_{2i,2i+1} + k^{\psi_2} a_{2i+1,2i} = k a_{2i,2i+1} + k^{\psi_1} a_{2i+1,2i} = k(a_{2i,2i+1} - a_{2i+1,2i})$, we also have $a_{2i,2i+1} = a_{2i+1,2i}$.

• We prove that $a_{00} \neq 0$ and $a_{0i} = a_{i0} = 0$ for every $i \in \{2,\dots,n\}$. Since $\langle \bar{e}_0 + \lambda^{\psi_1} \bar{e}_i \rangle \notin \mathcal{H}$ for every $\lambda \in \mathbb{K}$, we have $a_{00} + \lambda a_{0i} + \lambda^{\psi_1} a_{i0} = a_{00} + \lambda a_{0i} + (\lambda a_{0i})^{\psi_1} \neq 0$ for every $\lambda \in \mathbb{K}$. This implies that $a_{00} \neq 0$ and $a_{0i} = 0$. (Otherwise, we can take $\lambda = -\frac{a_{00}}{a_{0i}} \cdot \frac{a}{a - a^{\psi_1}}$ where a is an arbitrary element of $\mathbb{K} \setminus \mathbb{K}_{\psi_1}$.) Hence, also $a_{i0} = a_{0i}^{\psi_1} = 0$.

• We prove that $a_{11} \neq 0$ and $a_{1i} = a_{i1} = 0$ for every $i \in \{2,\dots,n\}$. Since $\langle \bar{e}_1 + \lambda^{\psi_1} \bar{e}_i \rangle \notin \mathcal{H}$ for every $\lambda \in \mathbb{K}$, we have $a_{11} + \lambda a_{1i} + \lambda^{\psi_1} a_{i1} = a_{11} + \lambda a_{1i} + (\lambda a_{1i})^{\psi_1} \neq 0$ for every $\lambda \in \mathbb{K}$. As in the previous case, this implies that $a_{11} \neq 0$ and $a_{1i} = 0$. Hence, also $a_{i1} = a_{1i}^{\psi_1} = 0$.

• We prove that $a_{00} = a_{2i,2i+1}$ for every $i \in \{1,\dots,\frac{n-1}{2}\}$. Let $k \in \mathbb{K} \setminus \mathbb{K}_{\psi_1}$. Since $\langle \bar{e}_0 - \frac{k}{k-k^{\psi_1}} \bar{e}_{2i} + \bar{e}_{2i+1} \rangle \in \mathcal{H}$, we have $a_{00} - \frac{k}{k-k^{\psi_1}} a_{2i,2i+1} + \frac{k^{\psi_1}}{k-k^{\psi_1}} a_{2i+1,2i} = a_{00} - a_{2i,2i+1} = 0$.

• We prove that $a_{11} = \alpha \cdot a_{2i,2i+1}$ for every $i \in \{1,\dots,\frac{n-1}{2}\}$. Let $k \in \mathbb{K} \setminus \mathbb{K}_{\psi_1}$. Since $\langle \bar{e}_1 - \frac{k\alpha}{k-k^{\psi_1}} \bar{e}_{2i} + \bar{e}_{2i+1} \rangle \in \mathcal{H}$, we have $a_{11} - \frac{k\alpha}{k-k^{\psi_1}} a_{2i,2i+1} + \frac{k^{\psi_1}\alpha}{k-k^{\psi_1}} a_{2i+1,2i} = a_{11} - \alpha \cdot a_{2i,2i+1} = 0$.

• We prove that $a_{01} = 0$. Let $k \in \mathbb{K} \setminus \mathbb{K}_{\psi_1}$. Since $\langle \lambda \bar{e}_0 + \bar{e}_1 - \frac{k(\lambda^{\psi_1+1}+\alpha)}{k-k^{\psi_1}} \bar{e}_{2i} + \bar{e}_{2i+1} \rangle \in \mathcal{H}$ for every $\lambda \in \mathbb{K}$, we have $a_{00}\lambda^{\psi_1+1} + \lambda a_{01} + \lambda^{\psi} a_{10} + a_{11} - (\lambda^{\psi_1+1} + \alpha) a_{2i,2i+1} = \lambda a_{01} + (\lambda a_{01})^{\psi_1} = 0$ for every $\lambda \in \mathbb{K}$. This implies that $a_{01} = 0$.

By the above, we know that $\sum_{0 \leq i,j \leq n} a_{ij} X_i X_j^{\psi_2} = a_{00}(X_0^{\psi_1+1} + \alpha X_1^{\psi_1+1} + X_2 X_3^{\psi_1} + X_3 X_2^{\psi_1} + \cdots + X_{n-1} X_n^{\psi_1} + X_n X_{n-1}^{\psi_1})$. Here, $a_{00} \in \mathbb{K} \setminus \{0\}$ satisfies $a_{00}^{\psi_1} = a_{00}$. □

4.3. Canonical equations for nonsingular Hermitian varieties of Witt index at least $\frac{n-1}{2}$.

Lemma 4.3. *Let ψ be an involutary automorphism of $\mathbb{K}$. If $\mathcal{H}$ is a nonsingular ψ-Hermitian variety of Witt index at least $\frac{n-1}{2}$ of $\mathrm{PG}(V)$, then there exists an $\alpha \in \mathbb{K} \setminus \{0\}$ satisfying $\alpha^{\psi} = \alpha$ and an ordered basis $(\bar{e}_0, \bar{e}_1, \dots, \bar{e}_n)$ of V with respect to which $\mathcal{H}$ has equation $X_0^{\psi+1} - \alpha X_1^{\psi+1} + X_2 X_3^{\psi} + X_3 X_2^{\psi} + \cdots + X_{n-1} X_n^{\psi} + X_n X_{n-1}^{\psi} = 0$. $\mathcal{H}$ has Witt index $\frac{n+1}{2}$ if and only if there exists a $\lambda \in \mathbb{K}$ satisfying $\lambda^{\psi+1} = \alpha$. In this case, there exists an ordered basis of V with respect to which $\mathcal{H}$ has equation $X_0 X_1^{\psi} + X_1 X_0^{\psi} + \cdots + X_{n-1} X_n^{\psi} + X_n X_{n-1}^{\psi} = 0$.*

Proof. The proof is similar to the proof of Lemma 3.3. Let $h : V \times V \to \mathbb{K}$ be a sesquilinear form satisfying:

• $h(\lambda_1 \bar{x}_1 + \lambda_2 \bar{x}_2, \bar{y}) = \lambda_1 \cdot h(\bar{x}_1, \bar{y}) + \lambda_2 \cdot h(\bar{x}_2, \bar{y})$ for all $\lambda_1, \lambda_2 \in \mathbb{K}$ and all $\bar{x}_1, \bar{x}_2, \bar{y} \in V$.

• $h(\bar{x}, \lambda_1 \bar{y}_1 + \lambda_2 \bar{y}_2) = \lambda_1^\psi \cdot h(\bar{x}, \bar{y}_1) + \lambda_2^\psi \cdot h(\bar{x}, \bar{y}_2)$ for all $\lambda_1, \lambda_2 \in \mathbb{K}$ and all $\bar{x}, \bar{y}_1, \bar{y}_2 \in V$.
• $h(\bar{y}, \bar{x}) = h(\bar{x}, \bar{y})^\psi$ for all $\bar{x}, \bar{y} \in V$.
• $\mathcal{H}$ consists of all points $\langle \bar{x} \rangle$ of $\mathrm{PG}(V)$ for which $H(\bar{x}) := h(\bar{x}, \bar{x}) = 0$.

Since $\mathcal{H}$ is nonsingular, h is nondegenerate, i.e. there exists no $\bar{x} \in V \setminus \{\bar{o}\}$ such that $h(\bar{x}, \bar{y}) = 0$ for all $\bar{y} \in V$. Suppose the Witt index m' of $\mathcal{H}$ is at least $m := \frac{n-1}{2}$.

Now, let π_1 and π_2 be two m'-dimensional subspaces of V satisfying (i) $\pi_1 \cap \pi_2 = \{\bar{o}\}$ and (ii) $H(\bar{x}) = 0$ for all $\bar{x} \in \pi_1 \cup \pi_2$. Let $\{\bar{e}_1, \ldots, \bar{e}_{m'}\}$ be a basis of π_1. Similarly as in the proof of Lemma 3.3, there exists a basis $\{\bar{f}_1, \ldots, \bar{f}_{m'}\}$ of π_2 such that $h(\bar{e}_i, \bar{f}_i) = 1$ and $h(\bar{e}_j, \bar{f}_i) = 0$ for all $i \in \{1, \ldots, m'\}$ and all $j \in \{1, \ldots, m'\} \setminus \{i\}$.

If $m' = m+1 = \frac{n+1}{2}$, then $\{\bar{e}_1, \ldots, \bar{e}_{m'}, \bar{f}_1, \ldots, \bar{f}_{m'}\}$ is a basis of V and $H(X_0 \bar{e}_1 + X_1 \bar{f}_1 + X_2 \bar{e}_2 + X_3 \bar{f}_2 + \cdots + X_{n-1} \bar{e}_{m'} + X_n \bar{f}_{m'}) = X_0 X_1^\psi + X_1 X_0^\psi + X_2 X_3^\psi + X_3 X_2^\psi + \cdots + X_{n-1} X_n^\psi + X_n X_{n-1}^\psi$.

Consider now the most general case $m' \in \{m, m+1\}$. Let W denote the subspace of V consisting of all vectors which are h-orthogonal with all vectors of $U := \langle \bar{e}_1, \ldots, \bar{e}_m, \bar{f}_1, \ldots, \bar{f}_m \rangle$. Since U has co-dimension 2 in V, W has dimension 2. We have $H(X_2 \bar{e}_1 + X_3 \bar{f}_1 + \cdots + X_{n-1} \bar{e}_m + X_n \bar{f}_m) = X_2 X_3^\psi + X_3 X_2^\psi + \cdots + X_{n-1} X_n^\psi + X_n X_{n-1}^\psi$ and $H(\bar{x}_1 + \bar{x}_2) = H(\bar{x}_1) + H(\bar{x}_2)$ for all vectors $\bar{x}_1 \in W$ and $\bar{x}_2 \in U$. It is easily verified that there exists a basis $\{\bar{e}'_{m+1}, \bar{f}'_{m+1}\}$ of W such that $H(X_0 \bar{e}'_{m+1} + X_1 \bar{f}'_{m+1}) = \beta(X_0^{\psi+1} - \alpha X_1^{\psi+1})$ for some $\alpha, \beta \in \mathbb{K}_\psi \setminus \{0\}$. With respect to the reference system $(\bar{e}'_{m+1}, \bar{f}'_{m+1}, \bar{e}_1, \beta \cdot \bar{f}_1, \ldots, \bar{e}_m, \beta \cdot \bar{f}_m)$, $\mathcal{H}$ has equation $X_0^{\psi+1} - \alpha X_1^{\psi+1} + X_2 X_3^\psi + X_3 X_2^\psi + \cdots + X_{n-1} X_n^\psi + X_n X_{n-1}^\psi = 0$. Now, $\langle \bar{e}_1, \ldots, \bar{e}_m \rangle$ is a subspace of $\mathcal{H}$. So, $\mathcal{H}$ has Witt index $\frac{n+1}{2}$ if and only if there exists an $\frac{n-1}{2}$-dimensional subspace of $\mathcal{H}$ containing $\langle \bar{e}_1, \ldots, \bar{e}_m \rangle$. This precisely happens when the equation $X_0^{\psi+1} - \alpha X_1^{\psi+1} = 0$ has a nonzero solution for (X_0, X_1). The claims of the lemma follow. □

By Lemmas 4.2 and 4.3, we have

Corollary 4.1. *Let $\mathcal{H}$ be a nonsingular ψ-Hermitian variety of Witt index $m \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$ of $\mathrm{PG}(V)$ and let B be an ordered basis of V. Then the left hand side of an equation $\sum_{0 \le i,j \le n} a_{ij} X_i X_j^\psi = 0$ $(a_{ji} = a_{ij}^\psi)$ which describes $\mathcal{H}$ with respect to B is uniquely determined up to a nonzero factor of $\mathbb{K}_\psi$.*

4.4. **Proofs of Theorems 1.3 and 1.4.** Let ψ be an involutory automorphism of $\mathbb{K}$ and let $\mathcal{H}$ be the nonsingular ψ-Hermitian variety of Witt index at least $\frac{n-1}{2}$ of $\mathrm{PG}(V)$ whose equation with respect to a given reference system is given by $\sum_{0 \le i,j \le n} a_{ij} X_i X_j^\psi = 0$. Here, $a_{ji} = a_{ij}^\psi$ for all $i, j \in \{0, \ldots, n\}$. By Lemma 4.3, we know that there exist b_{ij} and c_{ij} $(0 \le i, j \le n)$ such that

(i) $b_{ji} = b_{ij}^\psi$ and $c_{ji} = c_{ij}^\psi$ for all $i, j \in \{0, \ldots, n\}$;
(ii) $\sum_{0 \le i,j \le n} b_{ij} X_i X_j^\psi$ is related to $\sum_{0 \le i,j \le n} a_{ij} X_i X_j^\psi$ via a linear coordinate transformation;
(iii) $\sum_{0 \le i,j \le n} c_{ij} X_i X_j^\psi = \lambda_3 \Big(\sum_{0 \le i,j \le n} b_{ij} X_i X_j^\psi \Big) = X_0^{\psi+1} - \alpha X_1^{\psi+1} + X_2 X_3^\psi + X_3 X_2^\psi + \cdots + X_{n-1} X_n^\psi + X_n X_{n-1}^\psi$ for some $\lambda_3, \alpha \in \mathbb{K}_\psi \setminus \{0\}$.

Now, put $\lambda_2 = \lambda_3^{\frac{n+1}{2}}$. Then $\lambda_2^{\psi+1} = \lambda_3^{n+1}$ since $\lambda_2 \in \mathbb{K}_\psi$. Consider the matrices $A = (a_{ij})_{0 \le i,j \le n}$, $B = (b_{ij})_{0 \le i,j \le n}$ and $C = (c_{ij})_{0 \le i,j \le n}$. By Lemma 4.1, there exists a $\lambda_1 \in \mathbb{K} \setminus \{0\}$ such that $\det(B) = \lambda_1^{\psi+1} \det(A)$. So, we have $(-1)^{\frac{n+1}{2}} \alpha = \det(C) = \lambda_3^{n+1} \det(B) = (\lambda_1 \lambda_2)^{\psi+1} \det(A)$. Hence, $\alpha = (\lambda_1 \lambda_2)^{\psi+1} (-1)^{\frac{n+1}{2}} \det(A)$ and $\mathcal{H}$ has equation $X_0^{\psi+1} - (-1)^{\frac{n+1}{2}} \det(A) \cdot (\lambda_1 \lambda_2 X_1)^{\psi+1} + X_2 X_3^{\psi} + X_3 X_2^{\psi} + \cdots + X_{n-1} X_n^{\psi} + X_n X_{n-1}^{\psi} = 0$ with respect to a suitable reference system. This completes the proof of Theorem 1.3.

Now, let ψ_1 and ψ_2 be two involutary automorphisms of $\mathbb{K}$ and let $\mu_i \in \mathbb{K}_{\psi_i} \setminus \{0\}$, $i \in \{1, 2\}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hermitian varieties whose equations with respect to a given ordered basis $B = (\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$ of V are given by respectively $X_0^{\psi_1+1} + \mu_1 \cdot X_1^{\psi_1+1} + X_2 X_3^{\psi_1} + X_3 X_2^{\psi_1} + \cdots + X_{n-1} X_n^{\psi_1} + X_n X_{n-1}^{\psi_1} = 0$ and $X_0^{\psi_2+1} + \mu_2 \cdot X_1^{\psi_2+1} + X_2 X_3^{\psi_2} + X_3 X_2^{\psi_2} + \cdots + X_{n-1} X_n^{\psi_2} + X_n X_{n-1}^{\psi_2} = 0$. If $\psi_1 = \psi_2$ and there exists a $\lambda \in \mathbb{K} \setminus \{0\}$ such that $\mu_2 = \lambda^{\psi_1+1} \mu_1$, then $\mathcal{H}_1$ and $\mathcal{H}_2$ are projectively equivalent since $X_0^{\psi_2+1} + \mu_2 \cdot X_1^{\psi_2+1} + X_2 X_3^{\psi_2} + X_3 X_2^{\psi_2} + \cdots + X_{n-1} X_n^{\psi_2} + X_n X_{n-1}^{\psi_2} = X_0^{\psi_1+1} + \mu_1 \cdot (\lambda X_1)^{\psi_1+1} + X_2 X_3^{\psi_1} + X_3 X_2^{\psi_1} + \cdots + X_{n-1} X_n^{\psi_1} + X_n X_{n-1}^{\psi_1}$.

Conversely, suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are projectively equivalent. Then by Lemmas 4.1 and 4.2, $\psi_1 = \psi_2$. Also by these lemmas, there exist elements a_{ij}, $i, j \in \{0, \ldots, n\}$, of $\mathbb{K}$ such that:

(i) $a_{ji} = a_{ij}^{\psi_1}$ for all $i, j \in \{0, \ldots, n\}$;

(ii) $\sum_{0 \le i,j \le n} a_{ij} X_i X_j^{\psi_1}$ is related to $X_0^{\psi_1+1} + \mu_1 \cdot X_1^{\psi_1+1} + X_2 X_3^{\psi_1} + X_3 X_2^{\psi_1} + \cdots + X_{n-1} X_n^{\psi_1} + X_n X_{n-1}^{\psi_1}$ via a linear coordinate transformation;

(iii) $\sum_{0 \le i,j \le n} a_{ij} X_i X_j^{\psi_1} = \alpha \cdot (X_0^{\psi_1+1} + \mu_2 \cdot X_1^{\psi_1+1} + X_2 X_3^{\psi_1} + X_3 X_2^{\psi_1} + \cdots + X_{n-1} X_n^{\psi_1} + X_n X_{n-1}^{\psi_1})$ for some $\alpha \in \mathbb{K}_{\psi_1} \setminus \{0\}$.

Put $A = (a_{ij})_{0 \le i,j \le n}$. Then $\det(A) = \alpha^{n+1} \mu_2 (-1)^{\frac{n-1}{2}}$. Now, by (ii) and Lemma 4.1, there exist a $\lambda \in \mathbb{K} \setminus \{0\}$ such that $\det(A) = \lambda^{\psi_1+1} \mu_1 (-1)^{\frac{n-1}{2}}$. Since $\alpha^{n+1} = (\alpha^{\frac{n+1}{2}})^{\psi_1+1}$, it follows that $\mu_2 = \left(\frac{\lambda}{\alpha^{\frac{n+1}{2}}} \right)^{\psi_1+1} \cdot \mu_1$.

For every automorphism θ of $\mathbb{K}$, let $\mathcal{H}_1^{\theta}$ denote the $\theta^{-1}\psi_1\theta$-Hermitian variety of the projective space $\mathrm{PG}(V)$ whose equation with respect to B is given by $X_0^{\theta^{-1}\psi_1\theta+1} + \mu_1^{\theta} X_1^{\theta^{-1}\psi_1\theta+1} + X_2 X_3^{\theta^{-1}\psi_1\theta} + X_3 X_2^{\theta^{-1}\psi_1\theta} + \cdots + X_{n-1} X_n^{\theta^{-1}\psi_1\theta} + X_n X_{n-1}^{\theta^{-1}\psi_1\theta}$. Notice that a point $\langle \sum_{i=0}^{n} X_i \bar{e}_i \rangle$ of $\mathrm{PG}(V)$ belongs to $\mathcal{H}_1$ if and only if $\langle \sum_{i=0}^{n} X_i^{\theta} \bar{e}_i \rangle$ belongs to $\mathcal{H}_1^{\theta}$. Now, $\mathcal{H}_1$ and $\mathcal{H}_2$ are equivalent under an automorphism of $\mathrm{PG}(V)$ if and only if $\mathcal{H}_2$ and $\mathcal{H}_1^{\theta}$ are projectively equivalent for some automorphism θ of $\mathbb{K}$, i.e. if and only if there exists an automorphism θ of $\mathbb{K}$ and a $\lambda \in \mathbb{K} \setminus \{0\}$ such that $\psi_2 = \theta^{-1}\psi_1\theta$ and $\mu_2 = \lambda^{\psi_2+1} \mu_1^{\theta}$. This finishes the proof of Theorem 1.4.

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Department of Mathematics, Ghent University
E-mail address: bdb@cage.ugent.be