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Research Article

Reducible good representations of semisimple Lie algebras A_r and B_r Part I

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Abstract

Given a semisimple (preferably simple) complex Lie algebra L, we consider the monoid $\Gamma = \Gamma(L)$ of equivalence classes of the finite dimensional reducible complex representations of L. Here Γ is identified with the lattice of the corresponding highest weights. (This equips Γ with the monoid structure.) For $\pi \in \Gamma$ one considers the symmetric algebra $S(\pi) = \bigoplus_{n=0}^{\infty} S^n(\pi)$ (here regarded as a representation). The elements of Γ "occurring" in $S(\pi)$ – i.e., which are the highest weights of some irreducible component of the representation $S(\pi)$ – form a subsemigroup $M(\pi)$ of Γ. Such a $M(\pi)$ has a naturally defined rank $r(\pi)$ with $1 \leq r(\pi) \leq r =$ rank of L. In this paper we give a classification, for all the simple $L = A_r$ and $L = B_r$ of all the π with $r(\pi) < r$.

Mathematics Subject Classification (2010). 17B10

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1. Introduction

This work deals with the classification of certain finite dimensional representations of simple finite dimensional complex Lie algebras, which we will call good representations in the paper.

Let L be such a Lie algebra and let r be its rank. Let $\{\Pi_1, \Pi_2, \ldots, \Pi_r\}$ be the set of fundamental weights, $\Lambda = \mathbb{Z}\Pi_1 + \mathbb{Z}\Pi_2 + \cdots + \mathbb{Z}\Pi_r$ be the lattice of weights and $\Lambda^+ = \mathbb{N}_0\Pi_1 + \mathbb{N}_0\Pi_2 + \cdots + \mathbb{N}_0\Pi_r$ be the monoid of dominant weights. (The notation is with respect to a fixed Cartan subalgebra in L and a fixed basis of the root system, [4,8].) We consider the r-dimensional $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}} = \mathbb{Q}\Pi_1 + \mathbb{Q}\Pi_2 + \cdots + \mathbb{Q}\Pi_r$ ($\mathbb{Q}$ = the field of rational numbers).

We say, that $\lambda \in \Lambda^+$ occurs in a (finite dimensional) representation of L, if the representation has an irreducible component with highest weight λ.

Now let ρ be a finite dimensional representation of L with the representation space V. It induces a representation $S^n(\rho)$ of L in the symmetric power $S^n(V)$ of V for all $n = 0, 1, 2, \ldots$. For $\pi = \tau_1 + \tau_2 + \cdots + \tau_t$, $2 \leq t < r$, $S^n(\pi)$ is

$$S^n(\pi) = S^n(\tau_1) \otimes S^0(\tau_2 + \cdots + \tau_t) + S^{n-1}(\tau_1) \otimes S^1(\tau_2 + \cdots + \tau_t) + \cdots + S^0(\tau_1) \otimes S^n(\tau_2 + \cdots + \tau_t).$$

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We define

$$M(\rho) := \{\lambda \in \Lambda^+ \mid \text{there is } n \in \mathbb{N}_0, \text{ such that } \lambda \text{ occurs in } S^n(V)\}.$$

Then $M(\rho)$ is a submonoid of Λ^+ (see [2, page 8]). Let then $M_{\mathbb{Q}}(\rho)$ be the $\mathbb{Q}$-vector subspace of $\Lambda_{\mathbb{Q}}$ generated by $M(\rho)$. We call the representation ρ **good**, if $\dim_{\mathbb{Q}} M_{\mathbb{Q}}(\rho) < r$, otherwise we call it **bad**. The dimension $\dim_{\mathbb{Q}} M_{\mathbb{Q}}(\rho)$ is called $\text{rank}M(\rho)$ or $\text{rank}\rho$.

The most reducible representations of the simple Lie algebras $L = A_r$ and $L = B_r$ are shown to be bad. In this section of the work the symmetric powers of representations are used, which we have compiled in the R-list 3 at the end of the work. (The calculations were made with the aid of the Lie calculation package from [9]). In Theorem 3.9 the good reducible representations are listed and they are classified according to the size of their rank. In Corollaries 3.10 and 3.11 the results from Theorems 3.1 and 3.9 are summarized once again.

The numbering of the simple roots in the basis $B = \{a_1, a_2, \ldots, a_r\}$ is chosen like in Tits (see [4, 8]). A representation $\rho : L \to \text{gl}(V)$ induces in the dual space V^* of V a representation $\rho^* : L \to \text{gl}(V^*)$, which is called the contragradient of ρ. The contragradient of a fundamental representation Π_i is a fundamental representation Π_i^* Since $(\rho^*)^* = \rho$, this applies

$$(\Pi_1^{m_1} \cdots \Pi_r^{m_r})^* = (\Pi_1^*)^{m_1} \cdots (\Pi_r^*)^{m_r}.$$

Let as before ρ be a reducible representation of L. Let V be the representation space of ρ. Let G be a simple connected linear algebraic group with a Lie algebra L. A linear algebraic action of G is then induced on V. We identify $S(\rho)$ with $S(V) \equiv \mathbb{C}[V] \equiv O(V^*)$, i.e., with the complex algebra of regular functions on the dual space V^* of V (V^* is the representation space of the dual representation ρ^* of L, respectively of G). Note thereby

$$\rho \text{ is good } \Leftrightarrow \rho^* \text{ is good}.$$

Let ρ be a representation of L on the representation space V. Then L acts also on S_ρ^n, i.e., on the symmetric powers $S^n(V)$ and finally on the (infinite dimensional) symmetric algebra

$$S(V) = \coprod_{n=0}^{\infty} S^n(V) \quad (S^0(V) = \mathbb{C}).$$

Let $M(\rho)$ be the following defined subset of Λ^+: A $\lambda \in \Lambda^+$ is in $M(\rho)$, if there is a $k \geq 0$, such that λ occurs in $S^k(\rho)$:

$$M(\rho) = \{\lambda \in \Lambda^+ \mid \text{there is a } k \geq 0 \text{ such that } \lambda \text{ occurs in } S^k(\rho)\}.$$

Remark 1.1. $M(\rho)$ is a submonoid of Λ^+.

Proof. If λ occurs in $S^m(\rho)$ and μ occurs in $S^n(\rho)$, then $\lambda + \mu$ occurs in in $S^{n+m}(\rho)$. □

We have $\Lambda_{\mathbb{Q}} := \mathbb{Q}\Pi_1 + \mathbb{Q}\Pi_2 + \cdots + \mathbb{Q}\Pi_r \subseteq C^*$, where C^* is the dual space (over the complex numbers) of the Cartan subalgebra C of L and $\mathbb{Q}$ the field of rational numbers. Finally, let $M_{\mathbb{Q}}(\rho) :=$ the $\mathbb{Q}$-subspace of $\Lambda_{\mathbb{Q}}$ generated by $M(\rho)$.

Definition 1.2. We call the dimension of $M_{\mathbb{Q}}(\rho)$ over $\mathbb{Q}$ the rank of the representation ρ:

$$\text{rank}M(\rho) := \text{rank}\rho := \text{dimension of } M_{\mathbb{Q}}(\rho) \text{ over } \mathbb{Q}.$$

Definition 1.3. A representation ρ of L is **good**, if $\text{rank}M(\rho) < r$. Otherwise, i.e., if $\text{rank}M(\rho) = r$, ρ is **bad**. (Note that $\dim_{\mathbb{Q}} \Lambda_{\mathbb{Q}} = r$, because the $\Pi_1, \ldots, \Pi_r$ are linearly independent.)

Remark 1.4. A reducible representation $\rho = \tau_1 + \tau_2 + \cdots + \tau_t$ with, $\tau_1, \tau_2, \ldots, \tau_t$ irreducible representations is good, if the sum $\tau_1 + \tau_2 + \cdots + \tau_t$, $t < r$, is good.

2. Reducible bad representations

We considered the irreducible good representations of simple Lie algebras in our doctoral thesis [2]. Here we consider the reducible good representations of simple Lie algebras $L = A_r$ and $L = B_r$. The irreducible good representations of simple Lie algebras are given in the following lists (see [2, page 44, Theorem 2.1]).

(*i*) List(A): $\Pi_1, \Pi_r, \Pi_2, \Pi_{r-1}$ of A_r, $r \geq 2$ and Π_3 of A_5;

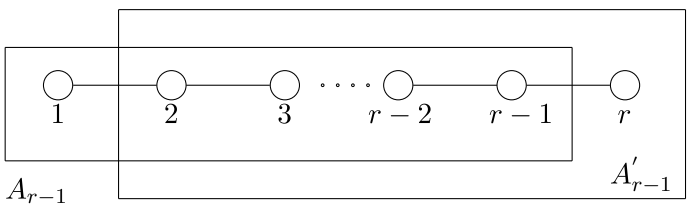
(ii) List(B): Π_1 of B_r, $r \geq 2$ and Π_r for $r = 2, 3, 4$.

Theorem 2.1. (R-bad) *Let L be a (semi-)simple Lie algebra and $\rho = \tau_1 + \tau_2$ be a reducible representation of L with τ_1 and τ_2 good. Then ρ is bad, if ρ is in the following list:*
(i) A_r*:*
$\Pi_1 + \Pi_2, \Pi_2 + \Pi_3$ *of* A_3*;*
$\Pi_1 + \Pi_2, \Pi_1 + \Pi_{r-1}, \Pi_2 + \Pi_2, \Pi_2 + \Pi_{r-1}, \Pi_2 + \Pi_r, \Pi_{r-1} + \Pi_{r-1}, \Pi_{r-1} + \Pi_r$ *of* A_r*,* $r \geq 4$*;*
$\Pi_1 + \Pi_3, \Pi_2 + \Pi_3, \Pi_3 + \Pi_3, \Pi_3 + \Pi_4, \Pi_3 + \Pi_5$ *of* A_5*;*
(ii) B_r*:*
$\Pi_1 + \Pi_3, \Pi_3 + \Pi_3$ *of* B_3*;*
$\Pi_1 + \Pi_4, \Pi_4 + \Pi_4$ *of* B_4*,* $r \geq 3$*.*

Proof. According to Lemmas 2.2 and 2.3. □

Lemma 2.2. (Case A_r) *Let $\rho = \tau_1 + \tau_2$ be a reducible representation of A_r, $r \geq 3$, where τ_1, τ_2 are from the List(A). Then ρ is bad if it is not in the list below:*
R-list (A-2):
$\Pi_1 + \Pi_1, \Pi_1 + \Pi_r, \Pi_r + \Pi_r$ *of* A_r*,* $r \geq 3$*;* $\Pi_2 + \Pi_2$ *of* A_3*.*
In other words,the following representations are bad:
$\Pi_1 + \Pi_2, \Pi_2 + \Pi_3$ *of* A_3*;*
$\Pi_1 + \Pi_2, \Pi_1 + \Pi_{r-1}, \Pi_2 + \Pi_2, \Pi_2 + \Pi_{r-1}, \Pi_2 + \Pi_r, \Pi_{r-1} + \Pi_{r-1}, \Pi_{r-1} + \Pi_r$ *of* A_r*,* $r \geq 4$*;*
$\Pi_1 + \Pi_3, \Pi_2 + \Pi_3, \Pi_3 + \Pi_3, \Pi_3 + \Pi_4, \Pi_3 + \Pi_5$ *of* A_5*.*

Proof. Let $n = 3$. The reducible representations $\Pi_1 + \Pi_2, \Pi_2 + \Pi_3$ of A_3 are bad according to R-list 3.1.1.
Let $n = 4$.
The reducible representations $\Pi_1 + \Pi_2, \Pi_1 + \Pi_3, \Pi_2 + \Pi_2, \Pi_2 + \Pi_3, \Pi_2 + \Pi_4, \Pi_3 + \Pi_3, \Pi_3 + \Pi_4$ are bad according to R-list 3.1.2.
Let $n = 5$.
The reducible representations $\Pi_1 + \Pi_2, \Pi_1 + \Pi_3, \Pi_1 + \Pi_4, \Pi_2 + \Pi_2, \Pi_2 + \Pi_3, \Pi_2 + \Pi_4, \Pi_2 + \Pi_5, \Pi_3 + \Pi_3, \Pi_3 + \Pi_4, \Pi_3 + \Pi_5, \Pi_4 + \Pi_4, \Pi_4 + \Pi_5$ of A_5 are bad according to R-list 3.1.3.
Let $n \geq 6$. Proof by induction.



There are regular subalgebras A_{r-1} and A'_{r-1} in A_r as in the picture. The representations $\Pi_1+\Pi_2, \Pi_1+\Pi_{r-1}, \Pi_2+\Pi_2, \Pi_2+\Pi_{r-1}, \Pi_{r-1}+\Pi_{r-1}$ are bad in A_{r-1} and $\Pi_2+\Pi_r, \Pi_r+\Pi_{r-1}$ are bad in A'_{r-1}.
$\rho = \Pi_1 + \Pi_2$
a) $L = A_{r-1} = A_{2m-1}$.
The trivial component Π_0 occurs with multiplicity 1 in $S^m(\Pi_2)$ of A_{2m-1} (see [2, page 71, Behauptung 1]).

$\rho = \Pi_1 + \Pi_2$ of A_{r-1} is also bad in A_r according to “Kriterium 1” (see [2, page 13]).
b) $L = A_{r-1} = A_{2m}$.

$$S^{m+1}(\Pi_1 + \Pi_2) = S^{m+1}(\Pi_1) + S^m(\Pi_1) \otimes \Pi_2 + \cdots + \Pi_1 \otimes S^m(\Pi_2) + S^{m+1}(\Pi_2).$$

The component Π_{2m} occurs in $S^m(\Pi_2)$ of A_{2m} (see [2, page 79, Behauptung 2]).
$\Pi_1 \otimes S^m(\Pi_2)$ contains the component $\Pi_1 \otimes \Pi_{2m}$ and $\Pi_1 \otimes \Pi_{2m} = \Pi_1\Pi_{2m} + \Pi_0$. The trivial component Π_0 occurs in $S^{m+1}(\Pi_1 + \Pi_2)$. According to Kriterium 1 $\rho = \Pi_1 + \Pi_2$ of A_{r-1} is bad in A_r.
$\rho = \Pi_2 + \Pi_2$.
a) $L = A_r = A_{2m-1}$. The case is similar to $\rho = \Pi_1 + \Pi_2$ of A_{2m}.
b) $L = A_r = A_{2m}$.
The zero weight δ occurs in A_{2m-1}. Hence it has Δ as its support in the rest diagram according to Facts 1.3 (see [2, page 11]). Therefore $\rho = \Pi_2 + \Pi_2$ of A_{2m} is bad.
The proof of the cases $\rho = \Pi_1 + \Pi_{r-1}, \Pi_2 + \Pi_{r-1}, \Pi_{r-1} + \Pi_{r-1}, \Pi_2 + \Pi_r, \Pi_r + \Pi_{r-1}$ is similar to the proof for $\rho = \Pi_1 + \Pi_2$ or $\rho = \Pi_2 + \Pi_2$ ($\Pi_1^* = \Pi_r$, $\Pi_2^* = \Pi_{r-1}$). □

Lemma 2.3. (Case B_r) *Consider the following list of reducible representations of B_r.*
R-list (B-2): $\Pi_1 + \Pi_1$ *of B_r, $r \geq 3$.*
All other reducible representations $\rho = \tau_1 + \tau_2$ are bad, where τ_1, τ_2 are from List(B) *(i.e., $\Pi_1 + \Pi_3, \Pi_3 + \Pi_3$ of B_3, $\Pi_1 + \Pi_4, \Pi_4 + \Pi_4$ of B_4).*

Proof. Let $n = 3$.
The reducible representations $\Pi_1 + \Pi_3, \Pi_3 + \Pi_3$ of B_3 are bad according to the R-list 3.2.1.
Let $n = 4$.
The reducible representations $\Pi_1 + \Pi_4, \Pi_4 + \Pi_4$ of B_4 are bad according to the R-list 3.2.2. □

3. Reducible good representations

Theorem 3.1. (R-good, 2-sum) *Let L be a simple Lie algebra and let $\rho = \tau_1 + \tau_2$ be a reducible representation with τ_1, τ_2 good. Then ρ is good, if ρ is in the following list:*

(i) *A_r: $\Pi_1 + \Pi_1, \Pi_1 + \Pi_r, \Pi_r + \Pi_r$ of A_r, $r \geq 3$ and $\Pi_2 + \Pi_2$ of A_3.*
(ii) *B_r: $\Pi_1 + \Pi_1$ of B_r, $r \geq 3$.*

Proof. The statement of the theorem follows in the Lemmas 3.2 and 3.3. □

Lemma 3.2. (Case A_r) *Let $L \cong A_r$, $r \geq 3$, and let $\rho = \tau_1 + \tau_2$ be a reducible representation of L with τ_1, τ_2 good.*

(1) *The irreducible components λ which occur in $S^n(\Pi_1 + \Pi_1)$ are exactly those from the type $\lambda = \Pi_1^{m_1}\Pi_2^{m_2}$, $m_1, m_2 \in \mathbb{N}_0$. Hence* $\mathrm{rank}(\Pi_1 + \Pi_1, A_r) = 2$ *and $\Pi_1 + \Pi_1$ of A_r is good.*
(2) *The irreducible components λ which occur in $S^n(\Pi_1 + \Pi_r)$ are exactly those from the type $\lambda = \Pi_1^{m_1}\Pi_r^{m_2}$, $m_1, m_2 \in \mathbb{N}_0$. Hence* $\mathrm{rank}(\Pi_1 + \Pi_r, A_r) = 2$ *and $\Pi_1 + \Pi_r$ of A_r is good.*
(3) *The irreducible components λ which occur in $S^n(\Pi_r + \Pi_r)$ are exactly those from the type $\lambda = \Pi_r^{m_1}\Pi_{r-1}^{m_2}$, $m_1, m_2 \in \mathbb{N}_0$. Hence* $\mathrm{rank}(\Pi_r + \Pi_r, A_r) = 2$ *and $\Pi_r + \Pi_r$ of A_r is good.*

Proof. The case (1): According to *Panyushev* (see [2, page 47], [7]) $\mathrm{rank}(\Pi_1 + \Pi_1, A_r) = \mathrm{rank} A_r - \mathrm{rank} K$, and $K = A_{r-2}$ (see [5],[6]).

$$\mathrm{rank}(\Pi_1 + \Pi_1, A_r) = \mathrm{rank} A_r - \mathrm{rank} A_{r-2} = r - (r-2) = 2, \tag{3.1}$$

$$\begin{gathered} S^2(\Pi_1 + \Pi_1) = S^2(\Pi_1) + S^1(\Pi_1) \otimes S^1(\Pi_1) + S^2(\Pi_1) = 2S^2(\Pi_1) + \Pi_1 \otimes \Pi_1 \\ \Pi_1 \otimes \Pi_1 = \Pi_1^2 + \Pi_2 \text{ with } \deg \Pi_2 = 2, \end{gathered} \tag{3.2}$$

According to (1) and (2) the irreducible components λ which occur in $S^n(\Pi_1+\Pi_1)$ are from the type $\lambda=\Pi_1^{m_1}\Pi_2^{m_2}$.

The case (2): According to *Panyushev* $\mathrm{rank}(\Pi_1+\Pi_r,A_r)=\mathrm{rank}A_r-\mathrm{rank}K$ and $\Pi_1^*=\Pi_r$.

$$\mathrm{rank}(\Pi_1+\Pi_r,A_r)=\mathrm{rank}A_r-\mathrm{rank}A_{r-2}=r-(r-2)=2. \tag{3.3}$$

According to (3) the irreducible components λ which occur in $S^n(\Pi_1+\Pi_r)$ are from the type $\lambda=\Pi_1^{m_1}\Pi_r^{m_2}$.

The case (3): According to *Panyushev* $\mathrm{rank}(\Pi_r+\Pi_r,A_r)=\mathrm{rank}A_r-\mathrm{rank}K$ and $\Pi_1^*=\Pi_r$.

$$\mathrm{rank}(\Pi_r+\Pi_r,A_r)=\mathrm{rank}A_r-\mathrm{rank}A_{r-2}=r-(r-2)=2, \tag{3.4}$$

$$\begin{gathered}S^2(\Pi_r+\Pi_r)=S^2(\Pi_r)+S^1(\Pi_r)\otimes S^1(\Pi_r)+S^2(\Pi_r)=2S^2(\Pi_r)+\Pi_r\otimes\Pi_r\\ \Pi_r\otimes\Pi_r=\Pi_r^2+\Pi_{r-1}\ \text{with}\deg\Pi_{r-1}=2,\end{gathered} \tag{3.5}$$

According to (4) and (5) the irreducible components λ which occur in $S^n(\Pi_r+\Pi_r)$ are from the type $\lambda=\Pi_r^{m_1}\Pi_{r-1}^{m_2}$. □

Lemma 3.3. (Case B_r) *Let $L\cong B_r$, $r\geq 3$, and let $\rho=\tau_1+\tau_2$ be a reducible representation with τ_1,τ_2 good. Then the irreducible components λ which occur in $S^n(\Pi_1+\Pi_1)$ are exactly those from the type $\lambda=\Pi_1^{m_1}\Pi_2^{m_2}$, $m_1,m_2\in\mathbb{N}_0$. Hence* $\mathrm{rank}(\Pi_1+\Pi_1,B_r)=2$ *and $\Pi_1+\Pi_1$ of B_r is good.*

Proof. The statement follows similarly to the case (1) of Lemma 3.2 for A_r with $K=B_{r-2}$. □

Theorem 3.4. (R-good, 3-sum) *Let L be a simple Lie algebra and let $\rho=\tau_1+\tau_2+\tau_3$ be a reducible representation with τ_1,τ_2,τ_3 good. Then ρ is good, if it is in the following list:*

(i) *A_r: $\Pi_1+\Pi_1+\Pi_1,\Pi_1+\Pi_1+\Pi_r,\Pi_1+\Pi_r+\Pi_r,\Pi_r+\Pi_r+\Pi_r$ of A_r, $r\geq 4$.*

(ii) *B_r: $\Pi_1+\Pi_1+\Pi_1$ of B_r, $r\geq 4$.*

Proof. The statement of the theorem follows from Lemmas 3.5 and 3.6. □

Lemma 3.5. (Case A_r) *Let $L\cong A_r$, $r\geq 4$, and let $\rho=\tau_1+\tau_2+\tau_3$ be a reducible representation of L with τ_1,τ_2,τ_3 good.*

(1) *The irreducible components λ which occur in $S^n(\Pi_1+\Pi_1+\Pi_1)$ are exactly those from the type $\lambda=\Pi_1^{m_1}\Pi_2^{m_2}\Pi_3^{m_3}$, $m_1,m_2,m_3\in\mathbb{N}_0$. Hence* $\mathrm{rank}(\Pi_1+\Pi_1+\Pi_1,A_r)=3$ *and $\Pi_1+\Pi_1+\Pi_1$ of A_r is good.*

(2) *The irreducible components λ which occur in $S^n(\Pi_r+\Pi_r+\Pi_r)$ are exactly those from the type $\lambda=\Pi_r^{m_1}\Pi_{r-1}^{m_2}\Pi_{r-2}^{m_3}$, $m_1,m_2,m_3\in\mathbb{N}_0$. Hence* $\mathrm{rank}(\Pi_r+\Pi_r+\Pi_r,A_r)=3$ *and $\Pi_r+\Pi_r+\Pi_r$ of A_r is good.*

(3) *The irreducible components λ which occur in $S^n(\Pi_1+\Pi_1+\Pi_r)$ are exactly those from the type $\lambda=\Pi_1^{m_1}\Pi_2^{m_2}\Pi_r^{m_3}$, $m_1,m_2,m_3\in\mathbb{N}_0$. Hence* $\mathrm{rank}(\Pi_1+\Pi_1+\Pi_r,A_r)=3$ *and $\Pi_1+\Pi_1+\Pi_r$ of A_r is good.*

(4) *The irreducible components λ which occur in $S^n(\Pi_1+\Pi_r+\Pi_r)$ are exactly those from the type $\lambda=\Pi_1^{m_1}\Pi_{r-1}^{m_2}\Pi_r^{m_3}$, $m_1,m_2,m_3\in\mathbb{N}_0$. Hence* $\mathrm{rank}(\Pi_1+\Pi_r+\Pi_r,A_r)=3$ *and $\Pi_1+\Pi_r+\Pi_r$ of A_r is good.*

Proof. The case (1): According to *Panyushev* (see [2, page 47], [7]) $\mathrm{rank}(\Pi_1+\Pi_1+\Pi_1,A_r)=\mathrm{rank}A_r-\mathrm{rank}K$, and $K=A_{r-3}$ (see [3,6]).

$$\mathrm{rank}(\Pi_1+\Pi_1+\Pi_1,A_r)=\mathrm{rank}A_r-\mathrm{rank}A_{r-3}=r-(r-3)=3. \tag{3.6}$$

In $S^2(\Pi_1+\Pi_1+\Pi_1)=S^2(\Pi_1+\Pi_1)+S^1(\Pi_1+\Pi_1)\otimes S^1(\Pi_1)+S^2(\Pi_1)$

$$S^1(\Pi_1+\Pi_1)\otimes S^1(\Pi_1)=2\Pi_1\otimes\Pi_1=2\Pi_1^2+2\Pi_2\ \text{with}\deg\Pi_2=2, \tag{3.7}$$

$$\begin{gathered}S^3(\Pi_1+\Pi_1+\Pi_1)=S^3(\Pi_1+\Pi_1)+S^2(\Pi_1+\Pi_1)\otimes S^1\Pi_1+S^1(\Pi_1+\Pi_1)\otimes S^2\Pi_1+S^3\Pi_1,\\ S^2(\Pi_1+\Pi_1)\otimes S^1\Pi_1=(3\Pi_1^2+\Pi_2)\otimes\Pi_1=3\Pi_1^2\otimes\Pi_1+\Pi_2\otimes\Pi_1\\ \Pi_2\otimes\Pi_1=\Pi_1\Pi_2+\Pi_3\ \text{with}\ \deg\Pi_3=3.\end{gathered} \tag{3.8}$$

According to (6), (7), and (8) the irreducible components λ which occur in $S^n(\Pi_1+\Pi_1+\Pi_1)$ are from the type $\lambda = \Pi_1^{m_1}\Pi_2^{m_2}\Pi_3^{m_3}$.
The proof for $(\Pi_r+\Pi_r+\Pi_r, A_r)$ is similar to $(\Pi_1+\Pi_1+\Pi_1, A_r)$, since $\Pi_1^* = \Pi_r$. The irreducible components λ which occur in $S^n(\Pi_r+\Pi_r+\Pi_r)$ are from the type $\lambda = \Pi_r^{m_1}\Pi_{r-1}^{m_2}\Pi_{r-2}^{m_3}$ $(\Pi_2^* = \Pi_{r-1},\ \Pi_3^* = \Pi_{r-2})$.
The proofs of the cases (3) and (4) are similar to the proofs of the cases (1) and (2). □

Lemma 3.6. (Case B_r) *Let $L \cong B_r$, $r \geq 4$, and let $\rho = \tau_1+\tau_2+\tau_3$ be a reducible representation with τ_1, τ_2, τ_3 good. The irreducible components λ which occur in $S^n(\Pi_1 + \Pi_1 + \Pi_1)$ are exactly those from the type $\lambda = \Pi_1^{m_1}\Pi_2^{m_2}\Pi_3^{m_3}$, $m_1, m_2, m_3 \in \mathbb{N}_0$. Hence* $\operatorname{rank}(\Pi_1+\Pi_1+\Pi_1, B_r) = 3$ *and $\Pi_1+\Pi_1+\Pi_1$ of B_r is good.*

Proof. The arguments for the proof are similar to those for the proof for $(\Pi_1+\Pi_1+\Pi_1, A_r)$ with $K = B_{r-3}$. □

Theorem 3.7. (R-good, t-sum) *Let L be a simple Lie algebra and let $\rho = \tau_1+\tau_2+\cdots+\tau_t$ be a reducible representation with $\tau_1, \tau_2, \ldots, \tau_t$ good. Then ρ is good, if it is in the following list:*

(i) A_r: $\underbrace{\Pi_1+\Pi_1+\cdots+\Pi_1}_{t \text{ times}}$, $\underbrace{\Pi_r+\Pi_r+\cdots+\Pi_r}_{t \text{ times}}$ *(t-sum), $t < r \geq 3$, and*

$$\underbrace{\Pi_1+\Pi_1+\cdots+\Pi_1}_{k \text{ times}} + \underbrace{\Pi_r+\Pi_r+\cdots+\Pi_r}_{s \text{ times}} \quad (k+s)\text{-sum},\ k+s = t < r \geq 3.$$

(ii) B_r: $\underbrace{\Pi_1+\Pi_1+\cdots+\Pi_1}_{t \text{ times}}$ *(t-sum), $t < r \geq 3$.*

Proof. The statement follows from Lemma 3.8. □

Lemma 3.8. (Case A_r) *Let L be a simple Lie algebra, let $r \geq 5$ and let $\rho = \tau_1+\tau_2+\cdots+\tau_t$ be a reducible representation of L with $\tau_1, \tau_2, \ldots, \tau_t$ good, $t < r$.*

(1) $L \cong A_r$ *(or $L \cong B_r$)*
The irreducible components λ which occur in $S^n(\underbrace{\Pi_1+\Pi_1+\cdots+\Pi_1}_{t \text{ times}})$ (t-sum) are exactly those from the type $\lambda = \Pi_1^{m_1}\Pi_2^{m_2}\cdots\Pi_t^{m_t}$ with $m_1, m_2, \ldots, m_t \in \mathbb{N}_0$. Hence

$$\operatorname{rank}[\underbrace{\Pi_1+\Pi_1+\cdots+\Pi_1)}_{t \text{ times}}, A_r(B_r)] = t < r$$

and $\underbrace{\Pi_1+\Pi_1+\cdots+\Pi_1)}_{t \text{ times}}$ of A_r (or of B_r) is good.

(2) (a) $L \cong A_r$
The irreducible components λ which occur in $S^n(\underbrace{\Pi_r+\Pi_r+\cdots+\Pi_r}_{t \text{ times}})$ (t-sum) are exactly those from the type $\lambda = \Pi_r^{m_1}\Pi_{r-1}^{m_2}\cdots\Pi_{r+1-t}^{m_t}$ with $m_1, m_2, \ldots, m_t \in \mathbb{N}_0$. Hence $\operatorname{rank}[\underbrace{\Pi_r+\Pi_r+\cdots+\Pi_r)}_{t \text{ times}}, A_r] = t < r$ *and $\underbrace{\Pi_r+\Pi_r+\cdots+\Pi_r)}_{t \text{ times}}$ of A_r is good.*

(b) *The irreducible components λ which occur in*
$S^n(\underbrace{\Pi_1+\Pi_1+\cdots+\Pi_1)}_{k \text{ times}}+\underbrace{\Pi_r+\Pi_r+\cdots+\Pi_r)}_{s \text{ times}})$ *$(k+s)$-sum are exactly those from the type $\lambda = \Pi_1^{m_1}\Pi_2^{m_2}\cdots\Pi_k^{m_k}\Pi_r^{n_1}\Pi_{r-1}^{n_2}\cdots\Pi_{r+1-s}^{n_s}$ with $m_1, m_2, \ldots, m_k$ and*
$n_1, n_2, \ldots, n_s \in \mathbb{N}_0$. Hence

$$\operatorname{rank}[\underbrace{\Pi_1+\Pi_1+\cdots+\Pi_1)}_{k \text{ times}}+\underbrace{\Pi_r+\Pi_r+\cdots+\Pi_r)}_{s \text{ times}}, A_r] = k+s = t < r,$$

$$\text{and } \underbrace{\Pi_1 + \Pi_1 + \cdots + \Pi_1)}_{k \text{ times}} + \underbrace{\Pi_r + \Pi_r + \cdots + \Pi_r)}_{s \text{ times}} \text{ of } A_r \text{ is good.}$$

Proof. The case (1): According to *Panyushev* (see [2, page 47], [7]) $\mathrm{rank}(\Pi_1 + \Pi_2 + \cdots + \Pi_t, A_r) = \mathrm{rank}A_r - \mathrm{rank}K$, and $K = A_{r-t}$ (see [1,6]).

$$\mathrm{rank}(\Pi_1 + \Pi_2 + \cdots + \Pi_t, A_r) = \mathrm{rank}A_r - \mathrm{rank}A_{r-t} = r - (r-t) = t, \tag{3.9}$$

$$\pi_i \otimes \pi_j = \pi_i\pi_j + \pi_{i-1}\pi_{j+1} + \cdots + \begin{cases} \pi_{j+i}, & r+1-j-i \geq 0, \\ \pi_{j+i-r-1}, & r+1-j-i \leq 0, \end{cases} \tag{3.10}$$

$r \geq 2$, $i \leq j$. According to (9) and (10) the irreducible components which occur in $S^n\underbrace{(\Pi_1 + \Pi_1 + \cdots + \Pi_1)}_{t \text{ times}}$ are from the type $\lambda = \Pi_1^{m_1}\Pi_2^{m_2}\cdots\Pi_t^{m_t}$.

The proof of the case (2a) is similar to (1) (since $\Pi_i^* = \Pi_{r+1-i}$, $i = 1, 2, \ldots, r$) .

The case (2b):

$$\mathrm{rank}(\underbrace{\Pi_1 + \Pi_1 + \cdots + \Pi_1}_{k \text{ times}} + \underbrace{\Pi_r + \Pi_r + \cdots + \Pi_r}_{s \text{ times}}, A_r) = \mathrm{rank}A_r - \mathrm{rank}A_{r-(k+s)} = k+s = t < r. \tag{3.11}$$

According to (10) and (11) the irreducible components which occur in

$$S^n(\underbrace{\Pi_1 + \Pi_1 + \cdots + \Pi_1}_{k \text{ times}} + \underbrace{\Pi_r + \Pi_r + \cdots + \Pi_r}_{s \text{ times}})$$

are from the type $\lambda = \Pi_1^{m_1}\Pi_2^{m_2}\cdots\Pi_k^{m_k}\Pi_r^{n_1}\Pi_{r-1}^{n_2}\cdots\Pi_{r+1-s}^{n_s}$. □

Theorem 3.9. (Classification) *The following lists of ρ and for $L = A_r$, $L = B_r$ contain all reducible good representations and their ranks.*

(1) (i) $L = A_r$ *and* $\rho = \Pi_1 + \Pi_1$, $\rho = \Pi_1 + \Pi_r$, $\rho = \Pi_r + \Pi_r$, $r \geq 3$,

(ii) $L = A_3$ *and* $\rho = \Pi_2 + \Pi_2$,

(iii) $L = B_r$ *and* $\rho = \Pi_1 + \Pi_1$, $r \geq 3$.

Then $\mathrm{rank}M(\rho) = 2$.

(2) (i) $L = A_r$, $r \geq 4$ *and* $\rho = \Pi_1 + \Pi_1 + \Pi_1$, $\rho = \Pi_1 + \Pi_1 + \Pi_r$, $\rho = \Pi_1 + \Pi_r + \Pi_r$, $\rho = \Pi_r + \Pi_r + \Pi_r$,

(ii) $L = B_r$, $r \geq 4$ *and* $\rho = \Pi_1 + \Pi_1 + \Pi_1$.

Then $\mathrm{rank}M(\rho) = 3$.

(3) (i) $L = A_r$, $r \geq 5$ *and* $\rho = \Pi_1 + \Pi_1 + \Pi_1 + \Pi_1$, $\rho = \Pi_1 + \Pi_1 + \Pi_1 + \Pi_r$, $\rho = \Pi_1 + \Pi_1 + \Pi_r + \Pi_r$, $\rho = \Pi_1 + \Pi_r + \Pi_r + \Pi_r$, $\rho = \Pi_r + \Pi_r + \Pi_r + \Pi_r$,

(ii) $L = B_r$, $r \geq 5$ *and* $\rho = \Pi_1 + \Pi_1 + \Pi_1 + \Pi_1$.

Then $\mathrm{rank}M(\rho) = 4$.

(4) (i) $L = A_r$ *or* $L = B_r$, $\rho = \underbrace{\Pi_1 + \Pi_1 + \cdots + \Pi_1}_{t \text{ times}}$ *(t-sum)*, $t < r$,

(ii) $L = A_r$, $r \geq 6$, $\rho = \underbrace{\Pi_r + \Pi_r + \cdots + \Pi_r}_{t \text{ times}}$ *(t-sum)*, $t < r$,

$$\rho = \underbrace{\Pi_1 + \Pi_1 + \cdots + \Pi_1}_{k \text{ times}} + \underbrace{\Pi_r + \Pi_r + \cdots + \Pi_r}_{s \text{ times}} \ ((k+s)\text{-sum}),\ k+s = t < r.$$

Then $\operatorname{rank} M(\rho) = t$.

Proof. Everything follows from Lemmas 2.2, 2.3, 3.2, 3.3, 3.5, 3.6, 3.8. □

Here we summarize our results again. We order the reducible good representations but on a different principle, namely after rank.

Corollary 3.10. *Let* $L \cong A_r$ *and* $\rho = \tau_1 + \tau_2$ *be a reducible representation of* A_r *with* τ_1, τ_2 *good.*

(1) *For* A_r, $r \geq 3$*:*

(a) $S^n(\Pi_1 + \Pi_1) = \sum_{m_1,m_2} (m_1+1)\Pi_1^{m_1}\Pi_2^{m_2}$ *with* $m_1 + 2m_2 = n$, $m_1, m_2 \in \mathbb{N}_0$,

(b) $S^n(\Pi_1 + \Pi_r) = \sum_{m_1,m_2} \Pi_1^{m_1}\Pi_r^{m_2}$ *with* $m_1 + m_2 = n - 2k$, $k = 0, 1, \ldots, \left[\frac{n}{2}\right]$,

(c) $S^n(\Pi_r + \Pi_r) = \sum_{m_1,m_2} (m_1+1)\Pi_r^{m_1}\Pi_{r-1}^{m_2}$ *with* $m_1 + 2m_2 = n$, $m_1, m_2 \in \mathbb{N}_0$,

(2) *For* A_3*:*

$$S^n(\Pi_2 + \Pi_2) = \frac{(k+1)(k+2)}{2} \sum_{m_1,m_2} (m_1+1)\Pi_2^{m_1}(\Pi_1\Pi_3)^{m_2}$$

with $m_1 + 2m_2 = n - 2k$, $k = 0, 1, \ldots, \left[\frac{n}{2}\right]$.

Corollary 3.11. *Let* $L \cong B_r$ *and* $\rho = \tau_1 + \tau_2$ *be a reducible representation of* B_r, $r \geq 3$, *with* τ_1, τ_2 *good. Then*

$$S^n(\Pi_1 + \Pi_1) = \frac{(k+1)(k+2)}{2} \sum_{m_1,m_2} (m_1+1)\Pi_1^{m_1}\Pi_2^{m_2}$$

with $m_1 + 2m_2 = n - 2k$, $k = 0, 1, \ldots, \left[\frac{n}{2}\right]$.

4. R-List 3

R-List 3.1 A_r:
R-List 3.1.1 A_3:
$S^2(\pi_1 + \pi_2) = \pi_1^2 + \pi_2^2 + \pi_3 + \pi_1\pi_2 + \pi_0$,
$S^2(\pi_2 + \pi_3) = \pi_2^2 + \pi_3^2 + \pi_1 + \pi_2\pi_3 + \pi_0$;

R-List 3.1.2 A_4:
$S^2(\pi_1 + \pi_2) = \pi_1^2 + \pi_2^2 + \pi_3 + \pi_4 + \pi_1\pi_2$,
$S^2(\pi_1 + \pi_3) = \pi_1^2 + \pi_3^2 + \pi_1 + \pi_4 + \pi_1\pi_3$,
$S^3(\pi_1 + \pi_3) = \pi_1^3 + \pi_3^3 + \pi_1\pi_3^2 + \pi_1^2\pi_3 + \pi_1\pi_3 + \pi_1\pi_4 + \pi_3\pi_4 + \pi_1^2 + \pi_2$,
$S^2(\pi_2 + \pi_2) = 3\pi_2^2 + 3\pi_4 + \pi_1\pi_3$,
$S^3(\pi_2 + \pi_2) = 4\pi_2^3 + 6\pi_2\pi_4 + 2\pi_1\pi_2\pi_3 + 2\pi_1$,
$S^4(\pi_2 + \pi_2) = 5\pi_2^4 + 9\pi_2^2\pi_4 + 6\pi_4^2 + 3\pi_1\pi_2^2\pi_3 + 3\pi_3 + 3\pi_1\pi_3\pi_4 + 4\pi_1\pi_2 + \pi_1^2\pi_3^2$,
$S^2(\pi_2 + \pi_3) = \pi_2^2 + \pi_3^2 + \pi_1 + \pi_4 + \pi_1\pi_4 + \pi_2\pi_3 + \pi_0$,
$S^2(\pi_2 + \pi_4) = \pi_2^2 + \pi_4^2 + \pi_1 + \pi_4 + \pi_2\pi_4$,
$S^3(\pi_2 + \pi_4) = \pi_2^3 + \pi_4^3 + \pi_2\pi_4^2 + \pi_2^2\pi_4 + \pi_1\pi_2 + \pi_1\pi_4 + \pi_2\pi_4 + \pi_3 + \pi_4^2$,
$S^2(\pi_3 + \pi_3) = \pi_3^2 + 3\pi_1 + \pi_2\pi_4$,
$S^3(\pi_3 + \pi_3) = 4\pi_3^3 + 6\pi_1\pi_3 + 2\pi_2\pi_3\pi_4 + 2\pi_4$,
$S^4(\pi_3 + \pi_3) = 5\pi_3^4 + 9\pi_1\pi_3^2 + 6\pi_1^2 + 3\pi_2\pi_3^2\pi_4 + 3\pi_2 + 3\pi_1\pi_2\pi_4 + 4\pi_3\pi_4 + \pi_2^2\pi_4^2$,
$S^2(\pi_3 + \pi_4) = \pi_3^2 + \pi_4^2 + \pi_2 + \pi_1 + \pi_3\pi_4$;

R-List 3.1.3 A_5:

$S^2(\pi_1+\pi_2) = \pi_1^2+\pi_2^2+\pi_3+\pi_4+\pi_1\pi_2,$
$S^3(\pi_1+\pi_2) = \pi_1^3+\pi_2^3+\pi_2\pi_4+\pi_2\pi_3+\pi_1\pi_3+\pi_1\pi_2^2+\pi_1^2\pi_2+\pi_1\pi_4+\pi_5+\pi_0,$
$S^2(\pi_1+\pi_3) = \pi_1^2+\pi_3^2+\pi_1\pi_5+\pi_1\pi_3+\pi_4,$
$S^3(\pi_1+\pi_3) = \pi_1^3+\pi_3^3+\pi_1^2\pi_3+\pi_1\pi_3^2+\pi_1^2\pi_5+\pi_1\pi_4+\pi_2\pi_5+\pi_3\pi_4+\pi_1\pi_3\pi_5+\pi_1+\pi_3,$
$S^4(\pi_1+\pi_3) = \pi_1^4+\pi_3^4+\pi_1\pi_3^2\pi_5+\pi_1^2\pi_5^2+\pi_2\pi_4+\pi_3^2+\pi_3^2\pi_4+\pi_1\pi_3^3+\pi_2\pi_3\pi_5+\pi_1\pi_4\pi_5+$
$2\pi_1\pi_3+\pi_1^2\pi_3\pi_5+\pi_4+\pi_4^2+\pi_1\pi_3\pi_4+\pi_1^2\pi_3^2+\pi_2+\pi_1\pi_2\pi_5+\pi_1^2+\pi_1^3\pi_5+\pi_1^2\pi_4+\pi_1^3\pi_3+\pi_0,$
$S^2(\pi_1+\pi_4) = \pi_1^2+\pi_4^2+\pi_2+\pi_5+\pi_1\pi_4,$
$S^3(\pi_1+\pi_4) = \pi_1^3+\pi_4^3+\pi_1^2\pi_4+\pi_1\pi_4^2+\pi_2\pi_4+\pi_1\pi_2+\pi_4\pi_5+\pi_1\pi_5+\pi_3+\pi_0,$
$S^2(\pi_2+\pi_2) = 3\pi_2^2+3\pi_4+\pi_1\pi_3,$
$S^3(\pi_2+\pi_2) = 4\pi_2^3+6\pi_2\pi_4+2\pi_1\pi_2\pi_3+2\pi_1\pi_5+4\pi_0,$
$S^4(\pi_2+\pi_2) = 5\pi_2^4+9\pi_2^2\pi_4+6\pi_4^2+9\pi_2+3\pi_1\pi_2^2\pi_3+3\pi_1\pi_3\pi_4+4\pi_1\pi_2\pi_5+\pi_1^2\pi_3^2+3\pi_3\pi_5,$
$S^2(\pi_2+\pi_3) = \pi_2^2+\pi_3^2+\pi_2\pi_3+\pi_1\pi_4+\pi_1\pi_5+\pi_4+\pi_5,$
$S^3(\pi_2+\pi_3) = \pi_2^3+\pi_3^3+\pi_1\pi_3\pi_5+\pi_3+\pi_2\pi_3^2+2\pi_3\pi_5+\pi_1\pi_3\pi_4+\pi_1\pi_2\pi_5+\pi_2+\pi_1^2+\pi_2^2\pi_3+$
$2\pi_2\pi_5+\pi_1\pi_2\pi_4+\pi_3\pi_4+\pi_2\pi_4+\pi_1+\pi_0,$
$S^2(\pi_2+\pi_4) = \pi_2^2+\pi_4^2+\pi_2+\pi_4+\pi_2\pi_4+\pi_1\pi_5+\pi_0,$
$S^3(\pi_2+\pi_4) = \pi_4^3+2\pi_2\pi_4+2\pi_0+\pi_2\pi_4^2+2\pi_4+\pi_1\pi_4\pi_5+\pi_2^2+\pi_1\pi_3+\pi_2^2\pi_4+2\pi_2+\pi_1\pi_2\pi_5+$
$\pi_4^2 4+\pi_3\pi_5+\pi_2^3,$
$S^2(\pi_2+\pi_5) = \pi_2^2+\pi_5^2+\pi_2\pi_5+\pi_1+\pi_4,$
$S^3(\pi_2+\pi_5) = \pi_2^3+\pi_5^3+\pi_2^2\pi_5+\pi_2\pi_5^2+\pi_1\pi_2+\pi_1\pi_5+\pi_2\pi_4+\pi_4\pi_5+\pi_3+\pi_0,$
$S^2(\pi_3+\pi_3) = 3\pi_3^2+3\pi_1\pi_5+\pi_2\pi_4+\pi_0,$
$S^3(\pi_3+\pi_3) = 4\pi_3^3+6\pi_1\pi_3\pi_5+6\pi_3+2\pi_2\pi_3\pi_4+2\pi_4\pi_5+2\pi_1\pi_2,$
$S^2(\pi_3+\pi_4) = \pi_3^2+\pi_4^2+\pi_3\pi_4+\pi_1\pi_5+\pi_2\pi_5+\pi_1+\pi_2,$
$S^3(\pi_3+\pi_4) = \pi_3^3+\pi_4^3+\pi_1\pi_3\pi_5+\pi_3+\pi_3^2\pi_4+2\pi_1\pi_3+\pi_2\pi_3\pi_5+\pi_1\pi_4\pi_5+\pi_4+\pi_5^2+\pi_3\pi_4^2+$
$2\pi_1\pi_4+\pi_2\pi_4\pi_5+\pi_2\pi_3+\pi_2\pi_4+\pi_5+\pi_0,$
$S^2(\pi_3+\pi_5) = \pi_3^2+\pi_5^2+\pi_1\pi_5+\pi_3\pi_5+\pi_2,$
$S^3(\pi_3+\pi_5) = \pi_3^3+\pi_5^3+\pi_3\pi_5^2+\pi_3^2\pi_5+\pi_1\pi_5^2+\pi_2\pi_5+\pi_1\pi_4+\pi_2\pi_3+\pi_1\pi_3\pi_5+\pi_3+\pi_5,$
$S^4(\pi_3+\pi_5) = \pi_3^4+\pi_5^4+\pi_1\pi_3^2\pi_5+\pi_1^2\pi_5^2+\pi_2\pi_4+\pi_3^2+\pi_2\pi_3^2 3+\pi_3^3\pi_5+\pi_1\pi_3\pi_4+\pi_1\pi_2\pi_5+$
$2\pi_3\pi_5+\pi_1\pi_3\pi_5^2+\pi_2+\pi_2^4+\pi_2\pi_3\pi_5+\pi_3^3\pi_5^3+\pi_4+\pi_1\pi_4\pi_5+\pi_5^2+\pi_1\pi_5^3+\pi_2\pi_5^2 5+\pi_3\pi_5^3+\pi_0,$
$S^2(\pi_4+\pi_4) = 3\pi_4^2+3\pi_2+\pi_3\pi_5,$
$S^3(\pi_4+\pi_4) = 4\pi_4^3+6\pi_2\pi_4+2\pi_3\pi_4\pi_5+2\pi_1\pi_5+4\pi_0,$
$S^4(\pi_4+\pi_4) = 5\pi_4^4+9\pi_2\pi_4^2+6\pi_2^2+9\pi_4+3\pi_3\pi_4^2\pi_5+3\pi_2\pi_3\pi_5+4\pi_1\pi_4\pi_5+\pi_3^2\pi_5^2+3\pi_1\pi_3,$
$S^2(\pi_4+\pi_5) = \pi_5^2+\pi_4^2+\pi_3+\pi_2+\pi_4\pi_5,$
$S^3(\pi_4+\pi_5) = \pi_5^3+\pi_4^3+\pi_2\pi_4+\pi_3\pi_4+\pi_3\pi_5+\pi_2\pi_5+\pi_4^2\pi_5+\pi_5^2\pi_4+\pi_1+\pi_0.$

R-List 3.2 B_r:
R-List 3.2.1 B_3:
$S^2(\pi_1+\pi_3) = \pi_1^2+\pi_3^2+\pi_1\pi_3+\pi_3+2\pi_0,$
$S^2(\pi_1+\pi_3) = \pi_1^3+\pi_3^3+\pi_1^2\pi_3+\pi_1\pi_3^2+\pi_1^2+\pi_2+\pi_1\pi_3+\pi_3^2+2\pi_1+2\pi_3,$
$S^2(\pi_3+\pi_3) = 3\pi_3^2+\pi_1+\pi_2+\pi_0;$
R-List 3.2.2 B_4:
$S^2(\pi_1+\pi_4) = \pi_1^2+\pi_4^2+\pi_1\pi_4+\pi_1+\pi_4+2\pi_0,$
$S^3(\pi_1+\pi_4) = \pi_1^3+\pi_4^3+\pi_1^2\pi_4+\pi_1\pi_4^2+\pi_1^2+\pi_4^2+2\pi_1\pi_4+\pi_3+\pi_2+2\pi_1+2\pi_4+\pi_0,$
$S^2(\pi_4+\pi_4) = \pi_4^2+\pi_1+\pi_2+\pi_3+\pi_0.$

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