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Betweenness Centrality of Some Complementary Prism Graphs

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Abstract: There are a lot of centrality measures that have been introduced for networks. One of them is betweenness centrality. It is a measure of the influence of a vertex over the flow of information between all pairs of vertices. This information flows over the shortest paths between these vertices. The fact that any vertex has a high value of centrality indicates that what level this vertex is in connection with vertices which are not adjacent with each other. Since this vertex controls flows of information, it has a potential role in the network. In this paper, we study on the betweenness centrality of some complementary prism graphs.

Bazı Tümleyen Prizma Grafların Arasındalık Merkezliği

Anahtar Kelimeler
Graf teori,
Ağ tasarımı ve iletişim,
Arasındalık merkezliği,
Tümleyen prizma graf

Özet: Literatürde ağlar için tanımlanmış birçok merkezlik ölçümü vardır. Bunlardan biri arasındalık merkezliğidir. Arasındalık merkezliği bir tepenin tüm tepe çiftleri arasındaki bilgi akışına etkisinin bir ölçümüdür. Bu bilgi akışı, tepeler arasındaki en kısa yollar üzerinde olmaktadır. Herhangi bir tepenin yüksek arasındalık merkezliğe sahip olması o tepenin birbiriyle komşu olmayan tepelerle ne düzeyde bağlantı içinde olduğunu göstermektedir. Bu tepe ağdaki bilgi akışını kontrol ettiğinden ağda önemli bir yere sahiptir. Bu makalede bazı tümleyen prizma grafların arasındalık merkezliği üzerine çalışılmıştır.

1. Introduction

There are a lot of important properties for a network. One of them is which vertices lie on the shortest paths (geodesics) among pairs of other vertices [1, 2]. Betweenness centrality is based on shortest paths enumeration. It determines the importance or the centrality of a vertex (or an edge) in a network and plays an important role in analysis of social or communication networks [3], computer networks [4] and many other types of network data models [5, 6]. For example, in a telecommunication network, vertices with the higher value of centrality are more important. Because, more information passes through these vertices than the others. Since they lie on the largest number of paths taken by messages, removing these vertices from the network cuts off communications between others. Hence, the betweenness centrality is related to a network's connectivity and therefore its reliability [7].

The concept of betweenness centrality was first introduced by Bavelas [8] in 1948. Particularly, this concept is used in human communication in this study and and it indicates that when a person in a group is located on the shortest communication path connecting pairs of others, that person is in central position [9].

Betweenness centrality $C_B(v)$ for a vertex v is defined as

$$C_B(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}},$$

where σ_{st} is the number of shortest paths with vertices s and t as their end vertices, while $\sigma_{st}(v)$ is the number of those shortest paths that include vertex v.
The betweenness centrality of a graph G on n vertices is defined as

$$C_B(G) = \frac{2\sum_{i=1}^{n}[C_B(v^*) - C_B(v_i)]}{(n-1)^2(n-2)},$$

where $C_B(v^*)$ is the largest value of $C_B(v_i)$ for any vertex v_i in the given graph G.

This paper determines betweenness centrality of some complementary prism graphs. In 2007, Haynes *et al.* in [10] introduced the *complementary product* as a generalization of the Cartesian product. *Complementary prisms* of a graph

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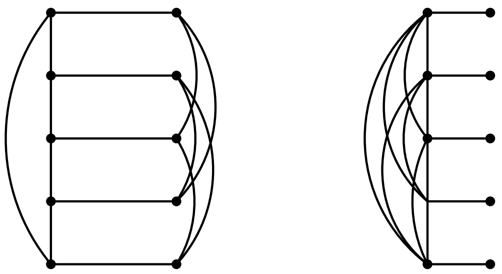

$C_5\overline{C}_5$ $K_5\overline{K}_5$

Figure 1. The Petersen graph $C_5\overline{C}_5$ and the corona $K_5 \circ K_1$

G is the subset of complementary products. Let G be a graph and $\overline{G}$ be the complement of G. The complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union $G \cup \overline{G}$ of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. In other words, for a graph G with vertex set $V(G)$ and edge set $E(G)$ the complementary prism of G is the graph with vertex set $V(G\overline{G}) = \{v_1, v_2, ..., v_n\} \cup \{\overline{v}_1, \overline{v}_2, ..., \overline{v}_n\}$ and edge set

$$E(G\overline{G}) = E(G) \cup \{\overline{v}_i\overline{v}_j : 1 \leqslant i < j \leqslant n \quad \text{and} \quad v_iv_j \notin E(G)\} \cup \{v_1\overline{v}_1, v_2\overline{v}_2, ..., v_n\overline{v}_n\},$$

in which for a vertex v of G, vertex $\overline{v}$ is the corresponding vertex in $\overline{G}$ [11, 12]. As demonstrated in Figure 1, the graph $C_5\overline{C}_5$ is the Petersen graph. Also, the graph $K_n\overline{K}_n$ is the corona $K_n \circ K_1$, where the corona $K_n \circ K_1$ is the graph obtained from K_n by attaching a pendant edge to each vertex of K_n. Complementary prisms are investigated in [11–15].

For notation and graph theory terminology we in general follow [16]. Before stating our results, we give some notations and formal definitions. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is the number of vertices of G. Given any two vertices $u, v \in V(G)$, the *distance* $d(u, v)$ is the length of the shortest path or geodesic path between u and v. The *diameter* $diam(G)$ of a graph G is the maximum distance between two vertices of G. The *degree* of a vertex v in a graph G is the number of edges of G incident to v and denoted by $deg_G(v)$. Throughout this paper, $deg(v)$ represents $deg_{G\overline{G}}(v)$ for any vertex v in $G\overline{G}$. The *center vertex* of a star or wheel graph is the only vertex that has a maximum degree.

2. Betweenness Centrality of Some Complementary Prism Graphs

In this section, we first state two known theorems that we use in the proof of our results. Next, we determine the betweenness centrality of some complementary prism graphs.

Theorem 2.1. *[2] The betweenness centrality of a vertex v in S_n is given by*

$$C_B(v) = \begin{cases} \binom{n-1}{2}, & \text{for center vertex,} \\ 0, & \text{for other vertices.} \end{cases}$$

Theorem 2.2. *[2] The betweenness centrality of a vertex v in a wheel graph W_n, $n > 5$ is given by*

$$C_B(v) = \begin{cases} \frac{(n-1)(n-5)}{2}, & \text{if } v \text{ is center vertex,} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Theorem 2.3. *Let $K_n\overline{K}_n$ be the complementary prism of a complete graph on $2n$ vertices. Then the betweenness centrality of a vertex v in $K_n\overline{K}_n$ is given by*

$$C_B(v) = \begin{cases} 2n-2, & \text{if } v \text{ is in } K_n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Take a vertex v in K_n. On K_n, there exists $n-1$ adjacent vertices of v and each pair of these vertices contributes 0 to v. Consider any adjacent vertex of v in K_n. There is only one geodesic path joining this vertex to corresponding vertex of v and it passes through v. Thus, each pair contributes centrality 1 to v and gives a total of $n-1$. Now, consider any vertex in $\overline{K}_n$ other than corresponding vertex of v. There is only one geodesic path from this vertex to corresponding vertex of v passing through v, and it contributes a betweenness centrality 1 to v. Since there are $n-1$ such pairs, they provide a betweenness centrality $n-1$ to v. Hence, the betweenness centrality of any vertex v in K_n is $2n-2$.

Take a vertex $\overline{v}$ in $\overline{K}_n$. Since $deg(\overline{v}) = 1$, there is one path joining vertex $\overline{v}$ and all other vertices of $K_n\overline{K}_n$. However, it does not pass through $\overline{v}$. Then the betweenness centrality of $\overline{v}$ is 0. □

Maximum value of the betweenness centrality of vertices in $K_n\overline{K}_n$ and the graph centrality are as follows:

$$C_B(v^*) = 2n-2,$$

$$C_B(K_n\overline{K}_n) = \frac{2\sum_{i=1}^{2n}[C_B(v^*) - C_B(v_i)]}{(2n-1)^2(2n-2)} = \frac{2n}{(2n-1)^2}.$$

Theorem 2.4. *Let $S_n\overline{S}_n$ be the complementary prism of a star on $2n$ vertices and c be the center vertex of S_n. Then the betweenness centrality of a vertex v in $S_n\overline{S}_n$ is given by*

$$C_B(v) = \begin{cases} \frac{(n-1)(n+2)}{2}, & \text{for center vertex } c, \\ 0, & \text{for } \overline{c}, \\ 2, & \text{for any vertex in } S_n - \{c\}, \\ n-2, & \text{for any vertex in } \overline{S}_n - \{\overline{c}\}, \end{cases}$$

in which $\overline{c}$ is the corresponding vertex of c.

Proof. Let v_1 be the center vertex of S_n in $S_n\overline{S}_n$. By Theorem 2.1, pairs of vertices in S_n contribute $\binom{n-1}{2}$ to v_1. Consider the pairs $(\overline{v}_1, v_i)$ and $(\overline{v}_1, \overline{v}_i)$ for all

$i \in \{2,3,...,n\}$. Each pair has a geodesic path which passes through v_1 and contributes 1 to the centrality of v_1. Since there are $2n-2$ such pairs, they give a total of $2n-2$. Thus, the betweenness centrality of v_1 is $\binom{n-1}{2}+2n-2=\frac{(n-1)(n+2)}{2}$.

Let $\overline{v}_1$ be the corresponding vertex of center vertex in $\overline{S}_n$. Since $deg(\overline{v}_1)=1$, there are no pairs in $S_n\overline{S}_n$ which pass through $\overline{v}_1$. Thus, the betweenness centrality of $\overline{v}_1$ is 0.

Consider any vertex v_i in S_n, where $i \in \{2,3,...,n\}$. None of $\binom{n-1}{2}$ pairs in S_n contains v_i. However, there is a geodesic path joining corresponding vertex $\overline{v}_i$ and v_1 and $\overline{v}_1$ passing through v_i. Since each one contributes 1 to the centrality of v_i, the betweenness centrality of v_i is 2.

For any vertex $\overline{v}_i$, there exists $n-2$ adjacent vertices of $\overline{v}_i$ in $\overline{S}_n$, in which $i \in \{2,3,...,n\}$. Then there is a geodesic path joining each adjacent vertex to v_i and it passes through $\overline{v}_i$. Thus, each pair contributes centrality 1 to $\overline{v}_i$ and they contribute a total of $n-2$. □

The largest value of the betweenness centrality of vertices of $S_n\overline{S}_n$ is

$$C_B(v^*)=\frac{(n-1)(n+2)}{2},$$

and the betweenness centrality of $S_n\overline{S}_n$ is

$$\begin{aligned}
C_B(S_n\overline{S}_n) &= \frac{2\sum_{i=1}^{2n}[C_B(v^*)-C_B(v_i)]}{(2n-1)^2(2n-2)} \\
&= \frac{2}{(2n-1)^2(2n-2)}\Bigg\{\left(\frac{(n-1)(n+2)}{2}-0\right) \\
&\quad +(n-1)\left(\frac{(n-1)(n+2)}{2}-2\right) \\
&\quad +(n-1)\left(\frac{(n-1)(n+2)}{2}-(n-2)\right)\Bigg\} \\
&= \frac{2n^2+n-2}{2(2n-1)^2}.
\end{aligned}$$

Theorem 2.5. *Let $P_n\overline{P}_n$ be the complementary prism of a path of order $2n$. Then the betweenness centrality of vertices of $P_n\overline{P}_n$ for $n>6$ is given as follows:*

If $v_i \in V(P_n)$, then

$$C_B(v_i)=\begin{cases}1, & \text{if } i \in \{1,n\},\\ \frac{7}{2}, & \text{if } i \in \{2,n-1\},\\ 4, & \text{if } i \in \{3,4,...,n-2\}.\end{cases}$$

If $\overline{v}_i \in V(\overline{P}_n)$, then

$$C_B(\overline{v}_i)=\begin{cases}2n+\frac{1}{n-3}-\frac{9}{2}, & \text{if } i \in \{1,n\},\\ 2n+\frac{1}{n-3}+\frac{n-5}{n-4}-\frac{15}{2}, & \text{if } i \in \{2,n-1\},\\ 2n+\frac{1}{n-3}+\frac{n-6}{n-4}-\frac{17}{2}, & \text{if } i \in \{3,n-2\},\\ 2n+\frac{2}{n-3}+\frac{n-7}{n-4}-9, & \text{if } i \in \{4,5,...,n-3\}.\end{cases}$$

Proof. Since $diam(P_n\overline{P}_n)=3$, there is a geodesic path of length at most 3 between two vertices in $P_n\overline{P}_n$. For the betweenness centrality of any vertex v in $P_n\overline{P}_n$, we have two cases: when $v \in V(P_n)$ and when $v \in V(\overline{P}_n)$.

Case 1. Let v_i be a vertex in P_n, where $i \in \{1,2,...,n\}$. Since the distance between any non-adjacent vertices of $\overline{P}_n$ is 2, none of $\binom{n}{2}$ pairs of vertices of $\overline{P}_n$ passes through v_i.

For all $i \geqslant 1$, consider the pair $(v_j,\overline{v}_k)$, where $j \in \{1,2,...,n\}$, $j \neq i$ and $k \in \{1,2,...,n\}$. There is a geodesic path joining $\overline{v}_i$ to adjacent vertices of v_i in P_n which passes through v_i. Since there are $deg(v_i)-1$ adjacent vertices of v_i in P_n, they contribute a betweenness centrality $deg(v_i)-1$ to v_i.

For contribution of any pair of vertices in P_n to the betweenness centrality of v_i, we partitioned the vertex set of P_n into the following three pairs.

- Let $i \in \{1,n\}$.

Since the distance between any two vertices of P_n is at most 3 in $P_n\overline{P}_n$, none of geodesic paths between these vertices except v_1 and v_n contains v_1 or v_n. Thus, these pairs contribute to v_1 or v_n the betweenness centrality zero.

As a consequence, we have

$$C_B(v_i)=deg(v_i)-1=1 \quad \text{for} \quad i \in \{1,n\},$$

in which $deg(v_i)=2$.

- Let $i \in \{2,n-1\}$.

For the pair (v_{i-1},v_{i+1}), there is a geodesic path of length 2 passing through v_i and it contributes a betweenness centrality 1 to v_i. Moreover, consider the paths of length 3 containing v_i. There are two geodesic paths joining v_1 and v_4, one of which passes through v_2. Similarly, there are two geodesic paths joining v_n and v_{n-3}, one of which passes through v_{n-1}. Hence, each pair contributes $\frac{1}{2}$ to v_i.

As a consequence, we have

$$C_B(v_i)=deg(v_i)-1+1+\frac{1}{2}=\frac{7}{2} \quad \text{for} \quad i \in \{2,n-1\},$$

in which $deg(v_i)=3$.

- Let $i \in \{3,4,5,...,n-2\}$.

There is a geodesic path of length 2 between adjacent vertices of v_i in P_n passing through v_i and it contributes a betweenness centrality 1 to v_i. Moreover, each pair (v_{i-2},v_{i+1}) and (v_{i-1},v_{i+2}) have two geodesic paths of length 3, one of which passes through v_i. These two pairs provide a total of 1 to the centrality of v_i.

As a consequence, we have

$$C_B(v_i)=deg(v_i)-1+1+1=4 \quad \text{for} \quad i \in \{3,4,...,n-2\},$$

in which $deg(v_i) = 3$.

Case 2. Let $\overline{v}_i$ be a vertex in $\overline{P}_n$, where $i \in \{1,2,...,n\}$. For all $i \geqslant 1$, consider the pair $(v_j, \overline{v}_k)$, where $j,k \in \{1,2,...,n\}$ and $k \neq i$. There is only one geodesic path joining v_i and adjacent vertices of $\overline{v}_i$ in $\overline{P}_n$ and it passes through $\overline{v}_i$. Since $\overline{v}_i$ has $deg(\overline{v}_i) - 1$ adjacent vertices in $\overline{P}_n$, it provides the betweenness centrality $deg(\overline{v}_i) - 1$ to $\overline{v}_i$.

We partitioned vertex set of $\overline{P}_n$ into three pairs for contribution of any pair of vertices in P_n and similarly any pair of vertices in $\overline{P}_n$ to the betweenness centrality of $\overline{v}_i$.

- Let $i \in \{1,2,3\}$.

Consider pairs of vertices in $\overline{P}_n$. Now, for each pair $(\overline{v}_{j+2}, \overline{v}_{j+3})$, where $j \in \{i, i+1, i+2, ..., n-3\}$, there is one geodesic path passing through $\overline{v}_i$. There are $n-i-2$ such pairs. While there are $n-3$ geodesic paths for the pair $(\overline{v}_{n-1}, \overline{v}_n)$, there are $n-4$ geodesic paths for the remaining $n-i-3$ pairs. Thus, these $n-i-2$ pairs contribute $\frac{1}{n-3} + \frac{n-i-3}{n-4}$ to the betweenness centrality of $\overline{v}_i$.

Consider pairs of vertices in P_n. If the distance between two vertices of P_n is 1 or 2 in P_n, there is a geodesic path joining these vertices but it does not contain $\overline{v}_i$. Thus, consider paths of length at least 3 in P_n. There are two geodesic paths of length 3 joining v_i and v_{i+3}, one of which passes through $\overline{v}_i$. Thereby, they contribute $\frac{1}{2}$ to the centrality of $\overline{v}_i$. Now, consider paths of length greater than 3 in P_n. Each pair (v_i, v_{i+3+k}) has only one geodesic path and it passes through $\overline{v}_i$ for each $k \in \{1,2,...,n-i-3\}$. Since there are $n-i-3$ such pairs, it contributes a betweenness centrality $n-i-3$ to $\overline{v}_i$.

As a consequence, for $i \in \{1,2,3\}$

$$C_B(\overline{v}_i) = deg(\overline{v}_i) + \frac{1}{n-3} + \frac{n-i-3}{n-4} + n - i - \frac{7}{2}.$$

- Let $i \in \{4,5,...,n-3\}$.

Consider the pair $(\overline{v}_{j+2}, \overline{v}_{j+3})$, where $j \in \{-1,0,1,...,i-5,i,i+1,...,n-3\}$. There are $n-5$ such pairs and two of them are $(\overline{v}_1, \overline{v}_2)$ and $(\overline{v}_{n-1}, \overline{v}_n)$. For these two pairs, there are $n-3$ geodesic paths of length 2, one of them passes through $\overline{v}_i$. Hence, they contribute centrality $\frac{2}{n-3}$ to $\overline{v}_i$. For the remaining $n-7$ pairs, there are $n-4$ geodesic paths of length 2, one of them passes through $\overline{v}_i$. Hence, each pair contributes centrality $\frac{1}{n-4}$ to $\overline{v}_i$ and they contribute a total of $\frac{n-7}{n-4}$. Therefore, $n-5$ pairs of vertices in $\overline{P}_n$ provides a betweenness centrality $\frac{2}{n-3} + \frac{n-7}{n-4}$ to $\overline{v}_i$.

Consider pairs of vertices in P_n. There is a geodesic path of length at least 3 passing through $\overline{v}_i$. Thus, we consider pairs (v_{i-3-k}, v_i) and (v_i, v_{i+3+l}), where $k \in \{0,1,...,i-4\}$ and $l \in \{0,1,...,n-i-3\}$. For each pair (v_{i-3}, v_i) and (v_i, v_{i+3}) where $k = 0$ and $l = 0$, there are two geodesic paths of length 3 and one of them passes through $\overline{v}_i$. Thus, they contribute a betweenness centrality 1 to $\overline{v}_i$. If $k > 0$ or $l > 0$, then each pair has only one geodesic path of length greater than 3 passing through $\overline{v}_i$. Since there are $n-7$ such pairs, they contribute a betweenness centrality $n-7$ to $\overline{v}_i$.

As a consequence, for $i \in \{4,5,...,n-3\}$

$$C_B(\overline{v}_i) = deg(\overline{v}_i) + \frac{2}{n-3} + \frac{n-7}{n-4} + n - 7.$$

- Let $i \in \{n-2, n-1, n\}$.

Consider the pair $(\overline{v}_j, \overline{v}_{j+1})$ for $j \in \{1,2,...,i-3\}$. There are $i-3$ such pairs and one of them is $(\overline{v}_1, \overline{v}_2)$. For this pair, there is $n-3$ geodesic paths joining $\overline{v}_1$ and $\overline{v}_2$, one of which passes through $\overline{v}_i$. Thus, it contributes $\frac{1}{n-3}$ to $\overline{v}_i$. For the remaining $i-4$ pairs, there are $n-4$ geodesic paths joining $\overline{v}_j$ and $\overline{v}_{j+1}$, one of which passes through $\overline{v}_i$ for each $j \in \{2,3,...,i-3\}$. Each pair contributes $\frac{1}{n-4}$ to $\overline{v}_i$ giving a total of $\frac{i-4}{n-4}$.

Consider pairs of vertices in P_n. Among the paths of length 3 in P_n, there are two geodesic paths joining v_i and v_{i-3}, one of which passes through $\overline{v}_i$. Hence, they contribute $\frac{1}{2}$ to $\overline{v}_i$. Now, consider all paths of length greater than 3 in P_n. There are $i-4$ geodesic paths joining v_i and v_{i-3-k}, one of which passes through $\overline{v}_i$ for $k \in \{1,2,...,i-4\}$. Thus, each pairs contributes centrality 1 to $\overline{v}_i$ giving a total of $i-4$.

As a consequence, for $i \in \{n-2, n-1, n\}$

$$C_B(\overline{v}_i) = deg(\overline{v}_i) + \frac{1}{n-3} + \frac{i-4}{n-4} - \frac{9}{2} + i.$$

It is clear that $deg(\overline{v}_1) = deg(\overline{v}_n) = n-1$ and $deg(\overline{v}_i) = n-2$ for $i \neq 1, n$. Simplifying the equations of Case 2,

$$C_B(\overline{v}_i) = \begin{cases} 2n + \frac{1}{n-3} - \frac{9}{2}, & \text{if } i \in \{1,n\}, \\ 2n + \frac{1}{n-3} + \frac{n-5}{n-4} - \frac{15}{2}, & \text{if } i \in \{2,n-1\}, \\ 2n + \frac{1}{n-3} + \frac{n-6}{n-4} - \frac{17}{2}, & \text{if } i \in \{3,n-2\}, \\ 2n + \frac{2}{n-3} + \frac{n-7}{n-4} - 9, & \text{if } i \in \{4,5,...,n-3\}. \end{cases}$$

is obtained. □

The maximum centrality value of Case 2 is at the first or end vertex of $\overline{P}_n$. Comparing the maximum values of Case 1 and Case 2 we have

$$C_B(v^*) = 2n + \frac{1}{n-3} - \frac{9}{2}.$$

Thus, the betweenness centrality of $P_n\overline{P}_n$ is as follows:

$$\begin{aligned} C_B(P_n\overline{P}_n) &= \frac{2}{(2n-1)^2(2n-2)} \sum_{i=1}^{2n} [C_B(v^*) - C_B(v_i)] \\ &= \frac{2n^3 - 11n^2 + 14n + 9}{(2n-1)^2(n-1)(n-3)}. \end{aligned}$$

Theorem 2.6. *The betweenness centrality of any vertex v in the complementary prism of a cycle with n > 6 is*

$$C_B(v) = \begin{cases} 4, & \text{if } v \in V(C_n) \\ 2n-8, & \text{if } v \in V(\overline{C}_n). \end{cases}$$

Proof. Since $diam(C_n\overline{C}_n) = 3$, there is a geodesic path of length at most 3 between two vertices in $C_n\overline{C}_n$. Since C_n and $\overline{C}_n$ are vertex transitive, it is sufficient to consider without loss of generality that the betweenness centrality of any vertex in C_n and similarly in $\overline{C}_n$.

Case 1. Let v_i be any vertex in C_n, where $i \in \{1,2,...,n\}$. The distance between non–adjacent vertices of $\overline{C}_n$ is 2 in $C_n\overline{C}_n$ and these vertices do not lie on any geodesic paths containing v_i. Then their betweenness centralities to v_i are zero.

Consider pairs of vertices in C_n. For each pairs (v_{i-1}, v_{i+2}) and (v_{i-2}, v_{i+1}), there are two geodesic paths of length 3, one of which passes through v_i. Note that, we take $v_0 = v_n$ and $v_{-1} = v_{n-1}$. Among the paths of length less than 3 in C_n, there is only one geodesic path joining two adjacent vertices of v_i in C_n passes through v_i. Thus, they provide a total of 2 to the centrality of v_i.

Now, consider the paths joining vertices of C_n and vertices of $\overline{C}_n$. There is only one geodesic path joining $\overline{v}_i$ and two adjacent vertices of v_i in C_n passing through v_i. Thus, each one contributes 1 to the centrality of v_i and they give a total of 2.

Consequently, the betweenness centrality of any vertex in C_n is 4.

Case 2. Let $\overline{v}_i$ be any vertex in $\overline{C}_n$, where $i \in \{1,2,...,n\}$. Consider pairs of vertices in C_n. There are two vertices in C_n at distance 3 with v_i, and there are two geodesic paths joining v_i and each of these two vertices. One of two geodesic paths passes through $\overline{v}_i$. Thus, each one contributes $\frac{1}{2}$ to the centrality of $\overline{v}_i$ and they give a total of 1. Furthermore, each geodesic path joining v_i and $n-7$ vertices of C_n that the distance with v_i in C_n is greater than 3, if any, contributes 1 to the centrality of $\overline{v}_i$. Then it gives a total of $n-7$.

Consider pairs of vertices between $V(C_n)$ and $V(\overline{C}_n)$. There is only one geodesic path joining vertex v_i and each adjacent vertex of $\overline{v}_i$ in $\overline{C}_n$. Since there are $n-3$ such adjacent vertices, they contribute a betweenness centrality $n-3$ to $\overline{v}_i$.

Consider pairs of vertices in $\overline{C}_n$. Since there are $n-3$ adjacent vertices of $\overline{v}_i$ in $\overline{C}_n$, there are $n-4$ non–adjacent pairs of vertices with these $n-3$ vertices. For each pair, there are $n-4$ geodesic paths joining its vertices, one of which passes through $\overline{v}_i$. Thus, $n-4$ pairs contribute 1 to the centrality of $\overline{v}_i$.

Consequently, the betweenness centrality of any vertex in $\overline{C}_n$ is $2n-8$. □

The largest value of the betweenness centrality of vertices of $C_n\overline{C}_n$

$$C_B(v^*) = 2n-8,$$

and the betweenness centrality of $C_n\overline{C}_n$ is

$$C_B(C_n\overline{C}_n) = \frac{2\sum_{i=1}^{2n}[C_B(v^*) - C_B(v_i)]}{(2n-1)^2(2n-2)} = \frac{2n(n-6)}{(2n-1)^2(n-1)}.$$

Theorem 2.7. *Let $K_{n,m}\overline{K}_{n,m}$ be the complementary prism of a complete bipartite graph with $n \leqslant m$. Then the betweenness centrality of a vertex v in $K_{n,m}\overline{K}_{n,m}$ is given by*

$$C_B(v) = \begin{cases} \frac{m^2-m+4mn}{2n}, & \textit{if } deg(v)=m+1, \\ \frac{n^2-n+4mn}{2m}, & \textit{if } deg(v)=n+1, \\ n-1, & \textit{if } deg(v)=n, \\ m-1, & \textit{if } deg(v)=m. \end{cases}$$

Proof. Let $V(K_{n,m}\overline{K}_{n,m}) = V_1^{(1)} \cup V_1^{(2)} \cup V_2^{(1)} \cup V_2^{(2)}$, where $V_1^{(1)} = \{v_1, v_2, ..., v_n\}$, $V_1^{(2)} = \{v_{n+1}, v_{n+2}, ..., v_{n+m}\}$ and $V_2^{(1)} = \{\overline{v}_1, \overline{v}_2, ..., \overline{v}_n\}$, $V_2^{(2)} = \{\overline{v}_{n+1}, \overline{v}_{n+2}, ..., \overline{v}_{n+m}\}$. The degree of any vertex of $V_1^{(1)}$, $V_1^{(2)}$, $V_2^{(1)}$ and $V_2^{(2)}$ in $K_{n,m}\overline{K}_{n,m}$ are $m+1$, $n+1$, n and m, respectively. Then we have following cases:

Case 1. Let v_i be a vertex in $V_1^{(1)}$, where $i \in \{1,2,...,n\}$. Consider pairs of vertices (v_j, v_k), where $j,k \in \{n+1, n+2, ..., n+m\}$ and $j \neq k$. For each pair, there are n geodesic paths joining the adjacent vertices of v_j in $V_1^{(1)}$ and v_k, one of which passes through v_i. Thus, each pair contributes centrality $\frac{1}{n}$ to v_i and they give a total of $\binom{m}{2}\frac{1}{n}$.

Consider pairs of vertices $(v_j, \overline{v}_k)$ for $v_j \in V_1^{(2)}$ and $\overline{v}_k \in V_2^{(1)}$. For these pairs there is only one geodesic path joining vertex v_j and corresponding vertex of v_i in $V_2^{(1)}$ containing v_i. Since there are m such pairs, they contribute a betweenness centrality m to v_i.

Consider pairs of vertices between $V_2^{(1)}$ and $V_2^{(2)}$. There is only one geodesic path joining corresponding vertex of v_i in $V_2^{(1)}$ and any vertex of $V_2^{(2)}$ passing through v_i. Since $|V_2^{(2)}| = m$, this contributes centrality m to v_i. The remaining pairs of vertices do not lie on any geodesic paths passing through v_i. Thus, their contribution of the betweenness centrality to v_i is zero.

As a consequence, the betweenness centrality of v_i in $V_1^{(1)}$ is

$$\binom{m}{2}\frac{1}{n} + 2m = \frac{m^2-m+4mn}{2n}.$$

Case 2. Let v_i be a vertex in $V_1^{(2)}$, where $i \in \{n+1, n+2, ..., n+m\}$. The proof is similar to that of Case 1 and is omitted. Hence, we have

$$C_B(v_i) = \frac{n^2-n+4mn}{2m}.$$

Case 3. Let $\overline{v}_i$ be a vertex in $V_2^{(1)}$, where $i \in \{1,2,...,n\}$. In this case, there is only one geodesic path from corresponding vertex of $\overline{v}_i$ in $V_1^{(1)}$ to any vertex $\overline{v}_k$ passing through $\overline{v}_i$ for each $k \in \{1,2,...,n\}$ and $k \neq i$. Since there are $n-1$ such pairs, they contribute $n-1$ to $\overline{v}_i$. None of other pairs lies on any geodesic paths. As a consequence, the betweenness centrality of any vertex $\overline{v}_i$ in $V_2^{(1)}$ is $n-1$.

Case 4. Let $\overline{v}_i$ be a vertex in $V_2^{(2)}$, where $i \in \{n+1,n+2,...,n+m\}$.
When this case is proved similar to Case 3, we have $C_B(v_i) = m-1$. □

The largest value of the betweenness centrality of vertices of $K_{n,m}\overline{K}_{n,m}$ is

$$C_B(v^*) = \frac{m^2 - m + 4mn}{2n},$$

and the betweenness centrality of $K_{n,m}\overline{K}_{n,m}$ is

$$\begin{aligned} C_B(K_{n,m}\overline{K}_{n,m}) &= \frac{2\sum\limits_{i=1}^{2(n+m)}[C_B(v^*) - C_B(v_i)]}{(2(n+m)-1)^2(2(n+m)-2)} \\ &= \frac{mn + 3n^2 + 7m^2n - 3n^3 + m^2(2m-2)}{2n(2n+2m-1)^2(n+m-1)}. \end{aligned}$$

Theorem 2.8. *For the complementary prism of a wheel graph $W_{1,n}\overline{W}_{1,n}$ of order $2n+2$, let c and $\overline{c}$ be center vertex of $W_{1,n}$ and corresponding vertex of c in $\overline{W}_{1,n}$, respectively. Then the betweenness centrality of a vertex v in $W_{1,n}\overline{W}_{1,n}$ for $n > 5$ is given by*

$$C_B(v) = \begin{cases} \frac{n^2}{2}, & \text{if } v = c \\ 0, & \text{if } v = \overline{c} \\ \frac{9}{2}, & \text{if } v \text{ is in } W_{1,n} - \{c\} \\ n-2, & \text{if } v \text{ is in } \overline{W}_{1,n} - \{\overline{c}\}. \end{cases}$$

*Proof.*Let $V(W_{1,n}) = \{c, v_1, v_2, ..., v_n\}$ and $V(\overline{W}_{1,n}) = \{\overline{c}, \overline{v}_1, \overline{v}_2, ..., \overline{v}_n\}$, where c is center vertex of $W_{1,n}$ and $\overline{c}$ is corresponding vertex of c in $\overline{W}_{1,n}$. Wheel graph $W_{1,n}$ contains a cycle of order n and center vertex is adjacent to each vertex of cycle C_n. Then we have following four cases:

Case 1. Take center vertex c of $W_{1,n}$.
Since $diam(W_{1,n}) = 2$, the distance between two vertices of C_n in $W_{1,n}\overline{W}_{1,n}$ is at most 2. Thus, by Theorem 2.2 vertices of $W_{1,n}$ in $W_{1,n}\overline{W}_{1,n}$ contribute a betweenness centrality $\frac{n(n-4)}{2}$ to c.

Among the paths between any vertex of $W_{1,n}$ and any vertex of $\overline{W}_{1,n}$, there is only one geodesic path of length 2 joining $\overline{c}$ and v_i for each $i \in \{1,2,...,n\}$ passing through c. Thus, each path contributes 1 to the centrality of c and they give a total of n.

Consider pairs of vertices in $\overline{W}_{1,n}$. Since the distance between any two vertices in $\overline{W}_{1,n} - \{\overline{c}\}$ is at most 2 in $W_{1,n}\overline{W}_{1,n}$, none of paths joining two vertices of $\overline{W}_{1,n} - \{\overline{c}\}$ contains c. Only there is one geodesic path joining vertex $\overline{c}$ and vertex $\overline{v}_i$ passing through c for each $i \in \{1,2,...,n\}$. Since there is n such pairs, they contribute centrality n to c.

As a consequence, the betweenness centrality of c is $\frac{n(n-4)}{2} + 2n = \frac{n^2}{2}$.

Case 2. Take corresponding vertex of center vertex.
Since $deg(\overline{c}) = 1$, there is one path joining $\overline{c}$ and all other vertices. However, no one contains $\overline{c}$. Thus, the betweenness centrality of $\overline{c}$ is zero.

Case 3. Consider any vertex v_i in $W_{1,n} - \{c\}$, where $i \in \{1,2,...,n\}$.
There are two geodesic paths joining adjacent vertices of v_i on C_n, one of them passes through v_i. Thus, its contribution to the centrality of v_i is $\frac{1}{2}$.

After examining the geodesic paths joining vertices of $W_{1,n}$ and vertices of $\overline{W}_{1,n}$, it is seen that there is a geodesic path of length 2 only joining corresponding vertex of v_i and adjacent vertex of v_i in $W_{1,n}$ passing through v_i. Since v_i has three adjacent vertices in $W_{1,n}$, they contribute a betweenness centrality 3 to v_i.

Now, consider pairs of vertices in $\overline{W}_{1,n}$. Since the distance between any two vertices of $\overline{W}_{1,n} - \{\overline{c}\}$ is at most 2 in $W_{1,n}\overline{W}_{1,n}$, no paths joining vertices of $\overline{W}_{1,n} - \{\overline{c}\}$ pass through v_i. However, there is a geodesic path of length 3 joining vertex $\overline{c}$ and vertex $\overline{v}_i$ passing through v_i. It contributes a betweenness centrality 1 to v_i.

As a consequence, the betweenness centrality of any vertex in $W_{1,n} - \{c\}$ is $\frac{1}{2} + 3 + 1 = \frac{9}{2}$.

Case 4. Consider any vertex $\overline{v}_i$ in $\overline{W}_{1,n} - \{\overline{c}\}$, where $i \in \{1,2,...,n\}$.
There is a geodesic path joining v_i and adjacent vertices of $\overline{v}_i$ in $\overline{W}_{1,n}$ and it contains $\overline{v}_i$. Since $\overline{v}_i$ has $n-3$ adjacent vertices in $\overline{W}_{1,n}$, their contribution is $n-3$ to v_i. The distance between all pair of vertices of $W_{1,n}$ is at most 2 in $W_{1,n}\overline{W}_{1,n}$. Thus, none of paths joining these vertices contains $\overline{v}_i$.

Among pairs of vertices in $\overline{W}_{1,n}$, consider the pair $(\overline{v}_{j(\text{mod } n)}, \overline{v}_{j+1(\text{mod } n)})$, where $j \in \{i+2, i+3, ..., n+i-3\}$ and $\overline{v}_0 = \overline{v}_n$. There are $n-4$ geodesic paths joining $\overline{v}_j$ and $\overline{v}_{j+1}$ for each $j \in \{i+2, i+3, ..., n+i-3\}$ and one of these contains $\overline{v}_i$. Since there are $n-4$ such pairs, they contribute a betweenness centrality 1 to $\overline{v}_i$.

As a consequence, the betweenness centrality of any vertex in $\overline{W}_{1,n} - \{\overline{c}\}$ is $n-3+1 = n-2$. □

The largest value $C_B(v^*)$ is $\frac{n^2}{2}$ and the graph centrality is as follows:

$$C_B(W_{1,n}\overline{W}_{1,n}) = \frac{2\sum\limits_{i=1}^{2(n+1)}[C_B(v^*) - C_B(v_i)]}{2n(2n+1)^2} = \frac{2n^2 - n - 5}{2(2n+1)^2}.$$

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