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A NEW TYPE OF STRUCTURE ON A DIFFERENTIABLE MANIFOLD

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Abstract. The objective of the present paper is to study a new type of structure named as almost quadratic ϕ-structure in an n-dimensional Riemannian manifold. Some results involving this structure have been established. Also conditions of being an almost contact and almost para-contact manifold have been deduced. Finally the existence for this type of structure is shown with an example.

1. Introduction

An odd dimensional differentiable manifold with structure tensors (ϕ, ξ, η), where ϕ is a (1, 1) type tensor, ξ is a vector field and η is a 1-form on the manifold, satisfying

$$\phi^2(X) + X = \eta(X)\xi, \quad \phi(\xi) = 0,$$

for any vector field X, is said to be an almost contact manifold [2].

I. Sato [1], introduced the concept of a structure similar to the almost contact structure which is known as almost para-contact structure. A differentiable manifold with structure tensors (ϕ, ξ, η) where ϕ is a (1, 1) type tensor, ξ is a vector field and η is a 1-form on the manifold, satisfying

$$\phi^2(X) = X - \eta(X)\xi, \quad \phi(\xi) = 0,$$

for any vector field X, is said to be an almost para-contact manifold [1]. In this paper, we also introduce a new type of structure named as *almost quadratic* $\phi-$ *structure* defined in the following manner:

Let M_n be an $n(>2)$ dimensional manifold and ϕ, ξ, η be a tensor field of type $(1, 1)$, a unit vector field and a 1-form respectively. If ϕ, ξ, η satisfy the conditions

$$\phi(\xi) = 0 \tag{1.1}$$

and

$$\phi^2(X) + a\phi(X) + bX = b\eta(X)\xi, \quad a^2 \neq 4b, \tag{1.2}$$

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for any vector field X and constants a, b $(\neq 0)$, then M_n is said to admit an *almost quadratic* $\phi - structure$, (ϕ, ξ, η) and such a manifold M_n is called an *almost quadratic* $\phi - manifold$.

2. Preliminaries

For any vector field X in an almost quadratic ϕ-manifold M_n, we have

$$\phi^2(X) + a\phi(X) + bX = b\eta(X)\xi \ . \tag{2.1}$$

Now, operating ϕ from left and using equation (1) we get

$$\phi^3(X) + a\phi^2(X) + b\phi(X) = 0 \ . \tag{2.2}$$

Again replacing X by $\phi(X)$, in equation (3), we get

$$\phi^3(X) + a\phi^2(X) + b\phi(X) = b\eta(\phi(X))\xi \ . \tag{2.3}$$

Now, comparing equation (4) and (5), we have $b\eta(\phi(X))\xi = 0$, but $b \neq 0$ and ξ is not a zero vector, thus we have

$$\eta(\phi(X)) = 0 \qquad i.e.\ \eta \circ \phi = 0 \ . \tag{2.4}$$

Now, $\phi(\xi) = 0 \Rightarrow \phi^2(\xi) = 0$. So, putting $X = \xi$ in equation (3), we get

$$\phi^2(\xi) + a\phi(\xi) + b\xi = b\eta(\xi)\xi \ ,$$

i.e. $b\xi = b\eta(\xi)\xi$ but $b \neq 0$ and ξ is also non zero, so

$$\eta(\xi) = 1 \ . \tag{2.5}$$

Now, for the transformation ϕ, $\phi(\xi) = 0$ but ξ is not a zero vector, so *Rank* $\phi \leq n - 1$. If there exist another vector α such that $\phi(\alpha) = 0$, then from equation (3), we get $b\alpha = b\eta(\alpha)\xi$, i.e. $\alpha = \eta(\alpha)\xi$ as $b \neq 0$.
So, α and ξ becomes linearly dependent. Therefore *kernel* of the transformation contains the only vector ξ and consequently the *Rank* $\phi = n - 1$. Thus, in view of equation (6) and (7), we have the following theorem:

Theorem 2.1. *In an almost quadratic* $\phi - manifold$ *we have*
a) $\eta \circ \phi = 0$
b) $\eta(\xi) = 1$ *and*
c) Rank $\phi = n - 1$.

We will now show that the almost quadratic ϕ-structure is not unique. Let f be a non singular vector valued linear function on M_n.
Let us define the $(1, 1)$ tensor field ϕ^*, the 1-form η^* and the unit vector field ξ^* as

$$f \circ \phi^* = \phi \circ f \tag{2.6}$$

$$\eta^* = \eta \circ f \tag{2.7}$$

$$f\xi^* = \xi \tag{2.8}$$

Now, post multiplying equation (8) by ϕ^* and using it, we get

$$f \circ \phi^{*2} = \phi \circ f \circ \phi^* = \phi \circ (f \circ \phi^*)$$

$$= \phi^2 \circ f$$

$$= (-a\phi - bI_n + b\eta \otimes \xi) \circ f$$

$$= -af \circ \phi^* - fbI_n + b\eta^* \otimes \xi \ .$$

Applying equation (10), we get

$$f \circ {\phi^*}^2 = f \circ (-a\phi^* - bI_n + b\eta^* \otimes \xi^*) \ .$$

Since f is non singular, we have

(2.9) $${\phi^*}^2 = -a\phi^* - bI_n + b\eta^* \otimes \xi^* \ .$$

Now, $f \circ \phi^* \xi^* = \phi \circ f\xi^* = \phi(\xi) = 0$, by equation (10) and since f is non singular

(2.10) $$\phi^* \xi^* = 0$$

Therefore, with the help of equation (11) and (12), we can state the following theorem:

Theorem 2.2. *The almost quadratic $\phi -$ structure in an almost quadratic $\phi -$ $manifold$ is not unique.*

3. Necessary and Sufficient Condition for Being an Almost Quadratic ϕ-Manifold

To find the necessary and sufficient condition for M_n to be an almost quadratic ϕ-manifold, we need the following results:

Theorem 3.1. *The eigen values of the structure tensor ϕ are the roots of the equation $\alpha(\alpha^2 + a\alpha + b) = 0$.*

Proof. Let α be the eigen value of ϕ and ζ be the corresponding eigen vector. Then

$$\phi(\zeta) = \alpha\zeta \text{ and } \phi^2(\zeta) = \alpha^2\zeta \ .$$

Now, using equation (3), we get

(3.1) $$(\alpha^2 + a\alpha + b)\zeta = b\eta(\zeta)\xi \ .$$

So, two cases arise

a) ζ and ξ are linearly dependent, i.e. $\zeta = c\xi$ for some non zero scalar c, or

b) ζ and ξ are linearly independent.

Case-a) Putting $\zeta = c\xi$ in equation (13), we get

$$c(\alpha^2 + a\alpha + b)\xi = bc\eta(\xi)\xi \ ,$$

i.e. $\alpha^2 + a\alpha + b = b$, since $c \neq 0, \eta(\xi) = 1$ and ξ is a non zero vector. Thus we get

$$\alpha = 0, -a \ .$$

Now, earlier, in this paper, during the proof of Theorem 2.1, we have seen that ξ is the only vector for which $\phi(\xi) = 0$ and we know that, for every eigen vector there corresponds only one eigen value but $\alpha = -a$ contradicts $\phi(\xi) = 0$ when $a \neq 0$. Therefore zero is the only eigen value of ϕ when ξ and ζ are linearly dependent.

Case-b) If ζ and ξ are linearly independent, then we have by equation (13)

$$\alpha^2 + a\alpha + b = 0 \ .$$

Therefore, combining Case-a and Case-b we see that, if α is an eigen value of ϕ, then α is a root of $\alpha(\alpha^2 + a\alpha + b) = 0$. ☐

Corollary 3.1. *If the vectors ξ and ζ are linearly independent then $\eta(\zeta) = 0$.*

Proof. Since $b \neq 0$, the proof is obvious from equation (13). □

Theorem 3.2. *The necessary and sufficient condition that a manifold M_n will be an almost quadratic ϕ-manifold is that at each point of the manifold M_n, it contains a tangent bundle Π_p of dimension p, a tangent bundle Π_q of dimension q and a real line Π_1 such that $\Pi_p \cap \Pi_q = \{\Phi\}, \Pi_p \cap \Pi_1 = \{\Phi\}, \Pi_q \cap \Pi_1 = \{\Phi\}$ (where $\{\Phi\}$ is the null set) and $\Pi_p \cup \Pi_q \cup \Pi_1 =$ a tangent bundle of dimension n, projection L, M, N on Π_p, Π_q and Π_1 respectively being given by*

$$a)\ \alpha L = -\phi^2 - (\tfrac{\sqrt{a^2-4b}\ +a}{2})\phi, \text{ where } \alpha = 2b - (\tfrac{a^2-a\sqrt{a^2-4b}}{2}) ,$$

$$b)\ \beta M = -\phi^2 + (\tfrac{\sqrt{a^2-4b}\ -a}{2})\phi, \text{ where } \beta = 2b - (\tfrac{a^2+a\sqrt{a^2-4b}}{2}) ,$$

$$c)\ bN = \phi^2 + a\phi + b = b\eta \otimes \xi .$$

Proof. To prove the above theorem, we need the help of the following lemma:

Lemma 3.1. *Let $\lambda_i, 1 \leq i \leq n$ be the eigen values of a square matrix A and ζ_i be the eigen vectors corresponding to λ_i, then*

$$\begin{aligned} f_j(A)\zeta_k &= (A^2 - \lambda_j A)\zeta_k \\ &= f_j(\lambda_k)\zeta_k \qquad \text{when } j \neq k \\ &= 0 \qquad \text{when } j = k . \end{aligned}$$

Proof. Since ζ_i are the eigen vectors corresponding to λ_i, for $1 \leq i \leq n$, we have

$$\begin{aligned} A\zeta_i &= \lambda_i \zeta_i \text{ and} \\ A^2\zeta_i &= {\lambda_i}^2 \zeta_i . \end{aligned}$$

Now,

$$\begin{aligned} f_j(A)\zeta_k &= (A^2 - \lambda_j A)\zeta_k \\ &= A^2\zeta_k - \lambda_j A\zeta_k \\ &= ({\lambda_k}^2 - \lambda_j\lambda_k)\zeta_k . \end{aligned}$$

Therefore, for $j \neq k$, $f_j(A)\zeta_k = f_j(\lambda_k)\zeta_k$ and for $j = k$, $f_j(A)\zeta_k = 0$ □

We now prove the main theorem:
Let P_i be the eigen vectors corresponding to the eigen value $\frac{-a+\sqrt{a^2-4b}}{2}$ of ϕ, Q_j be the eigen vectors corresponding to $\frac{-a-\sqrt{a^2-4b}}{2}$ and ξ be the eigen vector corresponding to the eigen value 0 respectively.
Now, let us consider the equation

$$c^i P_i + d^j Q_j + e\xi = 0 . \tag{3.2}$$

where c^i, d^j and e are scalars, $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$ and Einstein's summation convention is used. Applying ϕ on equation (14), we get

$$c^i\phi(P_i) + d^j\phi(Q_j) = 0 \tag{3.3}$$

$$\Rightarrow\ c^i[\frac{-a+\sqrt{a^2-4b}}{2}]P_i + d^j[\frac{-a-\sqrt{a^2-4b}}{2}]Q_j = 0$$

$$\Rightarrow\ -\frac{a}{2}[c^iP_i + d^jQ_j] + \frac{\sqrt{a^2-4b}}{2}[c^iP_i - d^jQ_j] = 0$$

Now, using equation (14), we get

$$\frac{a}{2}e\xi + \frac{\sqrt{a^2-4b}}{2}[c^iP_i - d^jQ_j] = 0 \ .$$

operating ϕ and since $a^2 \neq 4b$, we get

$$c^i\phi(P_i) - d^j\phi(Q_j) = 0 \ . \tag{3.4}$$

Thus, from equation (15) and (16), we get

$$c^i\phi(P_i) = d^j\phi(Q_j) = 0 \ .$$

Now, $c^i\phi(P_i) = c^i(\frac{-a+\sqrt{a^2-4b}}{2})(P_i) = 0$ and $b \neq 0$, so, $c^iP_i = 0$,
i.e. $c^i = 0$ for all i.
Similarly $d^j = 0$ for all j and thus by equation (14), $e = 0$.
Therefore $c^i = d^j = e = 0$, i.e. $\{P_i, Q_j, \xi\}$ is a linearly independent set.
Now, let L, M, N be projection maps on Π_p, Π_q and Π_1 respectively, then we must have

$$\begin{array}{lll} LP_i = P_i & LQ_j = 0 & L\xi = 0 \\ MP_i = 0 & MQ_j = Q_j & M\xi = 0 \\ NP_i = 0 & NQ_j = 0 & N\xi = \xi \end{array}$$

So, in view of Lemma 3.2.1, let us choose

$$\alpha L = -\phi^2 - (\tfrac{\sqrt{a^2-4b}\ +a}{2})\phi, \qquad \text{where } \alpha = 2b - (\tfrac{a^2-a\sqrt{a^2-4b}}{2}) \ ,$$

$$\beta M = -\phi^2 + (\tfrac{\sqrt{a^2-4b}\ -a}{2})\phi, \qquad \text{where } \beta = 2b - (\tfrac{a^2+a\sqrt{a^2-4b}}{2}) \ ,$$

$$bN = \phi^2 + a\phi + b = b\eta \otimes \xi \qquad \text{and } a^2 \neq 4b \ ,$$

Such that

$$\begin{aligned} \alpha LP_i &= -\phi^2P_i - (\tfrac{\sqrt{a^2-4b}\ +a}{2})\phi P_i \\ &= -(\tfrac{-a+\sqrt{a^2-4b}}{2})^2 P_i - (\tfrac{\sqrt{a^2-4b}\ +a}{2})(\tfrac{-a+\sqrt{a^2-4b}}{2})P_i \\ &= \alpha P_i \ , \end{aligned}$$

i.e., we get $LP_i = P_i$. Similarly, other results can be proved.
Thus, we prove that an almost quadratic ϕ-manifold M_n, at each of its point contains a tangent bundle Π_p of dimension p, a tangent bundle Π_q of dimension q and a real line Π_1 such that $\Pi_p \cap \Pi_q = \{\Phi\}, \Pi_p \cap \Pi_1 = \{\Phi\}, \Pi_q \cap \Pi_1 = \{\Phi\}$ (where $\{\Phi\}$ is the null set) and $\Pi_p \cup \Pi_q \cup \Pi_1 =$ a tangent bundle of dimension n, L, M, N are the projections on Π_p, Π_q and Π_1 respectively.

Conversely, suppose that, there is a tangent bundle Π_p ,Π_q and Π_1 of dimension p, q and 1(real line) respectively at each point of M_n such that $\Pi_p \cap \Pi_q = \Pi_p \cap \Pi_1 = \Pi_q \cap \Pi_1 = \{\Phi\}$, also $\Pi_p \cup \Pi_q \cup \Pi_1 =$ a tangent bundle of dimension n. Let P_i and Q_j be p and q linearly independent vectors in Π_p and Π_q respectively where $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$ and ξ be a vector in Π_1. Let, $\{P_i, Q_j, \xi\}$ span a tangent bundle of dimension n. Then $\{P_i, Q_j, \xi\}$ is a linearly independent set.

Let us define the inverse set $\left\{p^{'i}, q^{'j}, \eta\right\}$ such that

$$I_n = p^{'i} \otimes P_i + q^{'j} \otimes Q_j + \eta \otimes \xi \ . \tag{3.5}$$

We define

$$\phi = -\frac{a}{2}(p^{'i} \otimes P_i + q^{'j} \otimes Q_j) + \frac{\sqrt{a^2-4b}}{2}(p^{'i} \otimes P_i - q^{'j} \otimes Q_j) \ .$$

Therefore

$$\phi^2 = (\frac{a^2-2b}{2})(p^{'i} \otimes P_i + q^{'j} \otimes Q_j) - (\frac{a\sqrt{a^2-4b}}{2})(p^{'i} \otimes P_i - q^{'j} \otimes Q_j) \ .$$

Thus, we have

$$\phi^2 = -a\phi - b(p^{'i} \otimes P_i + q^{'j} \otimes Q_j) \ . \tag{3.6}$$

Now, by equation (17), we get

$$-b(p^{'i} \otimes P_i + q^{'j} \otimes Q_j) = b\eta \otimes \xi - bI_n \ ,$$

putting this in equation (18), we have

$$\phi^2 + a\phi + bI_n = b\eta \otimes \xi \ .$$

Thus, we see that M_n admits an *almost quadratic* ϕ *− structure*. Hence the condition is sufficient. □

Corollary 3.2. *If $a^2 < 4b$, then the dimension of almost quadratic ϕ-manifold is odd.*

Proof. The eigen values of ϕ are 0, $\frac{-a-\sqrt{a^2-4b}}{2}$ and $\frac{-a+\sqrt{a^2-4b}}{2}$. Now, if $a^2 < 4b$, then the eigen values $\frac{-a-\sqrt{a^2-4b}}{2}$ and $\frac{-a+\sqrt{a^2-4b}}{2}$ are complex conjugate to each other.
Since trace of ϕ, i.e. the sum of the eigen values of ϕ is real, the complex conjugate eigen values of ϕ occur in pairs. Therefore the tangent bundle Π_q becomes complex conjugate to Π_p, i.e. in this case $p = q$. So, by Theorem 3.2, the dimension of almost quadratic ϕ-manifold becomes $2p + 1$. □

4. Metric On almost quadratic ϕ-manifold

Let us now try to find a metric on almost quadratic ϕ-manifold. We first prove the following lemma:

Lemma 4.1. *Every almost quadratic ϕ-manifold M_n admits a Riemannian metric tensor field h such that $h(X, \xi) = \eta(X)$ for every vector field X on M_n.*

Proof. Since M_n admits a metric tensor field f (which exists provided M_n is para-compact), we obtain h by setting

$$h(X, Y) = f(a\phi X + bX - b\eta(X)\xi,\ a\phi Y + bY - b\eta(Y)\xi) + \eta(X)\eta(Y) \tag{4.1}$$

Now, putting $Y = \xi$, we get

$$h(X, \xi) = \eta(X) \ .$$

□

Theorem 4.1. *Every almost quadratic ϕ-manifold M_n admits a Riemannian metric tensor field g such that $g(X, \xi) = \eta(X)$ and $g(\phi X, \phi Y) = bg(X, Y) - b\eta(X)\eta(Y)$.*

Proof. Let us put

$$g(X,Y) = \frac{1}{2b}[bh(X,Y) + h(\phi X, \phi Y) + \frac{a}{2}(h(\phi X, Y) + h(X, \phi Y)) + b\eta(X)\eta(Y)]$$

where h is given by equation (19), then, it can be easily verified that $g(X,\xi) = \eta(X)$ and $g(\phi X, \phi Y) = bg(X,Y) - b\eta(X)\eta(Y)$. □

5. Relation of almost quadratic ϕ-manifold with Almost Contact and Almost Para-contact manifold.

Theorem 5.1. *An almost quadratic ϕ-manifold induces an almost contact manifold iff $a = 0$ and $b > 0$.*

Proof. We first prove that if an *almost quadratic $\phi-$structure* is an almost contact structure[2], then $a = 0$ and $b > 0$. We have the *almost quadratic $\phi-$structure* as

$$\phi^2 + a\phi + bI = b\eta \otimes \xi \ ,$$
$$\text{i.e., } (\phi + \tfrac{a}{2}I)^2 + (b - \tfrac{a^2}{4})I = b\eta \otimes \xi \ .$$

Now, let us choose a transformation F such that

$$\phi + \frac{a}{2}I = F \ .$$

Thus, we get

$$F^2 + (b - \frac{a^2}{4})I = b\eta \otimes \xi \ . \tag{5.1}$$

Now, we choose the 1-form η^* and the vector field ξ^* in such a manner that the equation (20) takes the form

$$F^2 + (b - \frac{a^2}{4})I = (b - \frac{a^2}{4})\eta^* \otimes \xi^* \ . \tag{5.2}$$

So, without loss of generality we may take for real transformation

$\eta^* = \sqrt{\frac{4b}{4b-a^2}} \ \eta$ and $\xi^* = \sqrt{\frac{4b}{4b-a^2}} \ \xi$, b and $(4b - a^2)$ are of same sign.

Again, equation (21) can be represented as

$$\left[\frac{1}{\sqrt{(b - \frac{a^2}{4})}}\right]^2 F^2 + I = \eta^* \otimes \xi^*$$

for $4b > a^2$. Let us now choose $\psi = \frac{1}{\sqrt{(b - \frac{a^2}{4})}} F$. Therefore, we get

$$\psi^2 + I = \eta^* \otimes \xi^* \tag{5.3}$$

Now, the structure (22) will be an almost contact structure if

$$\psi(\xi^*) = 0 \Rightarrow \frac{1}{\sqrt{(b - \frac{a^2}{4})}} F(\xi^*) = 0 \Rightarrow F(\xi^*) = 0 \ .$$

Since $a^2 \neq 4b \neq 0$, we get

$$F(\xi) = 0 \Rightarrow (\phi + \frac{a}{2}I)\xi = 0 \ .$$

Again we have $\phi(\xi) = 0$. Thus the structure (22) is an almost contact structure if $a = 0$, since ξ is not a zero vector.
Also the dimension of an almost contact manifold is odd, for which it is necessary that $a^2 < 4b$ (according to Corollary 3.2.1). Again, we have $a = 0$, therefore $b > 0$.

Conversely, if $a = 0$ and $b > 0$, then by Corollary 3.2.1, the dimension of the almost quadratic ϕ-manifold is odd and the almost quadratic ϕ-structure becomes

$$\phi^2 + bI = b\eta \otimes \xi \ . \tag{5.4}$$

Now, let $\psi = \frac{1}{\sqrt{b}}\phi$, therefore equation (23) becomes

$$\psi^2 + I = \eta \otimes \xi$$

Again, we have $\psi(\xi) = \frac{1}{\sqrt{b}}\phi(\xi) = 0$, since in an almost quadratic ϕ-manifold we have $\phi(\xi) = 0$. Therefore this structure is an almost contact structure when $a = 0$ and $b > 0$. □

Theorem 5.2. *An almost quadratic ϕ-manifold induces an almost para-contact manifold iff $a = 0$ and $b < 0$.*

Proof. We first prove that if an almost quadratic ϕ-structure is an almost para-contact structure[1], then $a = 0$ and $b < 0$. We have the almost quadratic ϕ-structure as

$$\phi^2 + a\phi + bI = b\eta \otimes \xi \ ,$$

$$\text{i.e., } (\phi + \tfrac{a}{2}I)^2 + (b - \tfrac{a^2}{4})I = b\eta \otimes \xi \ .$$

Now, let us choose a transformation F such that

$$\phi + \frac{a}{2}I = F \ .$$

Thus, we get

$$F^2 + (b - \frac{a^2}{4})I = b\eta \otimes \xi \ . \tag{5.5}$$

Now, we choose the 1-form η^* and the vector field ξ^* in such a manner that the equation (24) takes the form

$$F^2 + (b - \frac{a^2}{4})I = (b - \frac{a^2}{4})\eta^* \otimes \xi^* \ . \tag{5.6}$$

So, without loss of generality let us take

$$\eta^* = \sqrt{\tfrac{4b}{4b-a^2}} \ \eta \ \text{ and } \ \xi^* = \sqrt{\tfrac{4b}{4b-a^2}} \ \xi \ .$$

Since η^* and ξ^* are real, $4b$ and $(4b - a^2)$ are of same sign.
Again, equation (25) can be represented as

$$(\frac{4}{a^2 - 4b})F^2 = I - \eta^* \otimes \xi^*$$

for $a^2 > 4b$ and let us choose $\psi = \frac{2}{\sqrt{a^2-4b}} F$. Therefore, we get

$$\psi^2 = I - \eta^* \otimes \xi^* \tag{5.7}$$

Again, ϕ and ψ are real, so $a^2 - 4b > 0$, thus $(4b - a^2) < 0$ and consequently $b < 0$.

Now, the structure (26) will be an almost para-contact structure if

$$\psi(\xi^*) = 0 \Rightarrow \frac{2}{\sqrt{a^2 - 4b}} \ F(\xi^*) = 0 \Rightarrow F(\xi^*) = 0$$

Since $a^2 \neq 4b \neq 0$, we get

$$F(\xi) = 0 \Rightarrow (\phi + \frac{a}{2}I)\xi = 0$$

Again we have $\phi(\xi) = 0$. Thus the structure (26) is an almost para-contact structure if $a = 0$, since ξ is not a zero vector.

Conversely, if $a = 0$ and $b < 0$, the almost quadratic ϕ-structure becomes

$$\phi^2 + bI = b\eta \otimes \xi \ . \tag{5.8}$$

Now, let $\psi = \frac{1}{\sqrt{-b}}\phi$, therefore equation (27) becomes

$$\psi^2 = I - \eta \otimes \xi \ .$$

Again, we have $\psi(\xi) = \frac{1}{\sqrt{-b}}\phi(\xi) = 0$, since in an almost quadratic ϕ-manifold we have $\phi(\xi) = 0$. Therefore this structure is an almost para-contact structure when $a = 0$ and $b < 0$. □

6. Torsion Tensor Fields and Integrability Condition of almost quadratic ϕ-manifold

Let M_n be an n-dimensional differentiable almost quadratic ϕ-manifold and R be a real line. we construct a product manifold $M_n \times R$. If we denote the tangent space of $M_n \times R$ at a point $(P, Q), (P \in M_n, Q \in R)$ by T, then the tangent space $M_n(P)$ of M_n at P may be naturally identified with a subspace of T. Now, denoting the unit vector of R by τ, we define a linear map $F : T \to T$ by

$$F(X) = \frac{2}{\sqrt{a^2 - 4b}}(\phi(X) + \frac{a}{2}X), \eta(X) = 0, F(\xi) = \tau, F(\tau) = \xi \tag{6.1}$$

when $X \in M_n(P)$ and $a^2 > 4b$ and

$$F(X) = \frac{2}{\sqrt{4b - a^2}}(\phi(X) + \frac{a}{2}X), \eta(X) = 0, F(\xi) = \tau, F(\tau) = -\xi \tag{6.2}$$

when $X \in M_n(P)$ and $a^2 < 4b$.

Then we can easily see that $F^2(X) = X, F \neq I$, hold good for any vector X of T, when $a^2 > 4b$ and $F^2(X) = -X$, hold good for any vector X of T, when $a^2 < 4b$. So, F gives an almost product structure or almost complex structure on T, when $a^2 > 4b$ or $a^2 < 4b$ respectively. As $P \in M_n$ and $Q \in R$ are arbitrary, we see that an almost product structure or almost complex structure can be defined over $M_n \times R$ by means of the *almost quadratic* $\phi - structure$, depending on a and b. Let $U \times R$ be a coordinate neighbourhood and set (x^i, x^∞) its local coordinates in $U \times R.(i, j, k, h$ run over 1,2,$\cdots n$ and ∞ is just a symbol which means $n + 1)$. Then we can easily verify that the almost product structure F has

$$F_i^h = \lambda_p \ \phi_i^h + \mu_p \ \delta_i^h, \ \ F_\infty^h = \xi^h, \ \ F_i^\infty = \eta_i, \ \ F_\infty^\infty = 0 \tag{6.3}$$

(where λ_p and μ_p are constants, given by $\lambda_p = \frac{2}{\sqrt{a^2-4b}}$ and $\mu_p = \frac{a}{\sqrt{a^2-4b}}$) as its components with respect to the coordinate neighbourhood. If the indices A, B, C run over $1, 2 \cdots, n, \infty$ then surely $F^A_B F^B_C = \delta^A_C$ holds good.
Again, the almost complex structure F has

$$F^h_i = \lambda_c\, \phi^h_i + \mu_c\, \delta^h_i, \quad F^h_\infty = -\xi^h, \quad F^\infty_i = \eta_i, \quad F^\infty_\infty = 0\ , \tag{6.4}$$

(λ_c and μ_c are constants, given by $\lambda_c = \frac{2}{\sqrt{4b-a^2}}$ and $\mu_c = \frac{a}{\sqrt{4b-a^2}}$) as its components with respect to the coordinate neighbourhood. Also $F^A_B F^B_C = -\delta^A_C$ holds good.

Now, the Nijenhuis tensor N of the almost product structure or almost complex structure F on $M_n \times R$ is given by

$$N^A_{CB}(F) = F^E_C \partial_E F^A_B - F^E_B \partial_E F^A_C - (\partial_C F^E_B - \partial_B F^E_C) F^A_E$$

So, if we calculate the components of this tensor by grouping their indices in two groups (1,2,$\cdots n$) and ∞, on $M_n \times R$, we get for $a^2 > 4b$

$$\begin{aligned} N^h_{ji} &= \lambda_p^2\ \ \Phi^h_{ji} - (\partial_j \eta_i - \partial_i \eta_j)\xi^h\ , \\ N^\infty_{ji} = N_{ji} &= \lambda_p[\phi^k_j(\partial_k \eta_i - \partial_i \eta_k) - \phi^k_i(\partial_k \eta_j - \partial_j \eta_k)] + \mu_p(\partial_j \eta_i - \partial_i \eta_j)\ , \\ N^h_{\infty i} = N^h_i &= \lambda_p \pounds_\xi \phi^h_i\ , \quad N^\infty_{\infty i} = N_i = \pounds_\xi \eta_i\ , \end{aligned} \tag{6.5}$$

and when $a^2 < 4b$, we similarly have

$$\begin{aligned} N^h_{ji} &= \lambda_c^2\ \ \Phi^h_{ji} - (\partial_i \eta_j - \partial_j \eta_i)\xi^h, \\ N^\infty_{ji} = N_{ji} &= \lambda_c[\phi^k_j(\partial_k \eta_i - \partial_i \eta_k) - \phi^k_i(\partial_k \eta_j - \partial_j \eta_k)] + \mu_c(\partial_j \eta_i - \partial_i \eta_j)\ , \\ N^h_{\infty i} = N^h_i &= -\lambda_c \pounds_\xi \phi^h_i\ , \quad N^\infty_{\infty i} = N_i = -\pounds_\xi \eta_i\ , \end{aligned} \tag{6.6}$$

Where Φ^h_{ji} is the Nijenhuis tensor of ϕ^h_i and $\pounds_\xi$ means the Lie derivative with respect to the vector field ξ^h. In view of equation (32) and (33), we can immediately say that N^h_{ji}, N^h_i, N_{ji} and N_i are components of four tensor fields over M respectively. So, from this definition we get immediately the following:

Theorem 6.1. *The tensor fields N^h_i, N_i vanish if and only if ϕ^h_i, η_i are invariant under the local group of local transformation generated by ξ^h respectively.*

Lemma 6.1. *The Nijenhuis tensor N^A_{CB} of the almost product and the almost complex structure satisfies the following equations*

$$N^A_{CE} F^E_B + N^E_{CB} F^A_E = 0 \ \ \textit{and}\ \ N^A_{EB} F^E_C - N^A_{CE} F^E_B = 0. \tag{6.7}$$

Proof. It is easily seen by straight forward calculation. □

Now, we will discuss the situation in two cases, when $a^2 > 4b$ and when $a^2 < 4b$.

Case-1. ($a^2 > 4b$): If we calculate the components of equation (34), by grouping their indices in two groups (1,2,$\cdots n$) and ∞, we get the following relations:

$$\begin{aligned} &\lambda_p(N^k_{ji}\phi^h_k + N^h_{jk}\phi^k_i) + N_{ji}\xi^h - N^h_j \eta_i + 2\mu_p\ N^h_{ji} = 0\ , \\ &\lambda_p(N^h_k \phi^k_i + N^k_i \phi^h_k) + N_i \xi^h + 2\mu_p\ N^h_i = 0\ , \\ &N^h_{ik}\xi^k - \lambda_p N^k_i \phi^h_k - \mu_p\ N^h_i - N_i \xi^h = 0\ , \quad N^h_k \xi^k = 0\ , \\ &\lambda_p N_{jk}\phi^k_i + \mu_p N_{ji} - N_j \eta_i + N^k_{ji}\eta_k = 0\ , \quad N_k \xi^k = 0\ , \\ &\lambda_p N_k \phi^k_i + \mu_p N_i + N^k_i \eta_k = 0, \quad N_{ik}\xi^k - N^k_i \eta_k = 0\ , \end{aligned} \tag{6.8}$$

and with these we also have

$$\lambda_p(N^h_{ki}\phi^k_j - N^h_{jk}\phi^k_i) + N^h_i\eta_j + N^h_j\eta_i = 0 \ ,$$

$$N^h_{ki}\xi^k - \lambda_p\ N^h_k\phi^k_i - \mu_p N^h_i = 0, \quad N_{ki}\xi^k - \lambda_p\ N_k\phi^k_i - \mu_p N_i = 0 \ , \tag{6.9}$$

$$\lambda_p(N_{ki}\phi^k_j - N_{jk}\phi^k_i) + N_i\eta_j + N_j\eta_i = 0 \ .$$

From (35) and (36), we get the following

$$i)\ N^h_i = -\lambda_p N^h_{jk}\phi^j_i\xi^k$$

$$ii)\ N_i = -\lambda_p N^h_k\phi^k_i\eta_h - 2\mu_p N^h_i\eta_h \tag{6.10}$$

$$iii)\ N_i = -\lambda_p N_{kh}\phi^k_i\xi^h - 2\mu_p N_{ih}\xi^h$$

$$iv)\ N_{ji} = -\lambda_p(N^h_{jk}\phi^k_i\eta_h + N^h_{ek}\phi^e_j\xi^k\eta_h\eta_i) - 2\mu_p N^h_{ji}\eta_h \ .$$

Thus, we have the following:

Theorem 6.2. *In an almost quadratic ϕ-manifold, when $a^2 > 4b$, if any one of N_{ji} and N^j_i vanishes, then N_i vanishes. If N^h_{ji} vanishes, then all the other tensors N_{ji}, N^j_i and N_i vanishes.*

We, now define the tensors P^A_B and Q^A_B over $M_n \times R$ by

$$P^A_B = \frac{1}{2}(\delta^A_B + F^A_B), \quad Q^A_B = \frac{1}{2}(\delta^A_B - F^A_B) \tag{6.11}$$

Then we have

$$P^A_C P^C_B = P^A_B, \quad P^A_C Q^C_B = 0, \quad Q^A_C P^C_B = 0 \quad Q^A_C Q^C_B = Q^A_B, \quad P^A_B + Q^A_B = \delta^A_B \ .$$

Thus P^A_B and Q^A_B defines two complementary distributions P and Q globally. Now, in order that the distributions P and Q be completely integrable it is necessary and sufficient that $N^A_{CB} = 0$,[3]. Thus, by virtue of Theorem 6.2, we get the following:

Theorem 6.3. *Let M_n be an almost quadratic ϕ-manifold. Then the almost product structure F over $M_n \times R$ defined by (28), when $a^2 > 4b$, is completely integrable if and only if $N^h_{ji} = 0$ holds good over whole M_n.*

Case-2.($a^2 < 4b$): If we calculate the components of equation (34), by grouping their indices in two groups (1,2,$\cdots n$) and ∞, we get the following relations:

$$\lambda_c(N^k_{ji}\phi^h_k + N^h_{jk}\phi^k_i) - N_{ji}\xi^h - N^h_j\eta_i + 2\mu_c\ N^h_{ji} = 0 \ ,$$

$$\lambda_c(N^h_k\phi^k_i + N^k_i\phi^h_k) - N_i\xi^h + 2\mu_c\ N^h_i = 0 \ ,$$

$$N^h_{ik}\xi^k + \lambda_c N^k_i\phi^h_k + \mu_c\ N^h_i - N_i\xi^h = 0 \ , \quad N^h_k\xi^k = 0 \ , \tag{6.12}$$

$$\lambda_c N_{jk}\phi^k_i + \mu_c N_{ji} - N_j\eta_i + N^k_{ji}\eta_k = 0 \ , \quad N_k\xi^k = 0 \ ,$$

$$\lambda_c N_k\phi^k_i + \mu_c N_i + N^k_i\eta_k = 0, \quad N_{ik}\xi^k + N^k_i\eta_k = 0 \ ,$$

and with these we also have

$$\lambda_c(N^h_{ki}\phi^k_j - N^h_{jk}\phi^k_i) + N^h_i\eta_j + N^h_j\eta_i = 0 \ ,$$

$$N^h_{ki}\xi^k + \lambda_c\ N^h_k\phi^k_i + \mu_c N^h_i = 0, \quad N_{ki}\xi^k + \lambda_c\ N_k\phi^k_i + \mu_c N_i = 0 \ , \tag{6.13}$$

$$\lambda_c(N_{ki}\phi^k_j - N_{jk}\phi^k_i) + N_i\eta_j + N_j\eta_i = 0 \ .$$

From (39) and (40), we get the following

$$\begin{aligned} &i)\ N_i^h = -\lambda_c N_{jk}^h \phi_i^j \xi^k \\ &ii)\ N_i = \lambda_c N_k^h \phi_i^k \eta_h + 2\mu_c N_i^h \eta_h \\ &iii)\ N_i = -\lambda_c N_{kh} \phi_i^k \xi^h - 2\mu_c N_{ih} \xi^h \\ &iv)\ N_{ji} = \lambda_c (N_{jk}^h \phi_i^k \eta_h + N_{ek}^h \phi_j^e \xi^k \eta_h \eta_i) + 2\mu_c N_{ji}^h \eta_h\ . \end{aligned} \tag{6.14}$$

Thus, we have the following:

Theorem 6.4. *In an almost quadratic ϕ-manifold, when $a^2 < 4b$, if any one of N_{ji} and N_i^j vanishes, then N_i vanishes. If N_{ji}^h vanishes, then all the other tensors N_{ji}, N_i^j and N_i vanishes.*

Now, the necessary and sufficient condition for the integrability of the almost complex structure over $M_n \times R$ is $N_{CB}^A = 0$. Thus in view of Theorem 6.4 we have the following theorem:

Theorem 6.5. *Let M_n be an almost quadratic ϕ-manifold. Then the almost complex structure F over $M_n \times R$ defined by (29), when $a^2 < 4b$, is completely integrable if and only if $N_{ji}^h = 0$ holds good over whole M_n.*

Thus, by virtue of Theorem 6.3 and Theorem 6.5 we have

Theorem 6.6. *Let M_n be an almost quadratic ϕ-manifold. Then the almost product or the almost complex structure F over $M_n \times R$ is completely integrable if and only if $N_{ji}^h = 0$ holds good over whole M_n.*

Again, in view of Theorem 6.2 and Theorem 6.4, we have the following theorem

Theorem 6.7. *In an almost quadratic ϕ-manifold if any one of N_{ji} and N_i^j vanishes, then N_i vanishes. If N_{ji}^h vanishes, then all the other tensors N_{ji}, N_i^j and N_i vanishes.*

We shall call N_{ji}^h the *Torsion Tensor* of the almost quadratic ϕ-structure.

Remark 6.1. The Torsion tensor of the almost quadratic ϕ-structure of the two cases ($a^2 > 4b$ and $a^2 < 4b$) are just of opposite sign, as $\lambda_c^2 = -\lambda_p^2$.

7. Example of almost quadratic ϕ-structure in 4-dimensional Euclidean Space

Let R_4 be any 4-dimensional Euclidean space and let us define

$$\phi = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 9 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.1}$$

So, $\phi^2 = \begin{pmatrix} 13 & 4 & 0 & 0 \\ 36 & 13 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and therefore $\phi^2 - 4\phi = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Now, let us choose $\xi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$, thus $\eta \otimes \xi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Therefore

$$\phi^2 - 4\phi - 5I_4 = -5\eta \otimes \xi \ . \tag{7.2}$$

Again, $\phi(\xi) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 9 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Thus, we conclude that the structure defined by equation (42) is an almost quadratic ϕ-structure and R_4 is an almost quadratic ϕ-manifold.

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