Exact Travelling Wave Solutions of the Nonlinear Gardner-Kawahara Equation by the Standard $\left(\frac{G'}{G}\right)$ – Expansion Method

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Abstract — *The basic idea of this article is to study the solution of the Gardner-Kawahara equation which is modelled to investigate the waves in magnetized plasma. To follow standard* $\left(\frac{G'}{G}\right)$*-expansion method more common forms of solutions are obtained, if the parameters were taken at special values, periodic, solitary, and rational results will obtained.*

Keywords: $\left(\frac{G'}{G}\right)$–expansion method, Gardner-Kawahara equation, Solitary waves.
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1 Introduction

Here in this research, we will gain the solitary wave solution of nonlinear Gardner – Kawahara equation (1.1) in the shape [1,2]

$$\frac{\partial\psi}{\partial t}+a\frac{\partial\psi}{\partial x}+\lambda\psi\frac{\partial\psi}{\partial x}-\alpha\psi^2\frac{\partial\psi}{\partial x}+\mu\frac{\partial^3\psi}{\partial x^3}+\beta\frac{\partial^5\psi}{\partial x^5}=0 \tag{1.1}$$

It is one more particular case of equation extended KdV equation

$$\frac{\partial\psi}{\partial t}+a\frac{\partial\psi}{\partial x}+\lambda\psi\frac{\partial\psi}{\partial x}+\mu\frac{\partial^3\psi}{\partial x^3}+\beta\frac{\partial^5\psi}{\partial x^5}-\alpha\psi^2\frac{\partial\psi}{\partial x}+\gamma_1\psi\frac{\partial^3\psi}{\partial x^3}+\gamma_2\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}=0$$

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when $\gamma_1 = \gamma_2 = 0$. The extended KdV equation leads to the Kawahara equation, when $\alpha = \gamma_1 = \gamma_2 = 0$.

Eq. (1.1) happens in the notion in plasmas and in notion of shallow water waves with surface tension and notion of magneto-acoustic waves. Eq. (1.1) describing solitary-wave propagation in media, was first proposed by Kawahara in 1972 [3]. Lately, Wang et al. [4] introduced that the traveling wave results can be explained by a polynomial in $\left(\frac{G'}{G}\right)$, where G=G(η) satisfies the following second-order ordinary linear differential equation $G''(\eta) + \gamma G'(\eta) + \delta G(\eta) = 0$, where $\eta = x - kt$, and $\gamma, \delta,$ and k are constants. Actually, $\left(\frac{G'}{G}\right)$–standard method has been successfully stratified to acquire exact solution for an assortment of nonlinear evolution equations, see [5,6,7,8,9,10,11,12,13,14,15]. This report is systematized as follows: In part 2, we offer the synopsis of the $\left(\frac{G'}{G}\right)$ –expansion technique. In part 3, we explain the applications of the $\left(\frac{G'}{G}\right)$–standard method. Finally, the conclusions are present in part 4.

2 The Synopsis of the $\left(\frac{G'}{G}\right)$ –Standard Method

In this part, we explain the $\left(\frac{G'}{G}\right)$–standard method to explore traveling wave results of nonlinear equations, let us consider a nonlinear evolution equation in two variables xt in the form:

$$\mathcal{L}\,(\psi, \psi_t, \psi_x, \psi_{xt}, \psi_{tt}, \psi_{xx}, \dots \dots \dots) = 0, \tag{2.1}$$

where $\psi = \psi(x,t)$ is an unknown function and $\mathcal{L}$ is a polynomial in $\psi = \psi(x,t)$, in which highest order derivatives and nonlinear terms are involved. The major procedures of this method are presented in this research as follows:

Step 1. Collecting the separate variables x and t into one variable $\eta = x - kt$, we assume that

$$\psi(x,t) = \psi(\eta), \ \eta = x - kt \tag{2.2}$$

When k is constant. Replacing (2.2) into (2.1), then we will gain the following differential ordinary equation (ODE):

$$\mathcal{O}\,(\psi, k\psi', \psi', k\psi'', k^2\psi'', \psi'', \dots \dots \dots) = 0 \tag{2.3}$$

Step 2. In case of need, we integrate (2.3) as many times as possible and assume the solution of (2.3) which can be expressed of the form

$$\psi(\eta) = \sum_{i=0}^{N} b_i \left(\frac{G'}{G}\right)^i, \tag{2.4}$$

where $G = G(\eta)$ satisfies the second order linear equation (ODE)

$$G''(\eta) + \gamma G'(\eta) + \delta G(\eta) = 0, \tag{2.5}$$

Where b_i, γ and δ are real constant with $b_N \neq 0$. Next, the prime denotes the derivative respective to η .Using the general solutions of (2.5), we get

$$\left(\frac{G'}{G}\right) = \begin{cases} -\frac{\gamma}{2} + \frac{\sqrt{\gamma^2 - 4\delta}}{2}\left(\frac{r_1 \sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\} + r_2 \cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}{r_1 \cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\} + r_2 \sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}\right), & \text{when } \gamma^2 - 4\delta > 0 \\ -\frac{\gamma}{2} + \frac{\sqrt{4\delta - \gamma^2}}{2}\left(\frac{-r_1 \sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\} + r_2 \cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}{r_1 \cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\} + r_2 \sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}\right), & \text{when } \gamma^2 - 4\delta < 0 \\ \left(\frac{r_2}{r_1 + r_2\eta}\right) - \frac{\gamma}{2}, & \text{when } \gamma^2 - 4\delta = 0 \end{cases}$$

Step 3. We determine the positive integer N by considering the homogeneous balance between nonlinear terms and the highest order derivatives showing in ODE (2.3) and replacing (2.4) into (2.3), then we use the general solutions of (2.5), and summation all terms with the similar order of $\left(\frac{G'}{G}\right)$ together, next setting each coefficient of this polynomial to zero yields a group of algebraic equations for $b_i, k, y,$ and δ.

Step 4. We solve the nonlinear algebraic equations of step3 by maple to find the constants $b_i, k, \gamma,$ and δ. Substituting these values into (2.4) and using the general solutions of (2.5).

3 Applications of the Method

In this part, the $\left(\frac{G'}{G}\right)$–standard method has been used it to check the results leading to solitary wave solutions to the Gardner –Kawahara equation. In order to explore the solitary wave result of (1.1), we are applying the transformations

$$\psi(x,t) = \psi(\eta) \qquad , \qquad \eta = x - kt$$

Then, (1.1) for $\psi(x,t) = \psi(\eta)$ become

$$-k\psi' + a\psi' + \lambda\psi\psi' - \alpha\psi^2\psi' + \mu\psi''' - \beta\psi''''' = 0 \tag{3.1}$$

Integrating (3.1) with respect to η once and putting the integration constant equal to zero, we obtain

$$-k\psi + a\psi + \frac{\lambda}{2}\psi^2 - \frac{\alpha}{3}\psi^3 + \mu\psi'' - \beta\psi'''' = 0 \tag{3.2}$$

By balancing between $\psi^3$and ψ'''' we get N=2, then Eq. (3.2) has the following solution:

$$\psi(\eta) = b_0 + b_1\left(\frac{G'}{G}\right) + b_2\left(\frac{G'}{G}\right)^2, \qquad b_2 \neq 0 \tag{3.3}$$

where $b_0, b_1, and\ b_2$ are unknown constants. Substituting (3.3) along with (2.5) into (3.2) and summation each terms with the similar power of $\left(\frac{G'}{G}\right)$, the left side of (3.2) is transmuted into a polynomial in $\left(\frac{G'}{G}\right)$. Putting the coefficients of all powers of $\left(\frac{G'}{G}\right)$ to zero yields a group of nonlinear equations (3.4) for $b_0, b_1, b_2, a, \lambda, \alpha, \mu, \beta$ & γ as follows:

$$\left(\frac{G'}{G}\right)^6 : \frac{-1}{3}\alpha {b_2}^3 + 120\beta b_2 = 0$$

$$\left(\frac{G'}{G}\right)^5 : -\alpha b_1 {b_2}^2 + \beta(336 b_2\gamma + 24 b_1) = 0$$

$$\left(\frac{G'}{G}\right)^4 : \frac{-1}{3}\alpha\left(b_0{b_2}^2 + 2{b_1}^2 b_2 + b_2\left(2b_0b_2 + {b_1}^2\right)\right) + \frac{1}{2}\lambda {b_2}^2 + 6\mu b_2 + \beta(330 b_2\gamma^2 + 60 b_1\gamma + 240 b_2\delta) = 0$$

$$\left(\frac{G'}{G}\right)^3 : \lambda b_1 b_2 - \frac{1}{3}\alpha\left(4b_0b_1b_2 + b_1\left(2b_0b_2 + {b_1}^2\right)\right) + \mu(10b_2\gamma + 2b_1) + \beta(130b_2\gamma^3 + 50b_1\gamma^2 + 440b_2\gamma\delta + 40b_1\delta) = 0$$

$$\left(\frac{G'}{G}\right)^2 : \frac{-1}{3}\alpha\left(b_0\left(2b_0b_2 + {b_1}^2\right) + 2{b_1}^2b_2 + b_2{b_0}^2\right) - kb_2 + \beta(16b_2\gamma^4 + 15b_1\gamma^3 + b_2\gamma^2\delta + 60b_1\gamma\delta + 136b_2\delta^2) + ab_2 + \frac{1}{2}\lambda\left(2b_0b_2 + {b_1}^2\right) + \mu(4b_2\gamma^2 + 3b_1\gamma + 8b_2\delta) = 0$$

$$\left(\frac{G'}{G}\right)^1 : ab_1 - kb_1 - \alpha{b_0}^2b_1 + \lambda b_0 b_1 + \beta(b_1\gamma^4 + 30b_2\gamma^3\delta + 22b_1\gamma^2\delta + 120b_2\gamma\delta^2 + 16b_1\delta^2) + \mu(b_1\gamma^2 + 6b_2\gamma\delta + 2b_1\delta) = 0$$

$\left(\frac{G'}{G}\right)^0 : -b_0k + \mu(b_1\gamma\delta + 2b_2\delta^2) + \frac{1}{2}{b_0}^2\lambda + b_0a + \beta(b_1\gamma^3\delta + 14b_2\gamma^2\delta^2 + 8b_1\gamma\delta^2 +$ *16b2δ3+232−13b03α=0.*

Solving the above system of algebraic equations by Maple, we get the following result:

$$b_1 = \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma \quad , \quad b_2 = \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}} \quad , \qquad \delta = 0 \ , \ \ k = k$$

$$\lambda = \frac{2}{3}\frac{3{b_0}^2\alpha\gamma^2\sqrt{10}\sqrt{\frac{\beta}{\alpha}} - 12\beta\gamma^4 b_0 - 2{b_0}^3\alpha - 696\beta}{b_0\left(\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2 - b_0\right)} \ ,$$

$$\lambda = \frac{2}{3}\frac{-3{b_0}^2\alpha\gamma^2\sqrt{10}\sqrt{\frac{\beta}{\alpha}} - 12\beta\gamma^4 b_0 - 2{b_0}^3\alpha - 696\beta}{b_0\left(-\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2 - b_0\right)}$$

$$\mu = \frac{-1}{3}\ \frac{3\sqrt{10}\sqrt{\frac{\beta}{\alpha}}b_0\beta\gamma^4 + \sqrt{10}\sqrt{\frac{\beta}{\alpha}}{b_0}^3\alpha - 15{b_0}^2\beta\gamma^2 - 696\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\beta}{b_0\left(\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2 - b_0\right)}$$

$$\mu = \frac{-1}{3}\ \frac{-3\sqrt{10}\sqrt{\frac{\beta}{\alpha}}b_0\beta\gamma^4 - \sqrt{10}\sqrt{\frac{\beta}{\alpha}}{b_0}^3\alpha - 15{b_0}^2\beta\gamma^2 + 696\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\beta}{b_0\left(-\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2 - b_0\right)}$$

$$\mathrm{a} = \frac{-1}{3}\ \frac{2\sqrt{10}\sqrt{\frac{\beta}{\alpha}}{b_0}^3\alpha\gamma^2 - 12{b_0}^2\beta\gamma^4 - 3\sqrt{10}\sqrt{\frac{\beta}{\alpha}}b_0\gamma^2 k - {b_0}^4\alpha + 696\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2 + 3{b_0}^2 k - 1392 b_0\beta}{b_0\left(\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2 - b_0\right)}$$

$$\mathrm{a} = \frac{-1}{3}\ \frac{-2\sqrt{10}\sqrt{\frac{\beta}{\alpha}}{b_0}^3\alpha\gamma^2 - 12{b_0}^2\beta\gamma^4 + 3\sqrt{10}\sqrt{\frac{\beta}{\alpha}}b_0\gamma^2 k - {b_0}^4\alpha - 696\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2 + 3{b_0}^2 k - 1392 b_0\beta}{b_0\left(-\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma^2 - b_0\right)}$$

Now, Eq. (3.3) becomes

$$\psi(\eta) = b_0 \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma\left(\frac{G'}{G}\right) \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(\frac{G'}{G}\right)^2 \ , \qquad (3.5)$$

Using the public solutions of Eq. (2.5) into Eq. (3.5), we have three kinds of traveling wave solutions. When $\gamma^2 - 4\delta > 0$, we get the hyperbolic function solution of Eq. (1.1)

$$\psi_{1,2}(\eta) = b_0 \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma\left(-\frac{\gamma}{2}+\frac{\sqrt{\gamma^2-4\delta}}{2}\left(\frac{r_1 sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}+r_2 cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}{r_1 cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}+r_2 sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}\right)\right) \pm$$

$$6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(-\frac{\gamma}{2}+\frac{\sqrt{\gamma^2-4\delta}}{2}\left(\frac{r_1 sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}+r_2 cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}{r_1 cosh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}+r_2 sinh\left\{\frac{\sqrt{\gamma^2-4\delta}\,\eta}{2}\right\}}\right)\right)^2 , \qquad (3.6)$$

In particular, if $r_1 \neq 0$, $r_2 = 0$, $\gamma > 0$, $\delta = 0$, then Eq. (3.6) becomes

$$\psi_{1,2}(\eta) = b_0 \pm \frac{3}{2}\sqrt{\frac{10\beta}{\alpha}}\gamma^2 \pm \frac{3}{2}\sqrt{\frac{10\beta}{\alpha}}(\gamma^2-4\delta)tanh^2\left(\frac{1}{2}\gamma\eta\right), \qquad (3.7)$$

When $\gamma^2 - 4\delta < 0$, we get the trigonometric function solution of Eq. (1.1)

$$\psi_{3,4}(\eta) = b_0 \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma\left(-\frac{\gamma}{2}+\frac{\sqrt{4\delta-\gamma^2}}{2}\left(\frac{-r_1 sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}+r_2 cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}{r_1 cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}+r_2 sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}\right)\right) \pm$$

$$6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(-\frac{\gamma}{2}+\frac{\sqrt{4\delta-\gamma^2}}{2}\left(\frac{r_1 sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}+r_2 cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}{r_1 cos\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}+r_2 sin\left\{\frac{\sqrt{4\delta-\gamma^2}\,\eta}{2}\right\}}\right)\right)^2 , \qquad (3.8)$$

If $r_1 = 0$, $r_2 \neq 0$, $\gamma = 0$, $\delta > 0$, then Eq. (3.8) becomes

$$\psi_{3,4}(\eta) = b_0 \pm 6\sqrt{\frac{10\beta}{\alpha}}\delta\cot^2(\eta) , \qquad (3.9)$$

when $\gamma^2 - 4\delta = 0$, we get the rational function solution of Eq. (1.1)

$$\psi_{5,6}(\eta) = b_0 \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\gamma\left(\left(\frac{r_2}{r_1+r_2\eta}\right)-\frac{\gamma}{2}\right) \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(\left(\frac{r_2}{r_1+r_2\eta}\right)-\frac{\gamma}{2}\right)^2 , \qquad (3.10)$$

If $r_1 = 0$, $r_2 \neq 0$, $\gamma = 2$, $\delta = 1$, then Eq. (3.10) becomes

$$\psi_{5,6}(\eta) = b_0 \pm 12\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(\frac{1}{\eta} - 1\right) \pm 6\sqrt{10}\sqrt{\frac{\beta}{\alpha}}\left(\frac{1}{\eta} - 1\right)^2, \qquad (3.11)$$

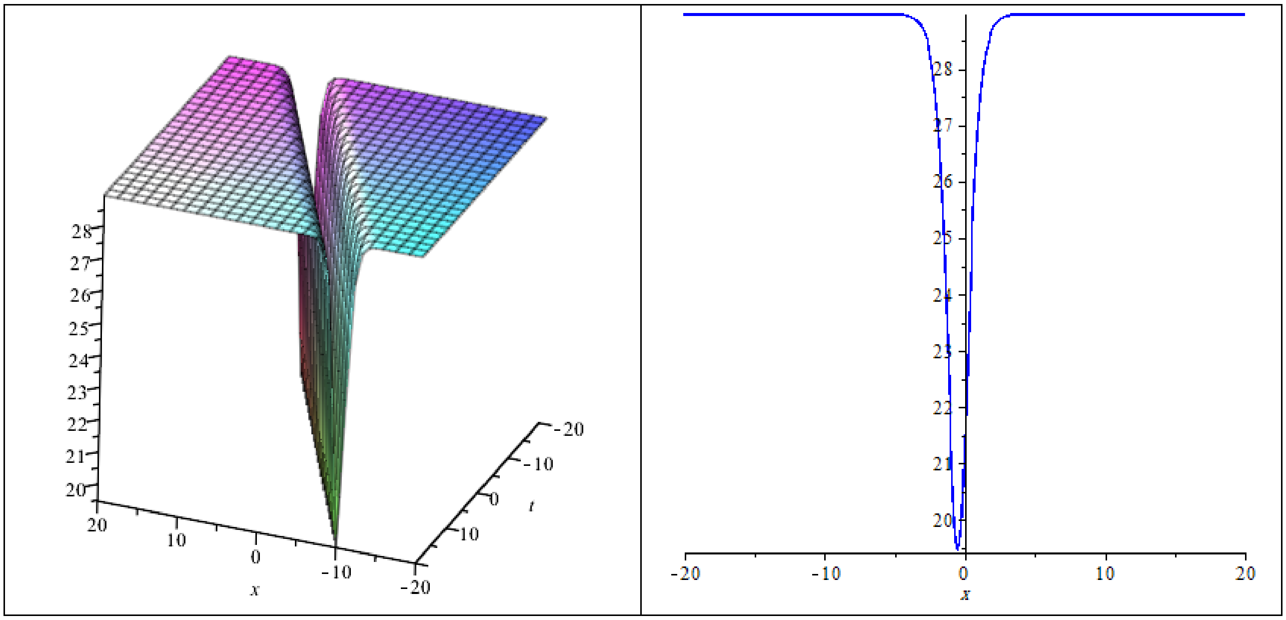

Figure 1. The wave solution given by (3.7) in 3D- and 2D-plots, when $k = -0.5$, $\beta = 1$ $\gamma = 2$, $\delta = 0.5$, $b_0 = 0.5$, $\alpha = 1$.

Remark 1. We have verified all the gained solutions by setting them back into the equations (3.4) with the aid of Maple.

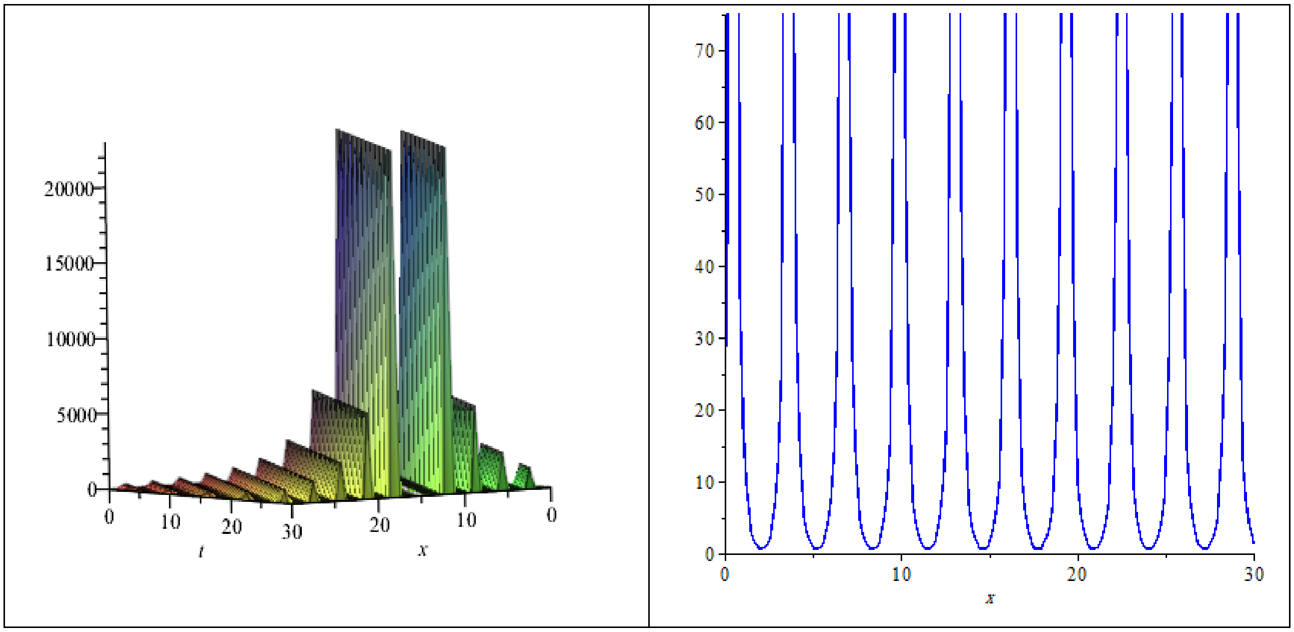

Figure 2. The wave solution given by (3.9) in 3D- and 2D-plots, when $k = 0.5, \beta = 1$ $\gamma = 2, \delta = \frac{1}{3}, b_0 = 0.5, \alpha = 1.$

4 Conclusion

In this research, the $\left(\frac{G'}{G}\right)$–standard method is efficiently and successfully utilized on the Gardner-Kawahara to find new solitary waves solutions. The kind of accurate solitary wave result is variety along with different value of appropriate choice of parameters (r_1 and r_2) . We note that the special case contains the trigonometric functions, the hyperbolic functions, and the rational functions. It is also a beneficial technique to solve other nonlinear evolution equations.

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