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Research Article

Constructions of helicoidal surfaces by using curvature functions in isotropic space

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Abstract

In the present paper, we study helicoidal surfaces in the three dimensional isotropic space $\mathbb{I}^3$ and construct helicoidal surfaces with prescribed Gaussian curvature or mean curvature given by smooth functions. Moreover, we give some examples of helicoidal surfaces with non-constant Gaussian curvature or mean curvature.

Mathematics Subject Classification (2010). 53C21, 53C42

Keywords. Isotropic space, helicoidal surface, mean curvature, Gaussian curvature

1. Introduction

Differential geometers have been interested in studying surfaces of constant mean curvature and constant Gaussian curvature in ambient spaces for a long time.
Rotational surfaces in the three dimensional Euclidean space $\mathbb{E}^3$ with constant mean curvature studied by Delaunay [5] and these surfaces with constant Gaussian curvature showed in [7]. Hano and Nomizu [8] extended it to the Lorentz version. On the other hand, Caddeo et. al. [4] studied rotational surfaces in the three dimensional Heisenberg group Nil_3 and they constructed these surfaces with constant Gaussian curvature.

Helicoidal surfaces in the three dimensional space forms arise as a natural generalization of rotational surfaces in such spaces. These surfaces are invariant by a subgroup of the group of isometries of the ambient space, called helicoidal group, whose elements can be seen as a composition of a translation with a rotation for a given axis. In the Euclidean space $\mathbb{E}^3$, do Carmo and Dajczer [6] described the space of all helicoidal surfaces that have constant mean curvature or constant Gaussian curvature. This space behaves as a circular cylinder, where a given generator corresponds to the rotational surfaces and each parallel corresponds to a periodic family of helicoidal surfaces. As a generalization of constant Gaussian curvature or mean curvature surfaces, helicoidal surfaces with prescribed mean curvature or Gaussian curvature are obtained by Baikoussis and Koufogiorgos [2] and they constructed a closed form of such a surface by integrating the second-order ordinary differential equation satisfied by the generating curve of the surface. After, Beneki et. al.

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[3], Ji and Hou [9] and the present author [12] described these surfaces in the Minkowski space and the Heisenberg group.

The main purpose of this paper is to construct helicoidal surfaces in the three dimensional isotropic space with prescribed mean or Gaussian curvature.

2. Preliminaries

The three dimensional isotropic space $\mathbb{I}^3$ has been developed by Strubecker in the 1940s, and it is based on the following group G_6 of an affine transformations $(x, y, z) \to (\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{R}^3$,

$$\begin{aligned} \bar{x} &= a + x\cos\phi - y\sin\phi, \\ \bar{y} &= b + x\sin\phi + y\cos\phi, \\ \bar{z} &= c + c_1 x + c_2 y + z, \end{aligned} \tag{2.1}$$

where $a, b, c, c_1, c_2, \phi \in \mathbb{R}$, which are called isotropic motions of $\mathbb{I}^3$ (cf. [10]). We see that isotropic motions appear as Euclidean motions (translation and rotation) in the projection onto the xy-plane; the result of this projection $\mathbf{p} = (x, y, z) \to \mathbf{p}' = (x, y, 0)$ is called top view. Hence an isotropic motion is composed of a Euclidean motion in the xy-plane and an affine shear transformation in z-direction.

On the other hand, the isotropic distance of two points $\mathbf{x} = (x_1, y_1, z_1)$ and $\mathbf{y} = (x_2, y_2, z_2)$ is defined as the Euclidean distance of the their top views, i.e.,

$$||\mathbf{x} - \mathbf{y}||_i := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{2.2}$$

As a fact, two points (x, y, z_i) $(i = 1, 2)$ with the same top views have isotropic distance zero, they called parallel points. Since the isotropic metric (2.2) degenerates along z-parallel lines, these are called isotropic lines. Isotropic angles between straight lines are measured as Euclidean angles in the top view. There are two types of planes in $\mathbb{I}^3$. Isotropic planes are planes parallel to the z-direction, and non-isotropic planes otherwise.

Let $\mathbf{x} = (x_1, y_1, z_1)$ and $\mathbf{y} = (x_2, y_2, z_2)$ be vectors in $\mathbb{I}^3$. The isotropic scalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \cdot \mathbf{y} = \begin{cases} z_1 z_2, & \text{if} \quad x_i = 0 \quad \text{and} \quad y_i = 0 \\ x_1 x_2 + y_1 y_2, & \text{if} \quad \text{otherwise}. \end{cases} \tag{2.3}$$

We call vector of the form $\mathbf{x} = (0, 0, z)$ in $\mathbb{I}^3$ isotropic vector, and non-isotropic vector otherwise.

A surface M immersed in $\mathbb{I}^3$ is called admissible if it has no isotropic tangent planes. For such a surface, the coefficients g_{ij} $(i, j = 1, 2)$ of its first fundamental form are calculated with respect to the induced metric and h_{ij} $(i, j = 1, 2)$ of the second fundamental form, with respect to the normal vector field of a surface which is always completely isotropic. The normal vector field of M is always the isotropic vector $(0, 0, 1)$, since it is perpendicular to all tangent vectors to M.

The isotropic Gaussian curvature K and the isotropic mean curvature H are defined by

$$K = \frac{\det(h_{ij})}{\det(g_{ij})}, \quad H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2\det(g_{ij})}. \tag{2.4}$$

The surface M is said to be isotropic flat (resp. isotropic minimal) if K (resp.H) vanishes (cf. [11]).

3. Main results

In this section, we completely construct helicoidal surfaces in $\mathbb{I}^3$ with the Gaussian curvature or the mean curvature given by smooth functions.

Now, we define helicoidal surfaces in an isotropic space $\mathbb{I}^3$. Considering the isotropic motion given by (2.1), the Euclidean rotations in the isotropic space $\mathbb{I}^3$ is given by the normal form (in affine coordinates)

$$\begin{cases} \bar{x} = x\cos\phi - y\sin\phi, \\ \bar{y} = x\sin\phi + y\cos\phi, \\ \bar{z} = z, \end{cases}$$

where $\phi \in \mathbb{R}$.

Let γ be a curve lying in the isotropic xz-plane or yz-plane without loss of generality. Then the profile curve γ is parameterized as

$$\gamma(u) = (f(u), 0, g(u)) \quad \text{or} \quad \gamma(u) = (0, f(u), g(u)),$$

where f and g are smooth functions.

By rotating the curve γ around z-axis and simultaneously followed by a translation, a helicoidal surface M in $\mathbb{I}^3$ can be parameterized by

$$\mathbf{r}(u,v) = (f(u)\cos v, f(u)\sin v, g(u) + hv), \tag{3.1}$$

or

$$\mathbf{r}(u,v) = (-f(u)\sin v, f(u)\cos v, g(u) + hv), \tag{3.2}$$

where $h \in \mathbb{R}$.

Since helicoidal surfaces given by (3.1) and (3.2) are locally isometric, the helicoidal surfaces given by (3.1) are meaningful for our study. Also, we take $f(u) = u$.

In this case, the components of the first fundamental form of M are given by

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = u^2,$$

and the components of the second fundamental form of M are

$$h_{11} = g'', \quad h_{12} = -\frac{h}{u}, \quad h_{22} = ug'.$$

Since M is a non-degenerate surface, $u \neq 0$. From now on, " $'$ " means the partial derivative with respect to the parameter u unless mentioned otherwise. Thus, the Gaussian curvature K and the mean curvature H of M are given by

$$K = \frac{u^3 g' g'' - h^2}{u^4}, \tag{3.3}$$

and

$$H = \frac{g'}{u} + g''. \tag{3.4}$$

The solution of equation (3.3).

The ODE (3.3) is written as

$$\left((g')^2\right)' = 2uK + \frac{2h^2}{u^3}. \tag{3.5}$$

A direct integration of (3.5) yields

$$g(u) = \pm \int \left(\int 2uK du - \frac{h^2}{u^2} + c_1 \right)^{\frac{1}{2}} du + c_2, \tag{3.6}$$

where c_1 and c_2 are constant.

Conversely, let h be a given non-zero real constant and $K(u)$ be a smooth function defined on an open interval $I \subset (0, +\infty)$. Let

$$F(u, c_1) = \int 2uK du - \frac{h^2}{u^2} + c_1$$

be a function defined on $I \times R \subset R^2$.

For any $u_0 \in I$, denote

$$c_1' = -\left(\int 2uK du\right)(u_0) + \frac{2h^2}{u_0^2}.$$

Thus, we can find an open subinterval $I' \subset I$ containing u_0 and an open interval B of R containing c_1' such that the function $F(u, c_1)$ is positive for any $(u, c_1) \in I' \times B$. In fact, because $F(u_0, c_1') = \frac{h^2}{u_0^2} > 0$, then by the continuity of F, it is positive in a subset of $I' \times B \subset R^2$. Therefore, for any $(u, c_1) \in I' \times B$, $h \in R$, $c_2 \in R$ and given the smooth function $K(u)$ we can define the two-parameter family of curves

$$\gamma(u, K(u), h, c_1, c_2) = \left(u, 0, \int \left(\int 2uK du - \frac{h^2}{u^2} + c_1\right)^{\frac{1}{2}} du + c_2\right).$$

Applying an isotropic motion on these curves we get a two-parameter family of helicoidal surfaces with the Gaussian curvature $K(u)$, $u \in I'$. So we have the following theorem.

Theorem 3.1. *Let $\gamma(u) = (u, 0, g(u))$, $u > 0$ be a profile curve of the helicoidal surface (3.1) of which the Gaussian curvature at the point $(u, 0, g(u))$ is given by $K(u)$. Then, for some constants c_1, c_2 and h, there exists the two-parameter family of the helicoidal surface generated by plane curves*

$$\gamma(u, K(u), h, c_1, c_2) = \left(u, 0, \int \left(\int 2uK du - \frac{h^2}{u^2} + c_1\right)^{\frac{1}{2}} du + c_2\right). \tag{3.7}$$

Conversely, let $K(u)$, $u > 0$ be a smooth function. Then for any $u_0 \in I$ we can construct the two-parameter family of curves $\gamma(u, K(u), h, c_1, c_2), u \in I' \subset I$ and so it is the two-parameter family of helicoidal surfaces with the Gaussian curvature $K(u), u \in I'$.

Remark 3.2. In [1] Aydin classified helicoidal surfaces with constant Gaussian curvature K_0 in $\mathbb{I}^3$. In this case a smooth function g is given by

$$g(u) = \sqrt{au^2 - h^2} + h\tan^{-1}\left(\frac{h}{\sqrt{au^2 - h^2}}\right)$$

if $K_0 = 0$,

$$g(u) = \frac{1}{4}(2\alpha(u) - 2h\tan^{-1}\left(\frac{-2h^2 + cu^2}{2h\alpha(u)}\right) + \frac{c}{\sqrt{K_0}} \ln|c + 2(K_0u^2 + \sqrt{K_0}\alpha(u))|$$

if $K_0 \neq 0$, where $\alpha(u) = \sqrt{K_0u^4 - h^2 + cu^2}$ and $c \in \mathbb{R}$.

In order to compute of a partial solution of (3.6), we put

$$\int 2uK du - \frac{h^2}{u^2} + c_1 = A^2, \tag{3.8}$$

where A is smooth function with variable u.
By differentiating (3.8) with respect to u, the Gaussian curvature K becomes

$$K = \frac{1}{u}AA' - \frac{h^2}{u^4}. \tag{3.9}$$

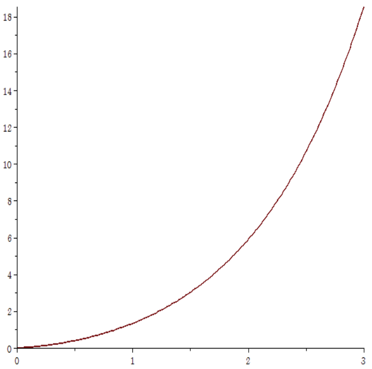

Figure 1.

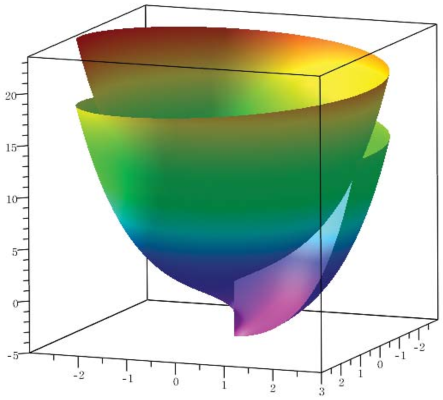

Figure 2.

Example 3.3. We consider a helicoidal surface with the Gaussian curvature $K = \frac{1}{u}e^{2u} - \frac{1}{u^4}$ for $A = e^u$. Then the parametrization of such a surface is given by

$$\mathbf{r}(u,v) = \left(u\cos v, u\sin v, \sqrt{e^{2u}-1} - \tan^{-1}(\sqrt{e^{2u}-1}) + v\right),$$

where $h = 1$, $c_1 = -1$ and $c_2 = 0$ in (3.6). In such case, Figure 1 and Figure 2 are shown the graphs of the profile curve and the surface, respectively.

Example 3.4. We take $A = \frac{1}{u}$ in (3.9). Then the Gaussian curvature K becomes $K = -\frac{2}{u^4}$ and a function g is given by

$$g(u) = \sqrt{1+u^2} - \tanh^{-1}\left(\frac{1}{\sqrt{1+u^2}}\right)$$

with $h = 1, c_1 = 1$ and $c_2 = 0$.

The solution of equation (3.4).

Put $p = g'$ in the ODE (3.4). Then (3.4) is written as

$$p'(u) + \frac{1}{u}p(u) = H(u),$$

and its solution is

$$p(u) = \frac{1}{u}\left(\int uH du + c_1\right).$$

Thus we can obtain a general solution of (3.4) and it is given by

$$g(u) = \int \frac{1}{u}\left(\int uH du + c_1\right) du + c_2, \tag{3.10}$$

where c_1 and c_2 are constant.

Conversely, because $H(u)$ is a smooth function we can define the two-parameter family of curves

$$\gamma(u, H(u), c_1, c_2) = \left(u, 0, \int \frac{1}{u}\left(\int uH du + c_1\right) du + c_2\right)$$

for any $c_1, c_2 \in \mathbb{R}$ and given the smooth function $H(u)$.

Applying an isotropic motion on these curves we get a two-parameter family of helicoidal surfaces with the mean curvature $H(u)$. So we have the following theorem.

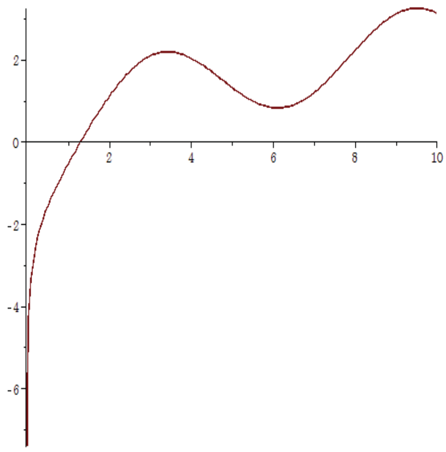

Figure 3.

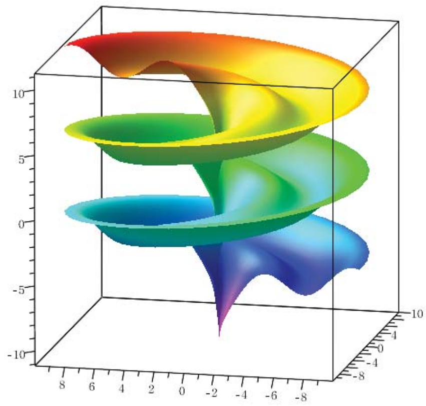

Figure 4.

Theorem 3.5. *Let $\gamma(u) = (u, 0, g(u)), u > 0$ be a profile curve of the helicoidal surface (3.1) of which the mean curvature at the point $(u, 0, g(u))$ is given by $H(u)$. Then, for some constants c_1, c_2, there exists the two-parameter family of helicoidal surfaces generated by plane curves*

$$\gamma(u, H(u), c_1, c_2) = \left(u, 0, \int \frac{1}{u}\left(\int uH du + c_1\right) du + c_2\right).$$

Conversely, let $H(u), u > 0$ be a smooth function. Then we can construct the two-parameter family of curves $\gamma(u, H(u), c_1, c_2)$ and so it is the two-parameter family of helicoidal surfaces with the mean curvature $H(u)$.

Example 3.6. We consider a helicoidal surface with the mean curvature $H(u) = \frac{1}{u}\sin u + \cos u$. Then the parametrization of such a surface is given by

$$\mathbf{r}(u, v) = (u\cos v, u\sin v, -\cos u + \ln u + v)\,,$$

where $h = 1$, $c_1 = 1$ and $c_2 = 0$ in (3.10). In such case, Figure 3 and Figure 4 are shown the graphs of the profile curve and the surface, respectively.

Example 3.7. We consider a helicoidal surface with the mean curvature $H(u) = 3u$. Then a profile curve $\gamma(u)$ of a helicoidal surface is given by

$$\gamma(u) = \left(u, 0, \frac{1}{3}u^3 + \ln u\right),$$

for $c_1 = 1$ and $c_2 = 0$.

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References

[1] M.E. Aydin, *Classification results on surfaces in the isotropic 3-space*, AKU J. Sci. Eng. **16**, 239-246, 2016.

[2] C. Baikoussis and T. Koufogiorgos, T., *Helicoidal surfaces with prescribed mean or Gaussian curvature*, J. Geom. **63**, 25-29, 1988.

[3] Chr.C. Beneki, G. Kaimakamis and B.J. Papantonios, *Helicoidal surfaces in three-dimensional Minkowski space*, J. Math. Anal. Appl. **275**, 586-614, 2002.

[4] R. Caddeo, P. Piu and A. Ratto, *Rotation surfaces in H_3 with constant Gauss curvature*, Bollettino U.M.I. **7**, 341-357, 1996.

[5] C. Delaunay, *Sur la surface de revolution dont la courbure moyenne est constante*, J. Math. Pures Appl. Series 1 **6**, 309-320, 1841.

[6] M.P. do Carmo and M. Dajczer, *Helicoidal surfaces with constant mean curvature*, Tôhoku Math. J. **34** (3), 425-435, 1982.

[7] A. Gray, *Modern differential geometry of curves and surfaces*, CRC Press 1993.

[8] J.-I. Hano and K. Nomizu, *Surfaces of revolution with constant mean curvature in Lorentz-Minkowski space*, Tôhoku Math. J. **36**, 427-437, 1984.

[9] F. Ji and Z.H. Hou, *Helicoidal surfaces under the cubic screw motion in Minkowski 3-space*, J. Math. Anal. Appl. **318**, 634-647, 2006.

[10] H. Pottmann, P. Grohs and N.J. Mitra, *Laguerre minimal surfaces, isotropic geometry and linear elasticity*, Adv. Comput. Math. **31**, 391-419, 2009.

[11] Ž.M. Šipuš, *Translation surfaces of constant curvatures in a simply isotropic space,* Period. Math. Hungar. **68**, 160-175, 2014.

[12] D.W. Yoon, D.-S. Kim, Y.H. Kim and J.W. Lee, *Helicoidal surfaces with prescribed curvature in Nil_3*, International J. Math. **24**, 1350107 (11 pages), 2013.