Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (4) (2019), 985 – 995
DOI : 10.15672/HJMS.2018.556

Research Article

Super-biderivations and linear super-commuting maps on the N=2 superconformal algebra

Xiao Cheng, Jiancai Sun*

Department of Mathematics, Shanghai University, Shanghai 200444, China

Abstract

In this paper, we first determine all the super-skewsymmetric super-biderivations of the N=2 superconformal algebra $\mathfrak{g}$. As an application of super-biderivations, we give the explicit form of each linear super-commuting map on the N=2 superconformal algebra $\mathfrak{g}$. We prove that every super-biderivation of the N=2 superconformal algebra $\mathfrak{g}$ is inner, but the corresponding super-commuting maps are non-standard.

Mathematics Subject Classification (2010). 17B05, 17B40, 17B68

Keywords. Lie super-algebra, super-biderivation, super-skewsymmetric, linear super-commuting map

1. Introduction

Lie superalgebras as a generalization of Lie algebras came from supersymmetry in mathematical physics. The theory of Lie superalgebras plays prominent roles in modern mathematics and physics. For basic concepts and usual notations on Lie superalgebras, we refer to [7] for details. Now, we first give the concept of super-commuting map on a Lie superalgebra, which has been given in [9] and [4], independently.

Let $\mathcal{L}$ be a Lie superalgebra with $\mathbb{Z}_2$-grading $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$, where $\mathcal{L}_{\bar{0}}$ and $\mathcal{L}_{\bar{1}}$ are even and odd parts of $\mathcal{L}$, respectively. We call a map $\psi : \mathcal{L} \to \mathcal{L}$ *super-commuting* if it preserves the $\mathbb{Z}_2$-grading of $\mathcal{L}$ and

$$[\psi(x), x] \quad = \quad 0 \qquad \text{for all} \quad x \in \mathcal{L}. \tag{1.1}$$

This concept can be also viewed as a parallel concept in the associative superalgebra setting, which seems to have first appeared in [5].

For a fixed linear super-commuting map ψ on a Lie superalgebra $\mathcal{L}$, we called ψ *standard* if it maps the even part $\mathcal{L}_{\bar{0}}$ of $\mathcal{L}$ to the center of $\mathcal{L}$, and maps the odd part $\mathcal{L}_{\bar{1}}$ of $\mathcal{L}$ to zero. All super-commuting maps of other forms are said to be *non-standard*. It was shown in [9] that all linear super-commuting maps on the super-Virasoro algebras are standard.

*Corresponding Author.
Email addresses: 717436110@qq.com (X. Cheng), jcsun@shu.edu.cn (J. Sun)
Received: 15.07.2017; Accepted: 05.02.2018

In this paper, we focus on another Lie superalgebra, i.e, the N=2 superconformal algebra, which is denoted by $\mathfrak{g}$. The N=2 superconformal algebra were discovered independently by Ademollo et al. [1] and by Kac [7], and it is the Lie superalgebra with the following (anti)commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c, \tag{1.2}$$

$$[L_m, G_r^+] = (\frac{m}{2} - r)G_{m+r}^+, \tag{1.3}$$

$$[L_m, G_r^-] = (\frac{m}{2} - r)G_{m+r}^-, \tag{1.4}$$

$$[L_m, H_n] = -nH_{m+n}, \tag{1.5}$$

$$[G_r^-, G_s^+] = 2L_{r+s} - (r-s)H_{r+s} + \frac{1}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}c, \tag{1.6}$$

$$[H_m, H_n] = \frac{1}{3}\delta_{m+n,0}c, \tag{1.7}$$

$$[H_m, G_r^+] = G_{m+r}^+, \tag{1.8}$$

$$[H_m, G_r^-] = -G_{m+r}^- \tag{1.9}$$

for $m, n \in \mathbb{Z}, r, s \in \frac{1}{2}\mathbb{Z}$. Moreover, $\mathfrak{g}$ is $\mathbb{Z}_2$-graded and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = \mathrm{span}\{L_m, H_n, c \mid m, n \in \mathbb{Z}\}$ and $\mathfrak{g}_{\bar{1}} = \mathrm{span}\{G_r^+, G_s^- \mid r, s \in \frac{1}{2}\mathbb{Z}\}$. We denoted the Neveu-Schwarz N=2 algebra and the Ramond N=2 algebra by NS and R, respectively, where $\mathrm{NS} = \mathrm{span}\{L_m, H_n, G_r^+, G_s^-, c \mid m, n \in \mathbb{Z}, r, s \in \frac{1}{2} + \mathbb{Z}\}$ and $\mathrm{R} = \mathrm{span}\{L_m, H_n, G_r^+, G_s^-, c \mid m, n, r, s \in \mathbb{Z}\}$.

In this paper, we mainly study the super-biderivations and the linear super-commuting maps on the N=2 superconformal algebra $\mathfrak{g}$. As Brešar [3] concluded, the concept of biderivation is an effective tool for studying commuting maps on associative algebras [2] and Lie algebras [6,8,9]. Similarly, the concept of super-biderivation plays the same roles in associative superalgebras and Lie superalgebras when studying the linear super-commuting maps. As a generalization of biderivations of Lie algebras and a parallel concept of super-biderivation of an associative superalgebra, the concept of super-biderivation of a Lie superalgebra was introduced in [9] and [4] indepently as follows.

We call a bilinear map $\varphi : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ a *super-biderivation* of $\mathcal{L}$ if for every $x_{\bar{0}} \in \mathcal{L}_{\bar{0}}$ the maps $x \mapsto \varphi(x_{\bar{0}}, x)$ and $x \mapsto \varphi(x, x_{\bar{0}})$ are even superderivations, and for every $x_{\bar{1}} \in \mathcal{L}_{\bar{1}}$ the maps $x \mapsto \varphi(x_{\bar{1}}, x)$ and $x \mapsto \varphi(x, x_{\bar{1}})^\sigma$ are odd superderivations, where σ is defined by $(x_{\bar{0}} + x_{\bar{1}})^\sigma = x_{\bar{0}} - x_{\bar{1}}$ for $x_{\bar{0}} \in \mathcal{L}_{\bar{0}}$ and $x_{\bar{1}} \in \mathcal{L}_{\bar{1}}$. One can easily check that this is equivalent to

$$\varphi([x, y], z) = [x, \varphi(y, z)] + (-1)^{|y||z|}[\varphi(x, z), y], \tag{1.10}$$

$$\varphi(x, [y, z]) = [\varphi(x, y), z] + (-1)^{|x||y|}[y, \varphi(x, z)] \tag{1.11}$$

for all $x, y, z \in \mathcal{L}$. Here, and in what follows, we use the nonation $|x|$ (0 or 1) to denote the $\mathbb{Z}_2$-degree of a homogeneous element $x \in \mathcal{L}$, and we always assume that x is homogeneous if $|x|$ appears in an expression. Clearly, the map φ_λ with $\lambda \in \mathbb{C}$ sending $(x, y) \in \mathcal{L} \times \mathcal{L}$ to $\lambda[x, y] \in \mathcal{L}$ is a super-biderivation of $\mathcal{L}$. We call φ_λ an *inner super-biderivation* of $\mathcal{L}$.

From the definition of super-biderivetions above, we note that $|\varphi(x, y)| = |x| + |y|$ for any homogeneous elements $x, y \in \mathcal{L}$. This is very important in the proof of Theorem 3.1 in Section 3.

This paper is organized as follows. In Section 2, we review some general results on super-biderivations of Lie superalgebras. Then, in Section 3, we determine the super-skewsymmetric super-biderivations of the N=2 superconformal algebra $\mathfrak{g}$ and we prove that every super-skewsymmetric super-biderivation of the N=2 superconformal algebra $\mathfrak{g}$ is inner. Based on the results of super-biderivations, finally, in Section 4, we give the explicit form of each linear super-commuting map on the N=2 superconformal algebra $\mathfrak{g}$ and find that each linear super-commuting on it is non-standard. Besides, comparing the main results with those for the Virasoro algebra in [8] and the super-Virasoro algebra in [9], we can get the following conclusion (see Table 1 and Table 2).

$\mathcal{L}$	Answer for question Q1	Reference
Virasoro algebra	Yes	[8, Theorem 2.3]
super-Virasoro algebra	Yes	[9, Theorem 3.1]
N=2 superconformal algebra	Yes	3.1 Theorem

Table 1: (Q1) Whether all the (super-)skewsymmetric (super-)biderivations of $\mathcal{L}$ are inner?

$\mathcal{L}$	Answer for question Q2	Reference
Virasoro algebra	Yes	[8, Theorem 3.1]
super-Virasoro algebra	Yes	[9, Theorem 4.1]
N=2 superconformal algebra	No	Theorem 4.1

Table 2: (Q2) Whether all the linear (super-)commuting maps on $\mathcal{L}$ are standard?

Throughout this paper, we denoted by $\mathbb{C}, \mathbb{Z}$ the sets of complex numbers, integers, respectively. We assume that all vector spaces are based on $\mathbb{C}$, unless otherwise stated.

2. General results on the N=2 superconformal algebra $\mathfrak{g}$

The definition of super-skewsymmetric bilinear maps on Lie superalgebras was introduced in [9] and [4], independently. Explicitly, let $\mathcal{S}$ be a Lie superalgebra with the center $Z(\mathcal{S})$. A bilinear map $\varphi : \mathcal{S}\times\mathcal{S} \to \mathcal{S}$ is called *super-skewsymmetric* (or *super-antisymmetric*) if

$$\varphi(x,y) = -(-1)^{|x||y|}\varphi(y,x) \qquad \text{for all } x,y \in \mathcal{S}.$$

Similarly, to avoid lengthy notations, we set

$$F(x,y,u,v) = (-1)^{|u||y|}([\varphi(x,y),[u,v]] - [[x,y],\varphi(u,v)]) \qquad \text{for } x,y,u,v \in \mathcal{S}.$$

The following two results are lifted from [9].

Lemma 2.1. *Let φ be a super-biderivation on $\mathcal{S}$. Then*

$$F(x,y,u,v) = (-1)^{|y||v|}F(x,v,u,y) \qquad \textit{for } x,y,u,v \in \mathcal{S}.$$

In particular, if φ is super-skewsymmetric, then $F(x,y,u,v) = 0$.

Corollary 2.2. *Let φ be a super-skewsymmetric super-biderivation on $\mathcal{S}$.*

(1) *For $x,y \in \mathcal{S}$, if $|x| + |y| = 0$, then $[\varphi(x,y),[x,y]] = 0$.*

(2) *Suppose $\mathcal{S}$ is perfect. For $x,y \in \mathcal{S}$, if $[x,y] = 0$, then $\varphi(x,y) \in Z(\mathcal{S})$.*

3. Super-biderivations of the N=2 superconformal algebra $\mathfrak{g}$

In this section, we give a description of the super-biderivations of the N=2 superconformal algebra $\mathfrak{g}$. First, let us list some useful facts on the N=2 superconformal algebra $\mathfrak{g}$, which can be easily checked by Eqs.(1.2)-(1.9).

- $\mathfrak{g}$ is perfect, namely, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.
- If $\mathfrak{g}$ is the Lie superalgebra of the Neveu-Schwarz N=2 type (denoted by NS), then $\mathfrak{g}$ can be generated by

$$A_{NS} := \{L_1, L_{\pm 2}, G^{+}_{\frac{1}{2}}, G^{-}_{\frac{1}{2}}\}.$$

- If $\mathfrak{g}$ is the Lie superalgebra of the Ramond N=2 type (denoted by R), then $\mathfrak{g}$ can be generated by
$$A_R := \{L_1, L_{\pm 2}, G_1^+, G_1^-\}.$$

The map $(x, y) \mapsto [x, y]$ is an inner super-biderivation as we mentioned in the introduction. Obviously, it is super-skewsymmetric. In [9], it was proved that the super-skewsymmetric super-biderivations of the super-Virasoro algebra SVir are inner. In this section, we will prove that every super-skewsymmetric super-biderivation of the N=2 superconformal algebra $\mathfrak{g}$ is inner. Next, we give the main result in this section as follows.

Theorem 3.1. *Let φ be a super-skewsymmetric super-biderivation of the N=2 superconformal algebra $\mathfrak{g}$. We have*
$$\varphi(x, y) = \lambda[x, y]$$
for $x, y \in \mathfrak{g}$, where $\lambda \in \mathbb{C}$.

Proof. The proof of the theorem will be carried out in several steps.

Claim 1. There exists $\lambda \in \mathbb{C}$ such that
$$\varphi(L_m, L_n) \equiv \lambda[L_m, L_n] \pmod{\mathbb{C}c} \qquad \text{for all } m, n \in \mathbb{Z}.$$

Because $|\varphi(L_m, L_n)| = |L_m| + |L_n| = \bar{0}$ for $m, n \in \mathbb{Z}$, we suppose that
$$\varphi(L_m, L_n) = k_1 c + \sum_{\alpha\in\mathbb{Z}} a_\alpha^{(1)} L_\alpha + \sum_{\beta\in\mathbb{Z}} b_\beta^{(1)} H_\beta,$$
where $k_1, a_\alpha^{(1)}, b_\beta^{(1)} \in \mathbb{C}$.

If $m = n$, then $[L_m, L_n] = 0$. By Corollary 2.2 (2), $\varphi(L_m, L_n) \in Z(\mathfrak{g}) = \mathbb{C}c$, where $Z(\mathfrak{g})$ is the center of the N=2 superconformal algebra $\mathfrak{g}$. Then this claim holds.

Next, we assume that $m \neq n$, by Corollary 2.2 (1), we have
$$\frac{1}{m-n}[\varphi(L_m, L_n), [L_m, L_n]] = 0,$$
we get that
$$\sum_{\alpha\in\mathbb{Z}} a_\alpha^{(1)}(\alpha - m - n) L_{m+n+\alpha} + \sum_{\beta\in\mathbb{Z}} \beta b_\beta^{(1)} H_{m+n+\beta} = 0,$$
from which it follows that
$$a_\alpha^{(1)}(\alpha - m + n) = 0; \qquad \beta b_\beta^{(1)} = 0.$$
So we have $a_\alpha^{(1)} = 0$ if $\alpha \neq m + n$; $b_\beta^{(1)} = 0$ if $\beta \neq 0$. We conclude that
$$\varphi(L_m, L_n) \equiv a_{m+n}^{(1)} L_{m+n} + b_0^{(1)} H_0 \pmod{\mathbb{C}c}.$$
By
$$[\varphi(L_m, L_n), [L_1, L_0]] = [[L_m, L_n], \varphi(L_1, L_0)],$$
we have that
$$(m + n - 1)a_{m+n}^{(1)} L_{m+n+1} = (m - n)(m + n - 1)a_1^{(1)} L_{m+n+1}.$$
Thus, $a_{m+n}^{(1)} = (m - n)a_1^{(1)}$ if $m + n \neq 1$. By
$$[\varphi(L_m, L_n), [L_2, L_0]] = [[L_m, L_n], \varphi(L_2, L_0)],$$
we have
$$(m + n - 2)a_{m+n}^{(1)} L_{m+n+2} = (m - n)(m + n - 2)a_1^{(1)} L_{m+n+2}.$$
So, $a_{m+n}^{(1)} = (m - n)a_1^{(1)}$ if $m + n \neq 2$. Then $a_{m+n}^{(1)} = (m - n)a_1^{(1)}$ for all $m, n \in \mathbb{Z}$. Taking $\lambda = a_1^{(1)}$, we obtain that
$$\varphi(L_m, L_n) \equiv \lambda[L_m, L_n] + b_0^{(1)} H_0 \pmod{\mathbb{C}c}.$$

In the Claim 2, we will prove that $b_0^{(1)} = 0$, thus

$$\varphi(L_m, L_n) \equiv \lambda[L_m, L_n] \ (\mathrm{mod} \ \ \mathbb{C}c).$$

Claim 2. There exists $\lambda \in \mathbb{C}$ such that

$$\varphi(L_m, G_r^+) \equiv \lambda[L_m, G_r^+] \ \ (\mathrm{mod} \ \ \mathbb{C}c) \qquad \text{for all } m \in \mathbb{Z}, r \in \tfrac{1}{2}\mathbb{Z}.$$

Because $|\varphi(L_m, G_r^+)| = |L_m| + |G_r^+| = \bar{1}$, for any fixed $m \in \mathbb{Z}, r \in \frac{1}{2}\mathbb{Z}$, we suppose that

$$\varphi(L_m, G_r^+) = \sum_{\alpha \in \frac{1}{2}\mathbb{Z}} a_\alpha^{(2)} G_\alpha^+ + \sum_{\beta \in \frac{1}{2}\mathbb{Z}} b_\beta^{(2)} G_\beta^-,$$

where $a_\alpha^{(2)}, b_\beta^{(2)} \in \mathbb{C}$.

When $\mathfrak{g} = NS$, we have

$$\varphi(L_m, G_r^+) = \sum_{\alpha \in \frac{1}{2}+\mathbb{Z}} a_\alpha^{(2)} G_\alpha^+ + \sum_{\beta \in \frac{1}{2}+\mathbb{Z}} b_\beta^{(2)} G_\beta^-.$$

From

$$[\varphi(L_m, G_r^+), [L_1, L_0]] = [[L_m, G_r^+], \varphi(L_1, L_0)],$$

we get that

$$\sum_{\alpha \in \frac{1}{2}+\mathbb{Z}} a_\alpha^{(2)} (\alpha - \frac{1}{2}) G_{\alpha+1}^+ + \sum_{\beta \in \frac{1}{2}+\mathbb{Z}} b_\beta^{(2)} (\beta - \frac{1}{2}) G_{\beta+1}^- = \tag{3.1}$$

$$a_1^{(1)} (\frac{m}{2} - r)(m + r - \frac{1}{2}) G_{m+r+1}^+ - b_0^{(1)} (\frac{m}{2} - r) G_{m+r}^+.$$

Using (3.1), we have $a_{m+r-1}^{(2)}(m+r-\frac{3}{2}) = -(\frac{m}{2}-r) b_0^{(1)}$ if $\alpha = m+r-1$. By the arbitrariness of m and r, we obtain that $b_0^{(1)} = 0$. That is to say Claim 1 holds when $\mathfrak{g} = NS$. By

$$[\varphi(L_m, G_r^+), [L_2, L_0]] = [[L_m, G_r^+], \varphi(L_2, L_0)],$$

we get that

$$2 \sum_{\alpha \in \frac{1}{2}+\mathbb{Z}} a_\alpha^{(2)} (\alpha - 1) G_{\alpha+2}^+ + 2 \sum_{\beta \in \frac{1}{2}+\mathbb{Z}} b_\beta^{(2)} (\beta - 1) G_{\beta+2}^- = 2 a_1^{(1)} (\frac{m}{2} - r)(n + r - 1) G_{n+r+2}^+.$$

Since $\alpha, \beta \in \frac{1}{2} + \mathbb{Z}$, we conclude that $b_\beta^{(2)} = 0$; $a_\alpha^{(2)} = 0$ if $\alpha \neq m + r$; $a_{m+r}^{(2)}(m + r - 1) = a_1^{(1)}(\frac{m}{2} - r)(m + r - 1)$ if $\alpha = m + r$. By the arbitrariness of m and r, we obtain that $a_{m+r}^{(2)} = a_1^{(1)}(\frac{m}{2} - r)$. Taking $a_1^{(1)} = \lambda$, then

$$\varphi(L_m, G_r^+) \equiv \lambda[L_m, G_r^+] \ \ (\mathrm{mod} \ \ \mathbb{C}c).$$

When $\mathfrak{g} = R$, we have

$$\varphi(L_m, G_r^+) = \sum_{\alpha \in \mathbb{Z}} a_\alpha^{(2)} G_\alpha^+ + \sum_{\beta \in \mathbb{Z}} b_\beta^{(2)} G_\beta^-.$$

By

$$[\varphi(L_m, G_r^+), [L_2, L_0]] = [[L_m, G_r^+], \varphi(L_2, L_0)],$$

we get that

$$2 \sum_{\alpha \in \mathbb{Z}} a_\alpha^{(2)} (\alpha - 1) G_{\alpha+2}^+ + 2 \sum_{\beta \in \mathbb{Z}} b_\beta^{(2)} (\beta - 1) G_{\beta+2}^- = 2 a_1^{(1)} (\frac{m}{2} - r)(n + r - 1) G_{n+r+2}^+. \tag{3.2}$$

Using (3.2), we have that $2a_{m+r-2}^{(2)}(m + r - 3) = -(\frac{m}{2} - r) b_0^{(1)}$ if $\alpha = m + r - 2$. By the arbitrariness of m and r, we obtain that $b_0^{(1)} = 0$. That is to say the Claim 1 holds when $\mathfrak{g} = R$. By

$$[\varphi(L_m, G_r^+), [L_1, L_0]] = [[L_m, G_r^+], \varphi(L_1, L_0)],$$

we get that

$$\sum_{\alpha\in\mathbb{Z}} a_\alpha^{(2)}(\alpha-\frac{1}{2})G^+_{\alpha+1}+\sum_{\beta\in\mathbb{Z}} b_\beta^{(2)}(\beta-\frac{1}{2})G^-_{\beta+1}=$$
$$a_1^{(1)}(\frac{m}{2}-r)(n+r-\frac{1}{2})G^+_{n+r+1}-b_0^{(1)}(\frac{m}{2}-r)G^+_{m+r}.$$

Since $\alpha,\beta\in\mathbb{Z}$, we conclude that $b_\beta^{(2)}=0$; $a_\alpha^{(2)}=0$ if $\alpha\neq m+r$; $a_{m+r}^{(2)}(m+r-\frac{1}{2})=a_1^{(1)}(\frac{m}{2}-r)(m+r-\frac{1}{2})$ if $\alpha=m+r$. By the arbitrariness of m and r, we obtain that $a_{m+r}^{(2)}=a_1^{(1)}(\frac{m}{2}-r)$. Taking $a_1^{(1)}=\lambda$, then

$$\varphi(L_m,G_r^+)\equiv\lambda[L_m,G_r^+]\pmod{\mathbb{C}c}.$$

Claim 3. There exists $\lambda\in\mathbb{C}$ such that

$$\varphi(L_m,G_r^-)\equiv\lambda[L_m,G_r^-]\pmod{\mathbb{C}c}\qquad\text{for all } m\in\mathbb{Z}, r\in\tfrac{1}{2}\mathbb{Z}.$$

The proof of this claim is similar to that of the Claim 2, we omit it.

Claim 4. There exists $\lambda\in\mathbb{C}$ such that

$$\varphi(L_m,H_n)\equiv\lambda[L_m,H_n]\pmod{\mathbb{C}c}\qquad\text{for all } m,n\in\mathbb{Z}.$$

Because $|\varphi(L_m,H_n)|=|L_m|+|H_n|=\bar{0}$, for any fixed $m,n\in\mathbb{Z}$, we suppose that

$$\varphi(L_m,H_n)=k_4c+\sum_{\alpha\in Z} a_\alpha^{(4)}L_\alpha+\sum_{\beta\in\mathbb{Z}} b_\beta^{(4)}H_\beta,$$

where $k_4, a_\alpha^{(4)}, b_\beta^{(4)}\in\mathbb{C}$.

When $\mathfrak{g}=NS$, we have that

$$\varphi(L_m,H_n)=k_4c+\sum_{\alpha\in\mathbb{Z}} a_\alpha^{(4)}L_\alpha+\sum_{\beta\in\mathbb{Z}} b_\beta^{(4)}H_\beta.$$

By

$$[\varphi(L_m,H_n),[L_m,H_n]]=0,$$

we have that

$$\sum_{\alpha\in\mathbb{Z}} n(m+n)a_\alpha^{(4)}H_{\alpha+m+n}\equiv 0\ (\mathrm{mod}\ \mathbb{C}c).$$

By the arbitrariness of m and n, we get that $a_\alpha^{(4)}=0$, we obtain

$$\varphi(L_m,H_n)\equiv\sum_{\beta\in\mathbb{Z}} b_\beta^{(4)}H_\beta\pmod{\mathbb{C}c}.$$

From

$$[\varphi(L_m,H_n),[L_0,G^+_{\frac{1}{2}}]]=[[L_m,H_n],\varphi(L_0,G^+_{\frac{1}{2}})],$$

we get that

$$-\sum_{\beta\in\mathbb{Z}}\frac{1}{2}b_\beta^{(4)}G^+_{\beta+\frac{1}{2}}=\frac{1}{2}\lambda nG^+_{m+n+\frac{1}{2}}.$$

Then $b_\beta^{(4)}=0$ if $\beta\neq m+n$; $b_{m+n}^{(4)}=-n\lambda$ if $\beta=m+n$. Thus

$$\varphi(L_m,H_n)\equiv\lambda[L_m,H_n]\pmod{\mathbb{C}c}.$$

When $\mathfrak{g}=R$, we have that

$$\varphi(L_m,H_n)=k_4c+\sum_{\alpha\in\mathbb{Z}} a_\alpha^{(4)}L_\alpha+\sum_{\beta\in\mathbb{Z}} b_\beta^{(4)}H_\beta.$$

By

$$[\varphi(L_m,H_n),[L_m,H_n]]=0,$$

we have that

$$\sum_{\alpha\in\mathbb{Z}} n(m+n)a_\alpha^{(4)}H_{\alpha+m+n} \equiv 0 \text{ (mod } \mathbb{C}c).$$

By the arbitrariness of m and n, we get that $a_\alpha^{(4)} = 0$. Then

$$\varphi(L_m, H_n) \equiv \sum_{\beta\in\mathbb{Z}} b_\beta^{(4)}H_\beta \quad (\text{mod } \ \mathbb{C}c).$$

From

$$[\varphi(L_m, H_n), [L_0, G_1^+]] = [[L_m, H_n], \varphi(L_0, G_1^+)]$$

for $m, n \in \mathbb{Z}$, we get that

$$-\sum_{\beta\in\mathbb{Z}} b_\beta^{(4)}G_{\beta+1}^+ = \lambda n G_{m+n+1}^+,$$

we immediately have that $b_\beta^{(4)} = 0$ if $\beta \neq m+n$; $b_{m+n}^{(4)} = -n\lambda$ if $\beta = m+n$. Thus we obtain

$$\varphi(L_m, H_n) \equiv \lambda[L_m, H_n] \quad (\text{mod } \ \mathbb{C}c).$$

Claim 5. There exists $\lambda \in \mathbb{C}$ such that

$$\varphi(G_r^-, G_s^+) \equiv \lambda[G_r^-, G_s^+] \quad (\text{mod } \ \mathbb{C}c) \qquad \text{for all } r, s \in \tfrac{1}{2}\mathbb{Z}.$$

Because $|\varphi(G_r^-, G_s^+)| = |G_r^-| + |G_s^+| = \bar{0}$, for any fixed r, s, we suppose that

$$\varphi(G_r^-, G_s^+) = k_5 c + \sum_{\alpha\in\mathbb{Z}} a_\alpha^{(5)}L_\alpha + \sum_{\beta\in\mathbb{Z}} b_\beta^{(5)}H_\beta,$$

where $k_5, a_\alpha^{(5)}, b_\beta^{(5)} \in \mathbb{C}$.

When $\mathfrak{g} = NS$, by

$$[\varphi(G_r^-, G_s^+), [L_0, H_1]] = [[G_r^-, G_s^+], \varphi(L_0, H_1)],$$

we have that

$$-\sum_{\alpha\in\mathbb{Z}} a_\alpha^{(5)}H_{\alpha+1} \equiv -2\lambda H_{r+s+1} \text{ (mod } \mathbb{C}c). \tag{3.3}$$

Thanks to (3.3), we have that $a_\alpha^{(5)} = 0$ if $\alpha \neq r+s$; $a_{r+s}^{(5)} = 2\lambda$ if $\alpha = r+s$. Thus we get

$$\varphi(G_r^-, G_s^+) = 2\lambda L_{r+s} + \sum_{\beta\in\mathbb{Z}} b_\beta^{(5)}H_\beta \text{ (mod } \mathbb{C}c).$$

From

$$[\varphi(G_r^-, G_s^+), [L_1, L_0]] = [[G_r^-, G_s^+], \varphi(L_1, L_0)],$$

we obtain that

$$\sum_{\beta\in\mathbb{Z}} \beta b_\beta^{(5)}H_{\beta+1} = -\lambda(r+s)(r-s)H_{r+s+1}. \tag{3.4}$$

Using (3.4), we get $b_\beta^{(5)} = 0$ if $\beta \neq r+s$ and $\beta \neq 0$; $(r+s)b_{r+s}^{(5)} = -\lambda(r+s)(r-s)$ if $\beta = r+s$. By the arbitrariness of r and s, we have that $b_{r+s}^{(5)} = -\lambda(r-s)$. Thus

$$\varphi(G_r^-, G_s^+) \equiv \lambda[G_r^-, G_s^+] + b_0^{(5)}H_0 \text{ (mod } \mathbb{C}c).$$

By

$$[\varphi(G_r^-, G_s^+), [L_0, G_{\frac{1}{2}}^+]] = [[G_r^-, G_s^+], \varphi(L_0, G_{\frac{1}{2}}^+)],$$

for $r,s\in\frac{1}{2}+\mathbb{Z}$, we have that

$$\lambda[L_{r+s},G^{+}_{\frac{1}{2}}]-\frac{1}{2}\lambda(r-s)[H_{r+s},G^{+}_{\frac{1}{2}}]+\frac{1}{2}b^{(5)}_0[H_0,G^{+}_{\frac{1}{2}}]=\lambda[L_{r+s},G^{+}_{\frac{1}{2}}]-\frac{1}{2}\lambda(r-s)[H_{r+s},G^{+}_{\frac{1}{2}}].$$

By the above equation, we have that $b^{(5)}_0=0$, then

$$\varphi(G^-_r,G^+_s)\equiv\lambda[G^-_r,G^+_s]\quad(\mathrm{mod}\ \ \mathbb{C}c).$$

When $\mathfrak{g}=R$, by

$$[\varphi(G^-_r,G^+_s),[L_0,H_1]]=[[G^-_r,G^+_s],\varphi(L_0,H_1)],$$

we have that

$$-\sum_{\alpha\in\mathbb{Z}}a^{(5)}_\alpha H_{\alpha+1}\equiv-2\lambda H_{r+s+1}\ (\mathrm{mod}\ \mathbb{C}c). \tag{3.5}$$

Thanks to (3.5), we have that $a^{(5)}_\alpha=0$ if $\alpha\neq r+s$; $a^{(5)}_{r+s}=2\lambda$ if $\alpha=r+s$. Thus

$$\varphi(G^-_r,G^+_s)=2\lambda L_{r+s}+\sum_{\beta\in\mathbb{Z}}b^{(5)}_\beta H_\beta\ (\mathrm{mod}\ \mathbb{C}c).$$

By

$$[\varphi(G^-_r,G^+_s),[L_1,L_0]]=[[G^-_r,G^+_s],\varphi(L_1,L_0)],$$

we obtain that

$$\sum_{\beta\in\mathbb{Z}}\beta b^{(5)}_\beta H_{\beta+1}=-\lambda(r+s)(r-s)H_{r+s+1}. \tag{3.6}$$

Using (3.6), we get that $b^{(5)}_\beta=0$ if $\beta\neq r+s$ and $\beta\neq 0$; $(r+s)b^{(5)}_{r+s}=-\lambda(r+s)(r-s)$ if $\beta=r+s$. By the arbitrariness of r and s, we have that $b^{(5)}_{r+s}=-\lambda(r-s)$. Thus

$$\varphi(G^-_r,G^+_s)\equiv\lambda[G^-_r,G^+_s]+b^{(5)}_0H_0\ (\mathrm{mod}\ \mathbb{C}c).$$

From

$$[\varphi(G^-_r,G^+_s),[L_0,G^+_1]]=[[G^-_r,G^+_s],\varphi(L_0,G^+_1)],$$

for $r,s\in\mathbb{Z}$, we have that

$$2\lambda[L_{r+s},G^+_1]-\lambda(r-s)[H_{r+s},G^+_1]+b^{(5)}_0[H_0,G^+_1]=2\lambda[L_{r+s},G^+_1]-\lambda(r-s)[H_{r+s},G^+_1]. \tag{3.7}$$

By (3.7), we obtain $b^{(5)}_0=0$, then

$$\varphi(G^-_r,G^+_s)\equiv\lambda[G^-_r,G^+_s]\quad(\mathrm{mod}\ \ \mathbb{C}c).$$

Claim 6. There exists $\lambda\in\mathbb{C}$ such that

$$\varphi(H_m,H_n)\equiv\lambda[H_m,H_n]\quad(\mathrm{mod}\ \ \mathbb{C}c)\qquad\text{for all } m,n\in\mathbb{Z}.$$

By

$$[H_m,H_n]=\frac{c}{3}\delta_{m+n,0},$$

we have that

$$[H_m,H_n]=0\quad(\mathrm{mod}\ \ \mathbb{C}c).$$

Then we obtain

$$\varphi(H_m,H_n)\equiv\lambda[H_m,H_n]\quad(\mathrm{mod}\ \ \mathbb{C}c).$$

This claim holds.

Claim 7. There exists $\lambda\in\mathbb{C}$ such that

$$\varphi(H_m,G^+_r)\equiv\lambda[H_m,G^+_r](\mathrm{mod}\ \mathbb{C}c)\qquad\text{for all } m\in\mathbb{Z},r\in\tfrac{1}{2}\mathbb{Z}.$$

Because $|\varphi(H_m, G_r^+)| = |H_m| + |G_r^+| = \bar{1}$, for any fixed $m \in \mathbb{Z}, r \in \frac{1}{2}\mathbb{Z}$, we suppose that

$$\varphi(H_m, G_r^+) = \sum_{\alpha \in \frac{1}{2}\mathbb{Z}} a_\alpha^{(7)} G_\alpha^+ + \sum_{\beta \in \frac{1}{2}\mathbb{Z}} b_\beta^{(7)} G_\beta^-,$$

where $a_\alpha^{(7)}, b_\beta^{(7)} \in \mathbb{C}$.

When $\mathfrak{g} = NS$, we have

$$\varphi(H_m, G_r^+) = \sum_{\alpha \in \frac{1}{2}+\mathbb{Z}} a_\alpha^{(7)} G_\alpha^+ + \sum_{\beta \in \frac{1}{2}+\mathbb{Z}} b_\beta^{(7)} G_\beta^-.$$

By

$$[\varphi(H_m, G_r^+), [L_0, H_1]] = [[H_m, G_r^+], \varphi(L_0, H_1)],$$

we have that

$$-\sum_{\alpha \in \frac{1}{2}+\mathbb{Z}} a_\alpha^{(7)} G_{\alpha+1}^+ + \sum_{\beta \in \frac{1}{2}+\mathbb{Z}} b_\beta^{(7)} G_{\beta+1}^- = -\lambda G_{m+r+1}^+. \tag{3.8}$$

Using (3.8), we get $b_\beta^{(7)} = 0$; $a_\alpha^{(7)} = 0$ if $\alpha \neq m + r$; $a_\alpha^{(7)} = \lambda$ if $\alpha = m + r$. Then

$$\varphi(H_m, G_r^+) \equiv \lambda[H_m, G_r^+] (\text{mod } \mathbb{C}c).$$

When $\mathfrak{g} = R$, we have

$$\varphi(H_m, G_r^+) = \sum_{\alpha \in \mathbb{Z}} a_\alpha^{(7)} G_\alpha^+ + \sum_{\beta \in \mathbb{Z}} b_\beta^{(7)} G_\beta^-.$$

From

$$[\varphi(H_m, G_r^+), [L_0, H_1]] = [[H_m, G_r^+], \varphi(L_0, H_1)],$$

we have that

$$-\sum_{\alpha \in \mathbb{Z}} a_\alpha^{(7)} G_{\alpha+1}^+ + \sum_{\beta \in \mathbb{Z}} b_\beta^{(7)} G_{\beta+1}^- = -\lambda G_{m+r+1}^+. \tag{3.9}$$

By (3.9), we obtain that $b_\beta^{(7)} = 0$; $a_\alpha^{(7)} = 0$ if $\alpha \neq m + r$; $a_\alpha^{(7)} = \lambda$ if $\alpha = m + r$. Then

$$\varphi(H_m, G_r^+) \equiv \lambda[H_m, G_r^+] (\text{mod } \mathbb{C}c).$$

Claim 8. There exists $\lambda \in \mathbb{C}$ such that

$$\varphi(H_m, G_r^-) \equiv \lambda[H_m, G_r^-] (\text{mod } \mathbb{C}c) \qquad \text{for all } m \in \mathbb{Z}, r \in \tfrac{1}{2}\mathbb{Z}.$$

Because $|\varphi(H_m, G_r^-)| = |H_m| + |G_r^-| = \bar{1}$, for any fixed $m \in \mathbb{Z}, r \in \frac{1}{2}\mathbb{Z}$, we suppose that

$$\varphi(H_m, G_r^-) = \sum_{\alpha \in \frac{1}{2}\mathbb{Z}} a_\alpha^{(8)} G_\alpha^+ + \sum_{\beta \in \frac{1}{2}\mathbb{Z}} b_\beta^{(8)} G_\beta^-,$$

where $a_\alpha^{(8)}, b_\beta^{(8)} \in \mathbb{C}$.

When $\mathfrak{g} = NS$, we have

$$\varphi(H_m, G_r^-) = \sum_{\alpha \in \frac{1}{2}+\mathbb{Z}} a_\alpha^{(8)} G_\alpha^+ + \sum_{\beta \in \frac{1}{2}+\mathbb{Z}} b_\beta^{(8)} G_\beta^-.$$

By

$$[\varphi(H_m, G_r^-), [L_0, H_1]] = [[H_m, G_r^-], \varphi(L_0, H_1)],$$

we have that

$$\sum_{\alpha \in \frac{1}{2}+\mathbb{Z}} a_\alpha^{(8)} G_{\alpha+1}^+ - \sum_{\beta \in \frac{1}{2}+\mathbb{Z}} b_\beta^{(8)} G_{\beta+1}^- = \lambda G_{m+r+1}^-. \tag{3.10}$$

Thanks to (3.10), we get $a_\alpha^{(8)} = 0$; $b_\beta^{(8)} = 0$ if $\beta \neq m + r$; $b_\beta^{(8)} = -\lambda$ if $\beta = m + r$. Then

$$\varphi(H_m, G_r^-) \equiv \lambda[H_m, G_r^-] (\text{mod } \mathbb{C}c).$$

When $\mathfrak{g} = R$, we have

$$\varphi(H_m, G_r^-) = \sum_{\alpha\in\mathbb{Z}} a_\alpha^{(8)} G_\alpha^+ + \sum_{\beta\in\mathbb{Z}} b_\beta^{(8)} G_\beta^-.$$

By

$$[\varphi(H_m, G_r^-), [L_0, H_1]] = [[H_m, G_r^-], \varphi(L_0, H_1)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}} a_\alpha^{(8)} G_{\alpha+1}^+ - \sum_{\beta\in\mathbb{Z}} b_\beta^{(8)} G_{\beta+1}^- = \lambda G_{m+r+1}^-. \tag{3.11}$$

Using (3.11), we obtain that $a_\alpha^{(8)} = 0$; $b_\beta^{(8)} = 0$ if $\beta \neq m + r$; $b_\beta^{(8)} = -\lambda$ if $\beta = m + r$. Then

$$\varphi(H_m, G_r^-) \equiv \lambda[H_m, G_r^-] (\text{mod } \mathbb{C}c).$$

Claim 9. There exists $\lambda \in \mathbb{C}$ such that

$$\varphi(x, y) = \lambda[x, y] \pmod{\mathbb{C}c} \qquad \text{for all } x, y \in \mathfrak{g}.$$

By Claim1 to Claim 8, it is easy to obtain Claim 9.

Claim 10. There exists $\lambda \in \mathbb{C}$ such that

$$\varphi(x, y) = \lambda[x, y] \qquad \text{for all } x, y \in \mathfrak{g}.$$

According Claim 9, we may assume that $\varphi(x, y) = \lambda[x, y]+f(x, y)c$, where f is a bilinear function from $\mathfrak{g} \times \mathfrak{g}$ to $\mathbb{C}$. By

$$\varphi([x, y], z) = [x, \varphi(y, z)] + (-1)^{|y||z|}[\varphi(x, z), y],$$

we have that $f([x, y], z) = 0$ for all $x, y, z \in \mathfrak{g}$. Since the N=2 superconformal algebra $\mathfrak{g}$ coincides with its derived subalgebra, one sees that f is exactly the zero function. Finally, we have that $\varphi(x, y) = \lambda[x, y]$ for all $x, y \in \mathfrak{g}$. Thus, this claim holds and we can find that every super-biderivation of the N=2 superconformal algebra $\mathfrak{g}$ is an inner super-biderivation. It concludes the proof. □

4. Linear super-commuting maps on the N=2 superconformal algebra $\mathfrak{g}$

In this section, we shall describe the linear super-commuting maps on the N=2 superconformal algebra $\mathfrak{g}$ based on Theorem 3.1, we prove that linear super-commuting maps on the N=2 superconformal algebra $\mathfrak{g}$ are non-standard.

Theorem 4.1. *Each linear super-commuting map ψ on the N=2 superconformal algebra $\mathfrak{g}$ has the form*

$$\psi(x) = \lambda x + f(x)c \qquad \textit{for all } x \in \mathfrak{g},$$

where $\lambda \in \mathbb{C}$, f is a linear function from $\mathfrak{g}$ to $\mathbb{C}$.

Proof. Let ψ be a linear super-commuting map on the N=2 superconformal algebra $\mathfrak{g}$. Define

$$\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

sending (x, y) to $[\psi(x), y]$ for all $x, y \in \mathfrak{g}$. Note that ψ preserves the $\mathbb{Z}_2$-grading of the N=2 superconformal algebra $\mathfrak{g}$. By fixing x, we get a map $y \mapsto \varphi(x, y)$. It is easy to verify that it is a super-derivation of the N=2 superconformal algebra $\mathfrak{g}$. So we get the following conclusion

$$\varphi(x, [y, z]) = [\varphi(x, y), z] + (-1)^{|x||y|}[y, \varphi(x, z)] \qquad \text{for } y, z \in \mathfrak{g}.$$

In other words, φ is a super-derivation with respect to the second component. Since $[\psi(x), y] = (-1)^{|x||y|}[x, \psi(y)]$($\psi$ is a linear super-commuting map), one can easily see that φ is also a super-derivation with respect to the first component. Thus, φ is a super-biderivation on the N=2 superconformal algebra $\mathfrak{g}$. Furthermore, φ is automatically super-skewsymmetric by the definition of φ above. Thus, φ is a super-skewsymmetric super-biderivation. Applying Theorem 3.1, there exists $\lambda \in \mathbb{C}$ such that

$$\varphi(x, y) = \lambda[x, y] \qquad \text{for } x, y \in \mathfrak{g}.$$

We have that

$$[\psi(x) - \lambda x, y] = 0 \qquad \text{for } y \in \mathfrak{g}.$$

Furthermore, we have

$$\psi(x) - \lambda x \in Z(\mathfrak{g}) = \mathbb{C}c,$$

that is,

$$\psi(x) = \lambda x + f(x)c.$$

The proof is concluded. □

Acknowledgment. J.-C. Sun would like to thank the support of NNSF grant 11671247 of China.

References

[1] M. Ademollo, L. Brink, A. d'Adda, R. Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino, *Supersymmetric strings andcolour connement*, Phys. Lett. **B 62**, 105-110, 1976.

[2] M. Brešar, *On generalized biderivations and related maps*, J. Algebra **172**, 764-786, 1995.

[3] M. Brešar, *Commuting maps: A survey*, Taiwanese J. Math. **8**, 361-397, 2004.

[4] G.-Z. Fan and X.-S. Dai, *Super-biderivations of Lie superalgebras*, Linear Multilinear Algebra **26** (1), 58-66, 2017.

[5] M. Fošner, *On the extended centroid of prime associative superalgebras with applications to superderivations*, Commun. Algebra **32** , 689-705, 2004.

[6] X. Han, D.-Y. Wang and C.-G. Xia, *Linear commuting maps and biderivations on the Lie algebra $\mathcal{W}$(a,b)*, J. Lie theory **26** (3), 777-786, 2016.

[7] V. Kac, *Lie superalgebras*, Adv. Phys. **26**, 8-96, 1977.

[8] D.-Y. Wang and X.-X. Yu, *Biderivation and linear commuting maps on the Schrödinger-Virasoro Lie algebra*, Comm. Algebra **41**, 2166-2173, 2013.

[9] C.-G. Xia, D.-Y. Wang and X. Han, *Linear super-commuting maps and super-biderivations on the super-Virasoro algebras*, Comm. Algebra **44** (12), 5342-5350, 2016.