On Positive Solution for a Non-Local Fractional Boundary Value Problem

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Abstract — *We discuss the existence and uniqueness of positive solution for a fractional boundary value problem by using some fixed point theorems and the upper and lower solutions method. An example is also given to illustrate the obtained results.*

Keywords: Fractional boundary value problem, Positivity of solution, Fixed point theorem, Upper and lower solutions.
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1 Introduction

Recently, fractional differential equations have revealed to be great worth in the modeling of many phenomena in several fields of sciences, economics and engineering. For this purpose, we find many applications in electrochemistry, viscoelasticity, control theory, electrical networks, signal process. see [10-13]. Significant developments in fractional differential equations can be found in the monographs [11,12,13,15]. Different methods are introduced in the investigation of fractional differential equations, such as the theory of fixed points, see [1-11,15-17].

In [7], the authors proved the existence of at least one or three positive solutions of the following problem, by applying the Guo-Krasnosel'skii and Avery-Peterson fixed-point theorems and under growing conditions on the nonlinear term f:

$$\begin{cases} D_{0+}^{q}u(t) = a\left(t\right) f\left(u(t)\right), \quad 0 \leq t \leq 1, 2 < q \leq 3 \\ u\left(0\right) = u'(0) = 0, \quad u''\left(0\right) = \alpha u\left(1\right), \end{cases}$$

here D_{0+}^{q} denotes the fractional derivative of Riemann-Liouville type, f is a given real function and the function a is continuous on $[0,1]$.

In [14], Matar studied the positivity of solution for the following boundary value problem:

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$$\begin{aligned} D_{0^+}^q u(t) &= f(t,u(t)), \quad 0<t<1, 1<q\le 2 \\ u(0) &= 0, \quad u'(0)=\theta>0, \end{aligned}$$

here the function f is continuous on $[0,1]\times\mathbb{R}$. By introducing the so-called upper and lower control functions and applying a fixed-point theorem on a cone, the author was able to establish the existence and uniqueness of the positive solution.

The purpose of this work is to establish sufficient conditions for the existence and uniqueness of the positive solution of the following fractional boundary value problem (P)

$$\begin{cases} {}^cD_{0^+}^q u(t) = f(t,u(t)), \quad 0\le t\le T, \ 2<q<3 \\ \qquad u(0)=u'(0)=0, \quad u''(0)=\alpha>0, \end{cases}$$

where the function f is continuous and nonnegative on $[0,T]\times\mathbb{R}$. We denote by ${}^cD_{0^+}^q$ the fractional derivative of Caputo type.

This work is organized as follows. We expose the tools that will be used later in the next section. The third section is devoted to the study of the existence of at least one positive solution of the problem (P) by the help of Schauder's theorem fixed on the cone, then we prove the uniqueness of positive solutions of the problem (P) by using Banach's contraction principle. We end this section with an example that elucidates the results obtained.

2 Preliminaries

In this section, we present some definitions and lemmas from fractional calculus theory, which will be needed later.

Definition 2.1. *For a continuous function g on $[a,b]$, we define the Riemann-Liouville fractional integral of order α by*

$$I_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} g(s)ds, \ \alpha>0$$

Definition 2.2. *The Caputo fractional derivative of order α of a function f is defined by*

$${}^cD_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{g^n(s)}{(t-s)^{\alpha-n+1}}ds$$

where $n=[\alpha]+1$, ($[\alpha]$ is the entire part of α).

Lemma 2.3. *The solution of the homogenous differential equation ${}^cD_{a^+}^{\alpha} g(t)=0$ is given by $g(t)=c_1+c_2t+c_3t^2+...+c_nt^{n-1}$, with $c_i\in\mathbb{R}$, $i=0,...,n$, if $g\in C([0,1])$.*

Lemma 2.4. *We have $I_{0^+}^p I_{0^+}^q f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^q I_{0^+}^p f(t)$ and ${}^cD_{a^+}^q I_{0^+}^q f(t) = f(t)$, for all $t\in[a,b]$, $p,q\ge 0$ and $f\in L_1[a,b]$.*

Now, we transform the problem (P) to an equivalent integral equation.

Lemma 2.5. *u is a solution of the problem (P) if and only if u is a solution of the integral equation*

$$u(t) = \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, u(s))\, ds.$$

Proof. The proof is standard, then we omit it. ☐

Define $E = C[0,T]$ equipped with the norm $\|u\| = \max_{t\in[0,T]} |u(t)|$. Define the subspace K of E as the set of nonnegative functions. Let a and b be two nonnegative real such that $b > a$. Define the upper control function and the lower control function of a function $u \in [a,b]$, respectively by

$$U(t,u) = \sup_{\lambda\in[a,u]} f(t,\lambda),\ L(t,u) = \inf_{\lambda\in[u,b]} f(t,\lambda)$$

Obviously, $U(t,u)$ and $L(t,u)$ are nondecreasing according to u, monotonous and satisfy $L(t,u) \le f(t,u) \le U(t,u)$.

We make the following hypotheses:

(H_1) There exist u_*, u^* two elements in K, verifying $a \le u_*(t) \le u^*(t) \le b$ and

$$\begin{cases} u^*(t) \ge \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} U(s, u^*(s))\, ds + \frac{\alpha}{2}t^2 \\ u_*(t) \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} L(s, u_*(s))\, ds + \frac{\alpha}{2}t^2. \end{cases}$$

(H_2) For any x, y belonging to E and $t \in [0,T]$, we can find a number $0 < \eta < 1$ such that

$$|f(t,y) - f(t,x)| \le \eta \|y - x\|.$$

The function u_* is called lower solution for problem (P) and u^* is called upper solutions.

Define the integral operator A on E as

$$Au(t) = \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, u(s))\, ds. \tag{1}$$

Definition 2.6. *We say that u is a positive solution of problem (P) if $u(t) > 0$, for all $t \in [0,T]$ and the boundary conditions in (P) are satisfied.*

Theorem 2.7. *Under the hypothesis (H_1) the fractional boundary value problem (P) has at least one positive solution u belonging to E and satisfying $u_*(t) \le u(t) \le u^*(t)$.*

Proof. Let

$$C = \{u \in K, u_*(t) \le u(t) \le u^*(t), 0 \le t \le T\},$$

remark that if $u \in C$, then $\|u\| \le b$. Hence, C is bounded, convex and closed subset of E.

Claim 1. A is uniformly bounded on C.

The operator A is continuous on C since f is continuous. Set

$$M = \max\{f(t, u(t)), t \in [0,T], \|u\| \le b\}.$$

Let $u \in C$, then $\|u\| \leq b$ and we have

$$|Au(t)| \leq \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, u(s))\, ds,$$
$$\leq \frac{\alpha}{2}T^2 + \frac{MT^q}{\Gamma(q+1)}.$$

Thus

$$\|Au\| \leq \frac{\alpha}{2}T^2 + \frac{MT^q}{\Gamma(q+1)}.$$

Hence A is uniformly bounded.

Claim 2. Au is equicontinuous. In fact, for $0 \leq t_1 < t_2 \leq T$, it yields

$$\begin{aligned}
|Au(t_2) - Au(t_1)| &\leq \frac{\alpha}{2}\left(t_2^2 - t_1^2\right) + \\
&\left|\frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-s)^{q-1} f(s, u(s)ds - \frac{1}{\Gamma(q)}\int_0^{t_2} (t_2-s)^{q-1} f(s, u(s)ds\right| \\
&\leq \alpha T(t_2 - t_1) + \frac{1}{\Gamma(q)}\int_0^{t_1} \left((t_2-s)^{q-1} - (t_1-s)^{q-1}\right) f(s, u(s)ds \\
&+ \frac{1}{\Gamma(q)}\int_{t_1}^{t_2} (t_2-s)^{q-1} f(s, u(s)ds \\
&\leq \alpha T(t_2 - t_1) + \frac{MT(t_2 - t_1)}{\Gamma(q-1)} + \frac{(t_2 - t_1)^q}{\Gamma(q+1)} \to 0, as\ t_1 \to t_2.
\end{aligned}$$

Thanks to Arzela-Ascoli Theorem we deduce the compacity of A.

Let $u \in C$, then by the definition of the control functions and the hypothesis (H1), it yields

$$\begin{aligned}
Au(t) &= \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, u(s))\, ds \\
&\leq \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} U(t, u^*(t))\, ds \\
&\leq u^*(t),
\end{aligned}$$

and

$$\begin{aligned}
Au(t) &= \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, u(s))\, ds \\
&\geq \frac{\alpha}{2}t^2 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} L(s, u_*(s))\, ds \\
&\geq u_*(t).
\end{aligned}$$

Hence, $u_*(t) \leq Au(t) \leq u^*(t)$, $0 \in t \leq T$, from which we deduce $A(C) \subseteq C$. Finally, we conclude by Schauder fixed point theorem, that A has at least one fixed point and consequently, the problem (P) has at least one positive solution u in E between the lower and upper solutions. □

The uniqueness of the positive solution of (P) is given in the following theorem.

Theorem 2.8. *The problem (P) has a unique positive solution $u \in E$, if the hypotheses (H_1) and (H_2) and the inequality*

$$\frac{\eta T^q}{\Gamma(q+1)} < 1, \tag{2}$$

are satisfied.

Proof. Since the hypothesis (H_1) is satisfied then, we conclude by Theorem 3.2 that the problem (P) has at least one positive solution in E. We claim that the operator A is a contraction on E. In fact, for any $u, v \in E$, we have

$$\begin{aligned} |Au(t) - Av(t)| &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left| f(s, u(s)) - f(s, v(s)) \right| ds \\ &\le \frac{\eta T^q}{\Gamma(q+1)} \| u - v \|, \end{aligned}$$

finally, taking (2) into account, then A is a contraction and thus the problem (P) has a unique positive solution $u \in C$. $\square$

Example 2.9. *Let us choose in the problem (P), $q = \frac{8}{3}$, $T = 1$, $f(t,u) = 1 + \frac{t}{2(u+1)}$, $0 \le t \le 1$, $u \ge 0$, $[a,b] = [0,1]$ and $\alpha = 1$. Since f is decreasing according to u, then*

$$U(t,u) = 1 + \frac{t}{2},\ L(t,u) = 1 + \frac{t}{4},$$

If we set

$$\begin{aligned} u^*(t) &= \frac{t^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \frac{t^{\frac{11}{3}}}{\Gamma\left(\frac{14}{3}\right)} + \frac{1}{2}t^2 \\ &\ge \frac{1}{\Gamma\left(\frac{8}{3}\right)} \int_0^t (t-s)^{\frac{5}{3}} U(s, u^*(s))\, ds + \frac{1}{2}t^2 \\ &= \frac{t^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \frac{t^{\frac{11}{3}}}{2\Gamma\left(\frac{14}{3}\right)} + \frac{1}{2}t^2 \end{aligned}$$

and

$$\begin{aligned} u_*(t) &= \frac{t^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \frac{t^{\frac{11}{3}}}{8\Gamma\left(\frac{14}{3}\right)} + \frac{1}{2}t^2 \\ &\le \frac{1}{\Gamma\left(\frac{8}{3}\right)} \int_0^t (t-s)^{\frac{5}{3}} L(s, u^*(s))\, ds + \frac{1}{2}t^2 \\ &= \frac{t^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \frac{t^{\frac{11}{3}}}{4\Gamma\left(\frac{14}{3}\right)} + \frac{1}{2}t^2. \\ 0 &\le u_*(t) \le u^*(t) \le 1. \end{aligned}$$

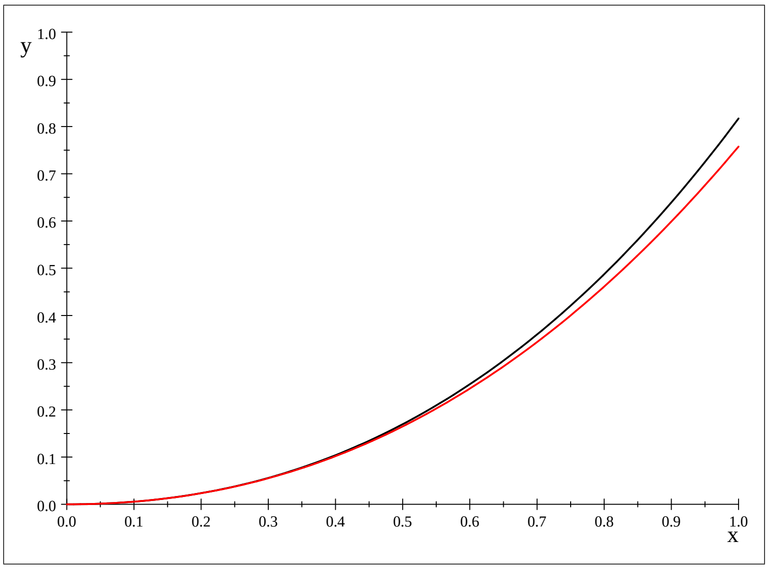

Figure 1: u_* in red, u^* in black.

hence assumption (H_1) *holds, then the problem* (P) *has at least one positive solution. Moreover, there exists* $\eta = \frac{1}{4}$, *such that hypothesis* (H_2) *is satisfied and*

$$\frac{\eta T^q}{\Gamma(q+1)} = \frac{1}{4\Gamma\left(\frac{11}{3}\right)} = 6.2310 \times 10^{-2} < 1.$$

We conclude by Theorem 3.3, the uniqueness of positive solution u *satisfying* $u_*(t) \leq u(t) \leq u^*(t)$, $0 \leq t \leq 1$.

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