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Research Article

Reiteration of a limiting real interpolation method with broken iterated logarithmic functions

Irshaad Ahmed*, Fakhra Umar

Department of Mathematics, Government College University, Faisalabad, Pakistan

Abstract

A reiteration theorem for a limiting real interpolation method with broken iterated logarithmic functions is established. An application to the generalized Lorentz-Zygmund spaces is given.

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1. Introduction

Let (A_0, A_1) be a compatible couple of quasi-normed spaces, $0 \le \theta \le 1$, $0 < q \le \infty$, and b be a slowly varying function. The real interpolation space $\bar{A}_{\theta,q;b} = (A_0, A_1)_{\theta,q;b}$ is formed by all those $f \in A_0 + A_1$ for which the quasi-norm

$$\|f\|_{\bar{A}_{\theta,q;b}} = \|t^{-\theta-1/q} b(t) K(t, f; A_0, A_1)\|_{q,(0,\infty)}$$

is finite, where $K(t, f; A_0, A_1)$ is the Peetre's K-functional and $\|\cdot\|_{q,(a,b)}$ is the standard L^q-quasi-norm on an interval $(a, b) \subset \mathbb{R}$.

Different reiteration theorems for the interpolation spaces $(\bar{A}_{\theta_0,q_0;b_0}, A_1)_{\theta,q;b}$, in limiting cases (when $\theta_0 \in \{0, 1\}$ or $\theta \in \{0, 1\}$), have been established in [16], [1] and [2]. The results in these papers generalize the earlier results in [8] and [9], where the case when b is a broken logarithmic function was treated. In papers [11–13], similar reiteration theorems have been derived for the extended scale $\bar{A}_{\theta,b,E}$, which is obtained by replacing Lebesgue space L^q by an arbitrary rearrangement invariant normed space E. However, the scale $\bar{A}_{\theta,b,E}$ does not cover the spaces $\bar{A}_{\theta,q;b}$ for the case $q < 1$.

In the present paper, we are interested in the limiting case $\theta_0 = \theta = 0$. In general, the spaces $(\bar{A}_{0,q_0;b_0}, A_1)_{0,q;b}$ might not belong to the original scale and a new interpolation scale is needed to describe them (see the reiteration formula (3.53) in [1]). The main result of our paper (see Theorem 3.1 below) asserts that the spaces $(\bar{A}_{0,q_0;b_0}, A_1)_{0,q;b}$ do belong to the original scale when b_0 and b are taken, in particular, to be the iterated logarithmic functions which are broken in the sense that they are raised to different powers near 0 and near infinity.

*Corresponding Author.
Email addresses: quantized84@yahoo.com (I. Ahmed), fakhra.umar@yahoo.com (F. Umar)
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The motivation for the use of iterated logarithmic functions mainly comes from the paper [6], where the reiteration theorem for the case $(\theta_0, \theta) \in \{1\} \times (0, 1]$ has been established for such functions. It should be mentioned that this reiteration theorem has subsequently been extended (for the case when $0 < q, q_0 < \infty$) in [2] to general slowly varying functions. The reader is also referred to [5], [10], [14] and [15], where limiting real interpolation methods involving iterated logarithmic functions have appeared.

When $q < q_0$, a Hardy-type inequality restricted to non-negative, non-increasing functions (see Lemma 2.4 below) has been applied to obtain reiteration theorems in [16], [1] and [2]. However, this inequality is not applicable in our limiting case ($\theta_0 = \theta = 0$) even when b_0 and b are just logarithmic functions. Therefore, we make use of another Hardy-type inequality restricted to non-negative, non-decreasing functions (see Lemma 2.5 below).

The paper is organised as follows. Section 2 contains all the necessary background along with the Hardy-type inequalities mentioned above. The main result of the paper is in Section 3 where we prove the reiteration theorem. Finally, in Section 4, we give an application of our main result to the interpolation of the generalized Lorentz-Zygmund spaces.

2. Preliminaries

In what follows, we will use the notation $A \lesssim B$ for non-negative quantities to mean that there is a positive constant c, which is independent of appropriate parameters involved in A and B, such that $A \le cB$. If $A \lesssim B$ and $B \lesssim A$, we put $A \approx B$.

Let $0 < q < \infty$, and $\bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$. Following [6], we say $\bar{\alpha} \in \mathbb{M}_{q,r,S}$ (or $\bar{\alpha} \in \mathbb{M}_{q,r,G}$) for some $2 \le r \le n$ if $\alpha_1 = ... = \alpha_{r-1} = -1/q$ and $\alpha_r < -1/q$ (or $\alpha_r > -1/q$). By $\bar{\alpha} \in \mathbb{M}_{q,1,S}$ (or $\bar{\alpha} \in \mathbb{M}_{q,1,G}$), we will mean $\alpha_1 < -1/q$ (or $\alpha_1 > -1/q$). Moreover, we will write $\bar{\alpha} =< \alpha >_n$ if $\alpha_1 = ... = \alpha_n = \alpha$. Define positive functions $\lambda_1, \lambda_2, ..., \lambda_n$ on $(0, \infty)$ by

$$\lambda_1(t) = 1 + |\ln t|, \quad \lambda_k(t) = \lambda_1(\lambda_{k-1}(t)), \quad k = 2, 3, ..., n,$$

and put $\lambda^{\bar{\alpha}}(t) = \lambda_1^{\alpha_1}(t)\lambda_2^{\alpha_2}(t)...\lambda_n^{\alpha_n}(t),\ t > 0$. It is easy to check that the iterated logarithmic function $\lambda^{\bar{\alpha}}$ is slowly varying in the sense of [16, Definition 2.1].

We omit the proof of the next lemma since it can be done as in [6, Lemma 2].

Lemma 2.1. *Let* $0 < q < \infty$.

(a) *If* $\bar{\alpha} \in \mathbb{M}_{q,r,G}$, *then*

$$1 + \left(\int_t^1 \lambda^{q\bar{\alpha}}(u)\frac{du}{u}\right)^{1/q} \approx \lambda^{\bar{\alpha} + <\frac{1}{q}>_r}(t), \quad 0 < t < 1. \tag{2.1}$$

(b) *If* $\bar{\alpha} \in \mathbb{M}_{q,r,S}$, *then*

$$\left(\int_t^\infty \lambda^{q\bar{\alpha}}(u)\frac{du}{u}\right)^{1/q} \approx \lambda^{\bar{\alpha} + <\frac{1}{q}>_r}(t), \quad t \ge 1. \tag{2.2}$$

Let (A_0, A_1) be a compatible couple of quasi-normed spaces, that is, we assume that both A_0 and A_1 are continuously embedded in the same Hausdorff topological vector space. The Peetre's K-functional is defined, for each $f \in A_0 + A_1$ and $t > 0$, by

$$\begin{aligned} K(t, f) &= K(t, f; A_0, A_1) \\ &= \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} :\ f_0 \in A_0,\ f_1 \in A_1,\ f = f_0 + f_1\}. \end{aligned}$$

Note that, as functions of t, $K(t, f)$ is non-decreasing and $K(t, f)/t$ is non-increasing.

Definition 2.2. Let $0 < q \le \infty$, $0 \le \theta \le 1$, $\bar{\alpha} \in \mathbb{R}^n$ and $\bar{\beta} \in \mathbb{R}^m$. The real interpolation space $\bar{A}_{\theta,q;\bar{\alpha},\bar{\beta}} = (A_0, A_1)_{\theta,q;\bar{\alpha},\bar{\beta}}$ consists of all $f \in A_0 + A_1$ for which the quasi-norm

$$\|f\|_{\bar{A}_{\theta,q;\bar{\alpha},\bar{\beta}}} = \|t^{-\theta-1/q}\lambda^{\bar{\alpha}}(t)K(t, f)\|_{q,(0,1)} + \|t^{-\theta-1/q}\lambda^{\bar{\beta}}(t)K(t, f)\|_{q,(1,\infty)}$$

is finite.

The spaces $\bar{A}_{\theta,q;\bar{\alpha},\bar{\beta}}$ are a particular case of the scale $\bar{A}_{\theta,q;b}$. When $\bar{\alpha} = \bar{\beta} = (0)$ with $0 < \theta < 1$, we get back the classical scale $\bar{A}_{\theta,q}$ (see [3, 4, 18]). For $\bar{\alpha}=(\alpha)$ and $\bar{\beta}=(\beta)$, we will use the notation $\bar{A}_{\theta,q;\alpha,\beta}$, and this is the case considered in [8] and [9].

In view of Proposition 2.5 (ii) in [16] and Lemma 2.1 (b), the condition $\bar{\beta} \in \mathbb{M}_{q,r,S}$, for some $1 \le r \le m$, will guarantee that the limiting spaces $\bar{A}_{0,q;\bar{\alpha},\bar{\beta}}$ are intermediate for the couple (A_0, A_1), that is,

$$A_0 \cap A_1 \hookrightarrow \bar{A}_{0,q;\bar{\alpha},\bar{\beta}} \hookrightarrow A_0 + A_1,$$

where the symbol $\hookrightarrow$ denotes the continuous embedding. A similar remark applies in the limiting case $\theta = 1$.

We conclude this section with the following weighted Hardy-type inequalities which will be the key ingredients in our proofs.

Lemma 2.3 ([1, Lemma 3.2])**.** *Let $1 \le s < \infty$, and assume that w, ϕ and h are non-negative functions on $(0, \infty)$. Then*

$$\int_0^\infty \left(\int_0^t \phi(u)h(u)du \right)^s w(t)dt \lesssim \int_0^\infty h^s(t)v(t)dt, \tag{2.3}$$

where

$$v(t) = (w(t))^{1-s} \left(\phi(t) \int_t^\infty w(u)du \right)^s . \tag{2.4}$$

Lemma 2.4 ([1, Lemma 3.3])**.** *Let $0 < s < 1$. Assume that w and ϕ non-negative functions on $(0, \infty)$, and h is a non-negative, non-increasing function on $(0, \infty)$. Then*

$$\int_0^\infty \left(\int_0^t \phi(u)h(u)du \right)^s w(t)dt \lesssim \int_0^\infty h^s(t)v_0(t)dt, \tag{2.5}$$

where

$$v_0(t) = \phi(t) \left(\int_0^t \phi(u)du \right)^{s-1} \int_t^\infty w(u)du. \tag{2.6}$$

Lemma 2.5 ([17, Theorem 3.3 (b)])**.** *Let $0 < s < 1$. Assume that w and v_1 are non-negative functions on $(0, \infty)$, and ψ is a non-negative function on $(0, \infty) \times (0, \infty)$. Then*

$$\int_0^\infty \left(\int_0^\infty \psi(t, u)h(u)du \right)^s w(t)dt \lesssim \int_0^\infty h^s(t)v_1(t)dt \tag{2.7}$$

holds for all non-negative, non-decreasing functions h on $(0, \infty)$ if and only if

$$\int_0^\infty \left(\int_x^\infty \psi(t, u)du \right)^s w(t)dt \lesssim \int_x^\infty v_1(t)dt \tag{2.8}$$

holds for all $x > 0$.

3. Reiteration

The following reiteration theorem is the main contribution of this paper.

Theorem 3.1. *Let $0 < q_0, q < \infty$. If $\bar{\alpha}_0 \in \mathbb{M}_{q_0,j,G}$, $\bar{\alpha} \in \mathbb{M}_{q,j,G}$, $\bar{\beta}_0 \in \mathbb{M}_{q_0,k,S}$, and $\bar{\beta} \in \mathbb{M}_{q,k,S}$, then*

$$(\bar{A}_{0,q_0;\bar{\alpha}_0,\bar{\beta}_0}, A_1)_{0,q;\bar{\alpha},\bar{\beta}} = \bar{A}_{0,q;\bar{\gamma},\bar{\eta}},$$

where $\bar{\gamma} = \bar{\alpha} + \bar{\alpha}_0 + < \frac{1}{q_0} >_j$ and $\bar{\eta} = \bar{\beta} + \bar{\beta}_0 + < \frac{1}{q_0} >_k$.

Proof. Put

$$b_0(t) = \begin{cases} \lambda^{\bar{\alpha}_0}(t), & 0 < t < 1, \\ \lambda^{\bar{\beta}_0}(t), & t \ge 1, \end{cases}$$

$$b(t)=\begin{cases}\lambda^{\bar{\alpha}}(t), & 0<t<1,\\ \lambda^{\bar{\beta}}(t), & t\geq 1,\end{cases}$$

and $\bar{A}=(\bar{A}_{0,q_0;\bar{\alpha}_0,\bar{\beta}_0},A_0)_{0,q;\bar{\alpha},\bar{\beta}}$. Let $f\in A_0+A_1$. According to the reiteration formula (3.53) in [1], we have

$$\|f\|^q_{\bar{A}}\approx C+D,$$

where

$$C=\int_0^\infty b^q(\rho_0(t))\left(\int_0^t b_0^{q_0}(u)K^{q_0}(u,f)\frac{du}{u}\right)^{q/q_0}\frac{dt}{t},$$

and

$$D=\int_0^\infty t^{-q}b^q(\rho_0(t))\rho_0^q(t)K^q(t,f)\frac{dt}{t},$$

with

$$\rho_0(t)=t\left(\int_t^\infty b_0^{q_0}(u)\frac{du}{u}\right)^{1/q_0},\quad t>0.$$

Thanks to the conditions $\bar{\alpha}_0\in\mathbb{M}_{q_0,j,G}$ and $\bar{\beta}_0\in\mathbb{M}_{q_0,k,S}$, we can apply Lemma 2.1 to obtain that

$$\rho_0(t)\approx\begin{cases}t\lambda^{\bar{\alpha}_0+<\frac{1}{q_0}>_j}(t), & 0<t<1,\\ t\lambda^{\bar{\beta}_0+<\frac{1}{q_0}>_k}(t), & t\geq 1.\end{cases}$$

In view of the observation $|\ln\rho_0(t)|\approx|\ln t|,\ \ t>0$, we get

$$C\approx\int_0^\infty b^q(t)\left(\int_0^t b_0^{q_0}(u)K^{q_0}(u,f)\frac{du}{u}\right)^{q/q_0}\frac{dt}{t},$$

and

$$D\approx\int_0^\infty t^{-q}b^q(t)\rho_0^q(t)K^q(t,f)\frac{dt}{t}.$$

Since

$$\|f\|^q_{\bar{A}_{0,q;\bar{\gamma},\bar{\eta}}}\approx D,$$

thus the proof will be complete if we show that $C\lesssim D$. Now $C\approx C_1+C_2+C_3$ and $D\approx D_1+D_2$, where

$$C_1=\int_0^1\lambda^{q\bar{\alpha}}(t)\left(\int_0^t\lambda^{q_0\bar{\alpha}_0}(u)K^{q_0}(u,f)\frac{du}{u}\right)^{q/q_0}\frac{dt}{t},$$

$$C_2=\int_1^\infty\lambda^{q\bar{\beta}}(t)\left(\int_0^1\lambda^{q_0\bar{\alpha}_0}(u)K^{q_0}(u,f)\frac{du}{u}\right)^{q/q_0}\frac{dt}{t},$$

$$C_3=\int_1^\infty\lambda^{q\bar{\beta}}(t)\left(\int_1^t\lambda^{q_0\bar{\beta}_0}(u)K^{q_0}(u,f)\frac{du}{u}\right)^{q/q_0}\frac{dt}{t},$$

$$D_1=\int_0^1\lambda^{q\bar{\alpha}+q\bar{\alpha}_0+<\frac{q}{q_0}>_j}(t)K^q(t,f)\frac{dt}{t},$$

and

$$D_2=\int_1^\infty\lambda^{q\bar{\beta}+q\bar{\beta}_0+<\frac{q}{q_0}>_k}(t)K^q(t,f)\frac{dt}{t}.$$

Since the condition $\bar{\beta}\in\mathbb{M}_{q,k,S}$ implies the convergence of the integral $\int_1^\infty\lambda^{q\bar{\beta}}(t)\frac{dt}{t}$, thus

$$C_2\approx\left(\int_0^1\lambda^{q_0\bar{\alpha}_0}(u)K^{q_0}(u,f)\frac{du}{u}\right)^{q/q_0}.$$

Next we observe that

$$t \longmapsto \frac{1}{t^{q_0}\lambda^{q_0\bar{\alpha}_0}(t)} \int_0^t \lambda^{q_0\bar{\alpha}_0}(u) K^{q_0}(u,f)\frac{du}{u}$$

is non-increasing since it is an integral average (with respect to the measure $t^{q_0-1}\lambda^{q_0\bar{\alpha}_0}(t)$) of a non-increasing function $t^{-q_0}K^{q_0}(t,f)$. As a consequence,

$$C_1 \geq \left(\int_0^1 \lambda^{q_0\bar{\alpha}_0}(u)K^{q_0}(u,f)\frac{du}{u}\right)^{q/q_0} \int_0^1 t^q \lambda^{q(\bar{\alpha}+\bar{\alpha}_0)}(t)\frac{dt}{t},$$

whence $C_2 \lesssim C_1$. Therefore, $C \approx C_1 + C_3$. Next we establish the estimates $C_1 \lesssim D_1 + D_2$ and $C_3 \lesssim D_2$. To this end, we distinguish two cases: $q \geq q_0$ and $q < q_0$. Assume first that $q \geq q_0$, and apply Lemma 2.3 with $s = q/q_0$, $h(t) = K^{q_0}(t,f)$, $\phi(t) = t^{-1}\lambda^{q_0\bar{\alpha}_0}(t)$ and $w(t) = t^{-1}\lambda^{q\bar{\alpha}}(t)\chi_{(0,1)}(t)$. We compute, with the aid of Lemma 2.1 (a), that

$$v(t) \lesssim t^{-1}\lambda^{q\bar{\alpha}+q\bar{\alpha}_0+<\frac{q}{q_0}>_j}(t),\ 0<t<1,$$

which implies that $C_1 \lesssim D_1$. Similarly, $C_3 \lesssim D_2$ follows from Lemma 2.3. Next assume that $q < q_0$, and take $s = q/q_0$, $h(t) = K^{q_0}(t,f)$, $w(t) = t^{-1}\lambda^{q\bar{\alpha}}(t)\chi_{(0,1)}(t)$, $\psi(t,u) = u^{-1}\lambda^{q_0\bar{\alpha}_0}(u)\chi_{(0,t)}(u)$, and

$$v_1(t) = \begin{cases} t^{-1}\lambda^{q\bar{\alpha}+q\bar{\alpha}_0+<\frac{q}{q_0}>_j}(t), & 0<t<1, \\ t^{-1}\lambda^{q\bar{\beta}+q\bar{\beta}_0+<\frac{q}{q_0}>_k}(t), & t \geq 1. \end{cases}$$

We observe that (2.8) holds trivially for all $x \geq 1$, and for $0 < x < 1$, we have

$$\begin{aligned} \int_0^\infty \left(\int_x^\infty \psi(t,u)du\right)^s w(t)dt &\approx \int_x^1 \left(\int_x^t \lambda^{q_0\bar{\alpha}_0}(u)\frac{du}{u}\right)^{q/q_0} \lambda^{q\bar{\alpha}}(t)\frac{dt}{t} \\ &\leq \left(\int_x^1 \lambda^{q_0\bar{\alpha}_0}(u)\frac{du}{u}\right)^{q/q_0} \int_x^1 \lambda^{q\bar{\alpha}}(t)\frac{dt}{t} \\ &\lesssim \lambda^{q\bar{\alpha}+q\bar{\alpha}_0+<\frac{q}{q_0}>_j+<1>_j}(t) \\ &\approx 1 + \int_x^1 \lambda^{q\bar{\alpha}+q\bar{\alpha}_0+<\frac{q}{q_0}>_j}(t)\frac{dt}{t} \\ &\approx \int_x^\infty v_1(t)dt, \end{aligned}$$

which establishes the validity of (2.8). Hence, the estimate $C_1 \lesssim D_1 + D_2$ follows from (2.7). Similarly, we can obtain $C_3 \lesssim D_2$ from Lemma 2.5. The proof of the theorem is complete. □

Writing down Theorem 3.1 in a particular case when $\bar{\alpha}_0 = (\alpha_0)$, $\bar{\alpha} = (\alpha)$, $\bar{\beta}_0 = (\beta_0)$ and $\bar{\beta} = (\beta)$, we get the following result which is not contained in [8] and [9].

Corollary 3.2. *Let* $0 < q_0, q < \infty$. *If* $\alpha_0 > -1/q_0$, $\alpha > -1/q$, $\beta_0 < -1/q_0$ *and* $\beta < -1/q$, *then*

$$(\bar{A}_{0,q_0;\alpha_0,\beta_0}, A_1)_{0,q;\alpha,\beta} = \bar{A}_{0,q;\gamma,\eta},$$

where $\gamma = \alpha + \alpha_0 + \frac{1}{q_0}$ *and* $\eta = \beta + \beta_0 + \frac{1}{q_0}$.

The next result is a symmetric counterpart of Theorem 3.1, and its proof can be derived from Theorem 3.1 by using the same symmetry argument as in the proof of Theorem 4.3 in [16].

Theorem 3.3. *Let* $0 < q_1, q < \infty$. *If* $\bar{\alpha}_1 \in \mathbb{M}_{q_1,j,S}$, $\bar{\alpha} \in \mathbb{M}_{q,j,S}$, $\bar{\beta}_1 \in \mathbb{M}_{q_1,k,G}$, *and* $\bar{\beta} \in \mathbb{M}_{q,k,G}$, *then*

$$(A_0, \bar{A}_{1,q_1;\bar{\alpha}_1,\bar{\beta}_1})_{1,q;\bar{\alpha},\bar{\beta}} = \bar{A}_{1,q;\bar{\gamma},\bar{\eta}},$$

where $\bar{\gamma} = \bar{\alpha} + \bar{\alpha}_0 + < \frac{1}{q_1} >_j$ *and* $\bar{\eta} = \bar{\beta} + \bar{\beta}_0 + < \frac{1}{q_1} >_k$.

4. Application

Let (Ω, μ) be a σ-finite measure space. Let f^* denotes the non-increasing rearrangement of a μ-measurable function f on Ω (see, for instance, [3]).

Definition 4.1. Let $0 < p, q \leq \infty$, $\bar{\alpha} \in \mathbb{R}^n$ and $\bar{\beta} \in \mathbb{R}^m$. The generalized Lorentz-Zygmund space $L_{p,q;\bar{\alpha},\bar{\beta}}$ consists of all μ-measurable functions f on Ω such that the quasi-norm

$$\|f\|_{L_{p,q;\bar{\alpha},\bar{\beta}}} = \|t^{1/p-1/q}\lambda^{\bar{\alpha}}(t)f^*(t)\|_{q,(0,1)} + \|t^{1/p-1/q}\lambda^{\bar{\beta}}(t)f^*(t)\|_{q,(1,\infty)}$$

is finite.

The spaces $L_{p,q;\bar{\alpha},\bar{\beta}}$ are a particular case of the more general scale of the Lorentz-Karamata space (see, for example, [16]). For $\bar{\alpha} = \bar{\beta}$, the spaces $L_{p,q;\bar{\alpha},\bar{\beta}}$ coincide with the spaces $L_{p,q;\bar{\alpha}}$ from [7]. If $\bar{\alpha} = (\alpha)$ and $\bar{\beta} = (\beta)$, then we obtain the spaces $L_{p,q;\mathbb{A}}$, $\mathbb{A} = (\alpha, \beta)$, considered in [8] and [9]. When $\bar{\alpha} = \bar{\beta} = (0)$, the spaces $L_{p,q;\bar{\alpha},\bar{\beta}}$ become the Lorentz spaces $L^{p,q}$, which coincide with the classical Lebesgue spaces L^p for $p = q$.

The next theorem provides an application of Theorem 3.1 to the interpolation of the generalized Lorentz-Zygumnd spaces in a limiting case which is not covered by the results, concerning the interpolation of the Lorentz-Karamata spaces, in [16].

Theorem 4.2. *Let* $0 < q_0, q < \infty$. *Assume that* $\bar{\alpha}_0 \in \mathbb{M}_{q_0,k,S}$, $\bar{\alpha} \in \mathbb{M}_{q,j,G}$, $\bar{\beta}_0 \in \mathbb{M}_{q_0,j,G}$ *and* $\bar{\beta} \in \mathbb{M}_{q,k,S}$. *Then*

$$(L_{\infty,q_0;\bar{\alpha}_0,\bar{\beta}_0}, L^1)_{0,q;\bar{\alpha},\bar{\beta}} = L_{\infty,q;\bar{\gamma},\bar{\eta}},$$

where $\bar{\gamma} = \bar{\beta} + \bar{\alpha}_0 + < 1/q_0 >_k$ *and* $\bar{\eta} = \bar{\alpha} + \bar{\beta}_0 + < 1/q_0 >_j$.

Proof. Take $A_0 = L^\infty$ and $A_1 = L^1$, and apply Theorem 3.1 to obtain

$$((L^\infty, L^1)_{0,q_0;\bar{\beta}_0,\bar{\alpha}_0}, L^1)_{0,q;\bar{\alpha},\bar{\beta}} = (L^\infty, L^1)_{0,q;\bar{\eta},\bar{\gamma}}.$$

Put $X_1 = (L^\infty, L^1)_{0,q_0;\bar{\beta}_0,\bar{\alpha}_0}$ and $X_2 = L_{\infty,q_0;\bar{\alpha}_0,\bar{\beta}_0}$, and let $f \in L^1 + L^\infty$. Since (see [4, Theorem 5.2.1])

$$K(t, f; L^\infty, L^1) = t\int_0^{1/t} f^*(u)du,$$

it turns out that

$$\|f\|_{X_1} = \left(\int_0^\infty t^{-q} b_0^q(t)\left(\int_0^t f^*(u)du\right)^q \frac{dt}{t}\right)^{1/q},$$

where

$$b_0(t) = \begin{cases} \lambda^{\bar{\alpha}_0}(t), & 0 < t < 1, \\ \lambda^{\bar{\beta}_0}(t), & t \geq 1. \end{cases}$$

Now the estimate $\|f\|_{X_1} \geq \|f\|_{X_2}$ is trivial, and the converse estimate follows from Lemmas 2.3 and 2.4, applied with $s = q$, $w(t) = t^{-q-1}b_0^q(t)$, $\phi(t) = 1$ and $h(t) = f^*(t)$, so that $v(t) \approx v_0(t) \approx t^{-1}b_0^q(t)$. Therefore, the space $(L^\infty, L^1)_{0,q;\bar{\beta}_0,\bar{\alpha}_0}$ coincides with the space $L_{\infty,q_0;\bar{\alpha}_0,\bar{\beta}_0}$. Similarly, we have $(L^\infty, L^1)_{0,q;\bar{\eta},\bar{\gamma}} = L_{\infty,q;\bar{\gamma},\bar{\eta}}$. The proof is complete. □

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