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Research Article

Restricted Liu estimator in generalized linear models: Monte Carlo simulation studies on gamma and Poisson distributed responses

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Abstract

In this study, we introduce iterative restricted Liu estimator to combat multicollinearity in generalized linear models. We also obtain necessary and sufficient conditions for the superiority of the first-order approximated restricted Liu estimator over the first-order approximated maximum likelihood and Liu estimators by the approximated mean squared error criterion. The results are illustrated by conducting simulation studies and numerical examples.

Keywords: Restricted Liu estimation, Multicollinearity, Generalized linear models, Mean squared error, Gamma distribution, Poisson distribution.

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1. Introduction

Collinearity has long been recognised as a potential source of problem in the estimation, computation and interpretation of linear model parameters [1]. Although maximum likelihood (ML) estimator is the most widely used method in linear regression, ML estimates become unstable and covariance matrix of the ML estimator inflates when collinearity exists among the explanatory variables. Therefore, in the existence of the multicollinearity, estimators such as restricted estimator, Liu estimator [9], the restricted Liu estimator [7] and others were proposed as alternative to the ML estimator to reduce the adverse effects of the multicollinearity. As in the case in linear regression, collinearity causes difficulty in the interpretation of estimated regression coefficients in generalized

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linear models (GLMs). Mackinnon and Puterman [10] study how to detect the collinearity in GLMs. Marx and Smith [12] present a principle component estimator for GLMs, and show that it can be useful with the presence of an ill-conditioned information matrix. Nyquist [15] considers the maximum likelihood estimation in GLMs under linear restrictions on the parameters. Segerstedt [17] introduces the ordinary ridge regression estimator for GLMs. Kurtoğlu and Özkale [8] derive Liu estimator for GLMs. In addition to observations on the response and explanatory variables, auxiliary information on the vector of regression coefficients can exist. Then, auxiliary information, especially exact linear restrictions, can be used to overcome multicollinearity. Therefore, it is the objective of this paper to present a restricted Liu estimator in GLMs. Both iterative and first-order approximated restricted Liu (FOARL) estimators are developed using quadratic penalized likelihood. Because there are different estimators used in the case of multicollinearity in linear regression, these estimators can also be defined for GLMs. As in linear regression, estimation in GLMs is also sensitive to multicollinearity.

This article is organized as follows. In Section 2, we propose iterative restricted Liu estimator in GLMs in the existence of multicollinearity. In Section 3, the mean squared error (MSE) properties of FOARL estimator in GLMs and the performance of the FOARL estimator are obtained, numerical examples are given in Section 4. Finally, two Monte Carlo simulation studies are presented in Section 5. In Section 6, the conclusions of the paper are presented.

2. The FOARL Estimation in GLMs

Suppose that $y_1, \ldots, y_n$ be the observations of independent random variables $Y_1, \ldots, Y_n$, each of which has the probability density function such that the mean of Y_i is μ_i and the corresponding canonical parameter is $\theta_i = g(\mu_i)$, $i = 1, \ldots, n$. The exponential family can be written in the form

$$f(y_i, \theta_i, \phi) = \exp\left[\frac{\theta_i y_i - b(\theta_i)}{a(\phi)} + c(y_i, \phi)\right], \quad i = 1, \ldots, n,$$

where $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are known functions and $\phi > 0$ is a dispersion parameter, which in some families takes on a fixed known value, while in other families it is an unknown parameter to be estimated from the data along with θ. For each observation values of a set of q explanatory variables, $x_i^\top = (x_{i1}, \ldots, x_{iq})$ is also recorded where $x_i^\top$ is the ith row of the design matrix $\mathbf{X} = (x_1, \ldots, x_n)^\top$ which denotes the $n \times q$ matrix of explanatory variables, $\eta_i = x_i^\top \boldsymbol{\beta}$ is the linear predictor where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q)^\top$ is the $q \times 1$ vector of parameters, also known as regression coefficients. The log-likelihood function is given by

$$l(\theta, \phi, y) = \sum_{i=1}^{n} \left[\frac{\theta_i y_i - b(\theta_i)}{a(\phi)} + c(y_i, \phi)\right], \tag{2.1}$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)^\top$, $\mathbf{y} = (y_1, \ldots, y_n)^\top$.

We suppose that in addition to the linear predictor, the set of q linearly independent restrictions on the parameter vector $\boldsymbol{\beta}$ exist: $\mathbf{R}_t^\top \boldsymbol{\beta} = \mathbf{r}_t$, $t = 1, \ldots, T$, where $\mathbf{R}_t^\top = (R_{t1}, \ldots, R_{tq})$ are $q \times 1$ vectors and r_t are scalars. Alternatively, we define the restricted Liu estimator for GLMs with the log-likelihood function in (2.1) with respect to quadratic penaly function $(\mathbf{R}_t^\top \boldsymbol{\beta} - r_t)^2$. The objective function is

$$\begin{aligned}\psi(\boldsymbol{\beta}, d, \boldsymbol{\lambda}) \quad &= \quad \left[\sum_{i=1}^{n} \frac{\theta_i y_i - b(\theta_i)}{a(\phi)} + c(y_i, \phi)\right] \\ &\quad - \frac{1}{2a(\phi)}(\boldsymbol{\beta} - d\hat{\boldsymbol{\beta}})^\top(\boldsymbol{\beta} - d\hat{\boldsymbol{\beta}}) - \frac{1}{2a(\phi)}\sum_{t=1}^{T} \lambda_t(\mathbf{R}_t^\top \boldsymbol{\beta} - \mathbf{r}_t)^2,\end{aligned}$$

where $0 < d < 1$ is Liu-biasing parameter, Lagrange multipliers and $\hat{\boldsymbol{\beta}}$ is the iteratively reweighted least squares (IRLS) estimator evaluated at the final iteration.

Differentiating $\psi(\boldsymbol{\beta}, d, \boldsymbol{\lambda})$ with respect to $\boldsymbol{\beta}$ and making use of Fisher's method of scoring, we get the $q \times 1$ vector

$$\begin{aligned}& Q(\boldsymbol{\beta}, d, \boldsymbol{\lambda})|_{\beta=\hat{\beta}_r(d)^{(m)}} = \left[\tfrac{\partial\psi(\boldsymbol{\beta},d,\boldsymbol{\lambda})}{\partial\boldsymbol{\beta}}\right]_{\beta=\hat{\beta}_r(d)^{(m)}} \\ & = \tfrac{1}{a(\phi)}[\mathbf{X}^\top \mathbf{W}\mathbf{D}(\mathbf{y} - \boldsymbol{\mu}) - \boldsymbol{\beta} + d\hat{\boldsymbol{\beta}} - \mathbf{R}^\top \boldsymbol{\Lambda}^*(\mathbf{R}\boldsymbol{\beta} - \mathbf{r})]_{\beta=\hat{\beta}_r(d)^{(m)}} \quad ,\end{aligned} \tag{2.2}$$

where $\boldsymbol{\Lambda}^* = diag(\lambda_1, \ldots, \lambda_T)$ is the matrix of Lagrange multipliers, $\mathbf{R}^\top = (R_1 \ldots R_T)$ is the $q \times T$ known auxiliary information matrix that expresses the structure of information on the individual parameters or some linear combinatory of these parameters, $\mathbf{r} = (r_1 \ldots r_T)^\top$ is the $1 \times T$ vector of scalars and $\mathbf{W} = diag\{w_{ii}\}$ is the $n \times n$ diagonal matrix with weights $w_{ii} = \{1/\text{var}(Y_i)\}\,(d\mu_i/d\eta_i)^2$. Both R and r are assumed to be known and in addition $rank(R) = T$ is assumed. The jth $(j = 1, \ldots, q)$ element of $Q(\boldsymbol{\beta}, d, \boldsymbol{\lambda})$ is

$$\begin{aligned}& q_j(\boldsymbol{\beta}, d, \boldsymbol{\lambda}) = \tfrac{\partial\psi(\boldsymbol{\beta},d,\boldsymbol{\lambda})}{\partial\beta_j} \\ & = \tfrac{1}{a(\phi)}[\sum_{i=1}^{n} \tfrac{y_i - \mu_i}{w_i} g'(\mu_i) x_{ij} - \beta_j + d\hat{\beta}_j - \sum_{t=1}^{T} \mathbf{R}_{tj}\lambda_t(\mathbf{R}_t^\top \boldsymbol{\beta} - r_t)]\ .\end{aligned}$$

Second-order partial derivatives of $\psi(\boldsymbol{\beta}, d, \boldsymbol{\lambda})$ gives the Hessian matrix $\mathrm{H}_l(\boldsymbol{\beta}, d, \boldsymbol{\lambda}) = \partial^2\psi(\boldsymbol{\beta}, d, \boldsymbol{\lambda})/\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'$ having the (j, v)th entry as

$$\begin{aligned}\frac{\partial^2\psi(\boldsymbol{\beta}, d, \boldsymbol{\lambda})}{\partial\beta_j\partial\beta_v} \quad &= \quad \frac{1}{a(\phi)}[\sum_{i=1}^{n}(y_i - \mu_i)\frac{\partial}{\partial\beta_v}\left(\frac{g'(\mu_i)}{w_i}x_{ij}\right) \\ & \quad - \sum_{i=1}^{n}\frac{x_{ij}}{w_i}x_{iv} - \delta_{jv} - \sum_{t=1}^{T}\lambda_t \mathbf{R}_{tj}\mathbf{R}_{tv}],\end{aligned} \tag{2.3}$$

where $\delta_{jv} = 1$ if $j = v$ and zero otherwise.

Taking the expected value of both sides of (2.3), we get

$$\begin{aligned}s_{jv}(\boldsymbol{\beta}, d, \boldsymbol{\lambda}) \quad &= \quad -E\left[\frac{\partial^2\psi(\boldsymbol{\beta}, d, \boldsymbol{\lambda})}{\partial\beta_j\partial\beta_v}\right] \\ &= \quad \frac{1}{a(\phi)}\left[\sum_{i=1}^{n}\frac{1}{w_i}x_{ij}x_{iv} + \delta_{jv} + \sum_{t=1}^{T}\lambda_t \mathbf{R}_{tj}\mathbf{R}_{tv}\right].\end{aligned}$$

Consequently, the expected value of $\mathrm{H}_l(\boldsymbol{\beta}, d, \boldsymbol{\lambda})$ is of the form

$$\mathrm{E}[\mathrm{H}_l(\boldsymbol{\beta}, d, \boldsymbol{\lambda})] = \frac{1}{a(\phi)}[(\mathbf{X}^\top \mathbf{W}\mathbf{X} + \mathbf{I}) + \mathbf{R}^\top \boldsymbol{\Lambda}^*\mathbf{R}]. \tag{2.4}$$

Application of the Fisher scoring method yields

$$\begin{aligned}\hat{\boldsymbol{\beta}}_r(d)^{(m+1)} &= \hat{\boldsymbol{\beta}}_r(d)^{(m)} + \left\{\{\mathrm{E}[\mathrm{H}_l(\boldsymbol{\beta}, d, \boldsymbol{\lambda})]\}^{-1}\left[\frac{\partial\psi(\boldsymbol{\beta}, d, \boldsymbol{\lambda})}{\partial\boldsymbol{\beta}}\right]\right\}_{\beta=\hat{\beta}_r(d)^{(m)}}\\ &= \hat{\boldsymbol{\beta}}(d)^{(m+1)} - (\mathbf{X}^\top\mathbf{W}^{(m)}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top[(\boldsymbol{\Lambda}^*)^{-1}\\ &+\mathbf{R}(\mathbf{X}^\top\mathbf{W}^{(m)}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}}(d)^{(m+1)}-\mathbf{r}),\end{aligned}$$

where $\hat{\boldsymbol{\beta}}(d)^{(m+1)} = (\mathbf{X}^\top\mathbf{W}^{(m)}\mathbf{X}+\mathbf{I})^{-1}(\mathbf{X}^\top\mathbf{W}^{(m)}\mathbf{X}+d\mathbf{I})\hat{\boldsymbol{\beta}}^{(m+1)}$ is in the form of the proposed Liu estimator by Kurtoğlu and Özkale [8] and $\hat{\boldsymbol{\beta}}^{(m+1)}$ is in the form of the ML estimator in GLMs and both $\hat{\boldsymbol{\beta}}(d)^{(m+1)}$ and $\hat{\boldsymbol{\beta}}^{(m+1)}$ are evaluated at the iteration algorithm (derivation details are given in Appendix A).

The $(m+1)$th approximation of the restricted Liu estimator is finally obtained as

$$\begin{aligned}\hat{\boldsymbol{\beta}}_r(d)^{(m+1)} &= \lim_{\lambda_1,\ldots,\lambda_q\to\infty}\hat{\boldsymbol{\beta}}_r(d)^{(m+1)}\\ &= \hat{\boldsymbol{\beta}}(d)^{(m+1)} - (\mathbf{X}^\top\mathbf{W}^{(m)}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top[\mathbf{R}(\mathbf{X}^\top\mathbf{W}^{(m)}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}\\ &\quad\times(\mathbf{R}\hat{\boldsymbol{\beta}}(d)^{(m+1)}-\mathbf{r}).\end{aligned} \tag{2.5}$$

In the special case of $d = 0$, $\hat{\boldsymbol{\beta}}_r(d)^{(m+1)}$ equals to the restricted estimator which is introduced by Nyquist [15]. The estimator in (2.5) is calculated iteratively. The estimator in the first step can be written as

$$\begin{aligned}\hat{\boldsymbol{\beta}}_r(d)^{(1)} &= \hat{\boldsymbol{\beta}}(d)^{(1)} - (\mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top[\mathbf{R}(\mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}\\ &\quad\times(\mathbf{R}\hat{\boldsymbol{\beta}}(d)^{(1)}-\mathbf{r}),\end{aligned}$$

where $\hat{\boldsymbol{\beta}}(d)^{(1)} = (\mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X}+\mathbf{I})^{-1}(\mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X}+d\mathbf{I})\hat{\boldsymbol{\beta}}^{(1)}$ is the first-order approximated Liu (FOAL) estimator, $\hat{\boldsymbol{\beta}}^{(1)} = (\mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{z}^{(0)}$ is the first-order approximated ML (FOAML) estimator and $\mathbf{z}^{(0)} = \mathbf{X}\boldsymbol{\beta}^{(0)} + [\mathbf{D}(\mathbf{y}-\boldsymbol{\mu})]_{\beta=\beta^{(0)}}$ is the working response. The first-order approximated restricted ML (FOARML) estimator in GLMs given by Nyquist [15] is

$$\hat{\boldsymbol{\beta}}_r^{(1)} = \hat{\boldsymbol{\beta}}^{(1)} - (\mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X})^{-1}\mathbf{R}^\top[\mathbf{R}(\mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X})^{-1}\mathbf{R}^\top]^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}}^{(1)}-\mathbf{r}).$$

Thus, the weight matrix and the working response of $\hat{\boldsymbol{\beta}}_r(d)^{(1)}$, $\hat{\boldsymbol{\beta}}(d)^{(1)}$, $\hat{\boldsymbol{\beta}}^{(1)}$ and $\hat{\boldsymbol{\beta}}_r^{(1)}$ are calculated in the same $\boldsymbol{\beta}^{(0)}$ initial value or real parameter and all these four estimators have the same working response.

3. Superiority of the FOARL estimator over other estimators by the matrix MSE criterion

In this section, we compare the FOARL estimator to the FOAML, FOARML and FOAL estimators according to the matrix MSE criterion. We compute approximated biasing vector and variance covariance matrix of the estimator $\hat{\boldsymbol{\beta}}_r(d)^{(1)}$ as

$$\mathrm{Bias}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}] = (d-1)\mathbf{M}_1\boldsymbol{\beta} - \mathbf{S}_1^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}_1^{-1}\mathbf{R}^\top)^{-1}(\mathbf{R}\boldsymbol{\beta}-\mathbf{r}),$$

$$\mathrm{Var}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}] = a(\phi)\mathbf{M}_1\mathbf{S}_d\mathbf{S}_1^{-1}\mathbf{S}_d\mathbf{M}_1,$$

where $\mathbf{M}_1 = \mathbf{S}_1^{-1} - \mathbf{S}_1^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}_1^{-1}\mathbf{R}^\top)^{-1}\mathbf{R}\mathbf{S}_1^{-1}$, $\mathbf{S}_d = \mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X}+d\mathbf{I}$, and $\mathbf{S}_1 = \mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X}+\mathbf{I}$. Then, the matrix MSE of the estimator $\hat{\boldsymbol{\beta}}_r(d)^{(1)}$ is

$$\begin{aligned}\mathrm{MSE}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}] &= a(\phi)\mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1 + [(d-1)\mathbf{M}_1\beta - \mathbf{S}_1^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}_1^{-1}\mathbf{R}^\top)^{-1}\\ &\quad\times(\mathbf{R}\boldsymbol{\beta}-\mathbf{r})][(d-1)\mathbf{M}_1\boldsymbol{\beta} - \mathbf{S}_1^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}_1^{-1}\mathbf{R}^\top)^{-1}(\mathbf{R}\boldsymbol{\beta}-\mathbf{r})]^\top.\end{aligned}$$

Similarly, the MSE matrices of the estimators $\hat{\boldsymbol{\beta}}^{(1)}$, $\hat{\boldsymbol{\beta}}_r^{(1)}$ and $\hat{\boldsymbol{\beta}}(d)^{(1)}$ are respectively as

$$\mathrm{MSE}[\hat{\boldsymbol{\beta}}^{(1)}] = a(\phi)\mathbf{S}^{-1},$$

$$\begin{aligned}\mathrm{MSE}[\hat{\boldsymbol\beta}_r^{(1)}] &= a(\phi)\mathbf{M}_0 + \mathbf{S}^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}^{-1}\mathbf{R}^\top)^{-1}\\ &\quad \times(\mathbf{R}\boldsymbol\beta-\mathbf{r})(\mathbf{R}\boldsymbol\beta-\mathbf{r})^\top[\mathbf{S}^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}^{-1}\mathbf{R}^\top)^{-1}]^\top,\end{aligned}$$

and

$$\mathrm{MSE}[\hat{\boldsymbol\beta}(d)^{(1)}] = a(\phi)\mathbf{S}_1^{-1}\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{S}_1^{-1} + (d-1)^2\mathbf{S}_1^{-1}\boldsymbol\beta\boldsymbol\beta^\top\mathbf{S}_1^{-1},$$

where $\mathbf{S} = \mathbf{X}^\top\mathbf{W}^{(0)}\mathbf{X}$ and $\mathbf{M}_0 = \mathbf{S}^{-1} - \mathbf{S}^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}^{-1}\mathbf{R}^\top)^{-1}\mathbf{R}\mathbf{S}^{-1}$.

3.1. Comparisons when the restrictions are true, i.e., $\boldsymbol{R\beta}$=$\boldsymbol{r}$. We examine the matrix MSE comparisons of $\hat{\boldsymbol\beta}_r(d)^{(1)}$ and $\hat{\boldsymbol\beta}_r^{(1)}$ in GLMs when the restrictions hold true. We see that the MSE matrices of the estimators in GLMs are similar to that of the corresponding estimators in linear regression models. However, the points that should be emphasized are that the response in GLMs is from the exponential family while the response in linear regression is usually from normal distribution. Furthermore, the MSE matrices depend on the matrix $\mathbf{X}^\top\mathbf{W}\mathbf{X}$ in GLMs while on the matrix $\mathbf{X}^\top\mathbf{X}$ in linear regression. As a consequence, although the results in their closed forms seem to be similar they are not exactly the same.

3.1.1. *The comparison between the FOARL estimator and the FOAL estimator.* For the superiority of $\hat{\boldsymbol\beta}_r(d)^{(1)}$ over $\hat{\boldsymbol\beta}(d)^{(1)}$ when $\mathbf{R}\boldsymbol\beta = \mathbf{r}$ holds true, Theorem 3.1 can be given. The difference $\boldsymbol\Delta_1 = \mathrm{MSE}[\hat{\boldsymbol\beta}(d)^{(1)}] - \mathrm{MSE}[\hat{\boldsymbol\beta}_r(d)^{(1)}]$ is given by $\boldsymbol\Delta_1 = \mathbf{B} - \mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1$ where $\mathbf{B} = \mathrm{MSE}[\hat{\boldsymbol\beta}(d)^{(1)}]$.

3.1. Theorem. *The estimator $\hat{\boldsymbol\beta}_r(d)^{(1)}$ is superior to the estimator $\hat{\boldsymbol\beta}(d)^{(1)}$ by the criterion of matrix MSE if and only if $\lambda_{\max}(\mathbf{B}^{-1}\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1) \le 1$ where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1\mathbf{B}^{-1}$.*

Proof. For the superiority of $\hat{\boldsymbol\beta}_r(d)^{(1)}$ over $\hat{\boldsymbol\beta}(d)^{(1)}$, we examine the difference $\boldsymbol\Delta_1 = \mathrm{MSE}[\hat{\boldsymbol\beta}(d)^{(1)}] - \mathrm{MSE}[\hat{\boldsymbol\beta}_r(d)^{(1)}]$ is given by $\boldsymbol\Delta_1 = \mathbf{B} - \mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1$. By applying Theorem B.2 in Appendix B from Graybill [4] we can derive the necessary and sufficient condition for $\boldsymbol\Delta_1$ to be nonnegative definite (nnd). Since B is positive definite (pd) and $\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1$ is a symmetric matrix, there exists a nonsingular matrix $\mathbf{Q}$ such that $\mathbf{Q}^\top\mathbf{B}\mathbf{Q} = \mathbf{I}$ and $\mathbf{Q}^\top\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1\mathbf{Q} = \boldsymbol\Lambda$, where $\boldsymbol\Lambda$ is a diagonal elements are the roots of the polynomial equation $|\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1 - \lambda\mathbf{B}| = 0$. Since

$$|\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1 - \lambda\mathbf{B}| = \left|\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1\mathbf{B}^{-1} - \lambda\mathbf{I}_q\right||\mathbf{B}| = 0,$$

λ_i is an eigenvalue of $\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1\mathbf{B}^{-1}$. $\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1\mathbf{B}^{-1}$ has the same eigenvalues with $\mathbf{B}^{-1}\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1$. If $\mathbf{B}-\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1$ is an nnd matrix, $\mathbf{Q}^\top\mathbf{B}\mathbf{Q}-\mathbf{Q}^\top\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1\mathbf{Q} = \mathbf{I}_q - \boldsymbol\Lambda$ is an nnd matrix. Then $1-\lambda_i \ge 0$, so we get $\lambda_{\max}(\mathbf{B}^{-1}\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1) \le 1$.

Let $\lambda_{\max}(\mathbf{B}^{-1}\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1) \le 1$. Since $\mathbf{B}$ is pd and $\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1$ is a symmetric matrix, we have get

$$\lambda_q \le \frac{x^\top\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1x}{x^\top\mathbf{B}x} \le \lambda_1,$$

where $\lambda_1(\mathbf{B}^{-1}\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1) \ge \ldots \ge \lambda_q(\mathbf{B}^{-1}\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1)$ are the roots of $|\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1 - \lambda\mathbf{B}| = 0$. Then we get $x^\top\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1x \le x^\top\mathbf{B}x$, so $\mathbf{B} - \mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1$ is an nnd matrix. It is obvious that $\boldsymbol\Delta_1$ is nnd if and only if $\lambda_{\max}(\mathbf{B}^{-1}\mathbf{M}_1\mathbf{S}_1\mathbf{B}\mathbf{S}_1\mathbf{M}_1) \le 1$. □

3.1.2. *The comparison between the FOARL estimator and the FOARML estimator.* For the superiority of $\hat{\boldsymbol{\beta}}_r(d)^{(1)}$ over $\hat{\boldsymbol{\beta}}_r^{(1)}$, we consider the matrix MSE difference $\boldsymbol{\Delta}_2 = \text{MSE}[\hat{\boldsymbol{\beta}}_r^{(1)}] - \text{MSE}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}]$. In the case of $\mathbf{R}\boldsymbol{\beta} = \mathbf{r}$, the matrix MSE difference $\boldsymbol{\Delta}_2$ equals to

$$\boldsymbol{\Delta}_2 = a(\phi)\mathbf{M}_0 - a(\phi)\mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1 - (d-1)^2\mathbf{M}_1\boldsymbol{\beta}\boldsymbol{\beta}^\top\mathbf{M}_1.$$

The result can be presented by Theorem 3.2.

3.2. Theorem. *Under the conditions that* $a(\phi)[\mathbf{M}_0 - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1]$ *is nnd, the estimator* $\hat{\boldsymbol{\beta}}_r(d)^{(1)}$ *is superior to the estimator* $\hat{\boldsymbol{\beta}}_r^{(1)}$ *by the matrix MSE criterion if and only if*

$$(d-1)^2\boldsymbol{\beta}^\top\mathbf{M}_1(\mathbf{M}_0 - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1)^-\mathbf{M}_1\boldsymbol{\beta} \leq 1,$$

when the restrictions are true.

Proof. Let $\mathbb{C}(\mathbf{D})$ denotes the column space of $\mathbf{D}$ where $\mathbf{D} = a(\phi)[\mathbf{M}_0 - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d \mathbf{M}_1]$. Then if $(1-d)\mathbf{M}_1\boldsymbol{\beta} \in \mathbb{C}(\mathbf{D})$, $\boldsymbol{\Delta}_2$ is nnd if and only if $(d-1)^2\boldsymbol{\beta}^\top\mathbf{M}_1\mathbf{D}^-\mathbf{M}_1\boldsymbol{\beta} \leq 1$ where $\mathbf{D}^-$ is a generalized inverse of $\mathbf{D}$ under the condition that $\mathbf{D}$ is nnd, see [18].

Since $\mathbf{C}(\mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1) \subset \mathbb{C}(\mathbf{M}_0)$ and $\Re(\mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1) \subset \Re(\mathbf{M}_0)$ where $\Re(\ldots)$ denotes the row space of a matrix, applying Theorem B.1 in Appendix B from Harville [6], and by following Özkale [16], the g-inverse of $\mathbf{D}$ can be written as

$$\mathbf{D}^- = \frac{1}{a(\phi)}[\mathbf{M}_0^- + \mathbf{M}_0^-\mathbf{M}_1(\mathbf{S}_d^{-1}\mathbf{S}\mathbf{S}_d^{-1} - \mathbf{M}_1\mathbf{M}_0^-\mathbf{M}_1)^-\mathbf{M}_1\mathbf{M}_0^-].$$

Furthermore, $-(d-1)\mathbf{M}_1\boldsymbol{\beta} \in \mathbb{C}(\mathbf{D})$ if and only if $-(d-1)\mathbf{M}_1\boldsymbol{\beta} = -(d-1)\mathbf{D}\mathbf{D}^-\mathbf{M}_1\boldsymbol{\beta}$. The proof of the equality $-(d-1)\mathbf{M}_1\boldsymbol{\beta} = -(d-1)\mathbf{D}\mathbf{D}^-\mathbf{M}_1\boldsymbol{\beta}$ can be followed from Özkale [16]. □

3.1.3. *The comparison between the FOARL estimator and the FOAML estimator.* We consider the difference $\boldsymbol{\Delta}_3 = \text{MSE}[\hat{\boldsymbol{\beta}}^{(1)}] - \text{MSE}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}] = a(\phi)\mathbf{S}^{-1} - a(\phi)\mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1 - (d-1)^2\mathbf{M}_1\boldsymbol{\beta}\boldsymbol{\beta}^\top\mathbf{M}_1$ to compare the FOAML estimator and the FOARL estimator. Then, we give Theorem 3.3.

3.3. Theorem. *Under the condition that* $a(\phi)(\mathbf{S}^{-1} - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1)$ *is nnd,* $\hat{\boldsymbol{\beta}}_r(d)^{(1)}$ *is superior to* $\hat{\boldsymbol{\beta}}^{(1)}$ *in the sense of MSE matrix criterion if and only if*

$$(d-1)^2\boldsymbol{\beta}^\top\mathbf{M}_1(\mathbf{S}^{-1} - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1)^-\mathbf{M}_1\boldsymbol{\beta} \leq a(\phi),$$

when the restrictions are true.

Proof. Let $\mathbb{C}(\mathbf{A})$ denotes the column space of $\mathbf{A}$ where $\mathbf{A} = a(\phi)[\mathbf{S}^{-1} - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d \mathbf{M}_1]$. Then if $(1-d)\mathbf{M}_1\boldsymbol{\beta} \in \mathbb{C}(\mathbf{A})$, $\boldsymbol{\Delta}_3$ is nnd if and only if $(d-1)^2\boldsymbol{\beta}^\top\mathbf{M}_1\mathbf{A}^-\mathbf{M}_1\boldsymbol{\beta} \leq a(\phi)$ where $\mathbf{A}^-$ is a g-inverse of $\mathbf{A}$ under the condition that $\mathbf{A}$ is nnd (see [18]).

Since $\mathbb{C}(\mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1) \subset \mathbb{C}(\mathbf{S}^{-1})$ and $\Re(\mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1) \subset \Re(\mathbf{S}^{-1})$ where $\Re(\cdot)$ denotes the row space of a matrix, applying Theorem B.1 in Appendix B from Harville [6], and by following Özkale [16], the g-inverse of $\mathbf{A}$ can be written as

$$\mathbf{A}^- = \frac{1}{a(\phi)}[\mathbf{S} + \mathbf{S}\mathbf{M}_1(\mathbf{S}_d^{-1}\mathbf{S}\mathbf{S}_d^{-1} - \mathbf{M}_1\mathbf{S}\mathbf{M}_1)^-\mathbf{M}_1\mathbf{S}].$$

Furthermore, $-(d-1)\mathbf{M}_1\boldsymbol{\beta} \in \mathbb{C}(\mathbf{A})$ if and only if $-(d-1)\mathbf{M}_1\boldsymbol{\beta} = -(d-1)\mathbf{A}\mathbf{A}^-\mathbf{M}_1\boldsymbol{\beta}$. The proof of the equality $-(d-1)\mathbf{M}_1\boldsymbol{\beta} = -(d-1)\mathbf{A}\mathbf{A}^-\mathbf{M}_1\boldsymbol{\beta}$ can be followed from Özkale [16]. □

3.2. Comparisons when the restrictions are not true, i.e., $\mathbf{R}\beta \neq \mathbf{r}$.

3.2.1. *The comparison between the FOARL estimator and the FOARML estimator.* In this case, the performances of the FOARL and FOARML estimators depend upon $\mathbf{R}\boldsymbol{\beta} \neq \mathbf{r}$ and both of them ore biased estimators of $\boldsymbol{\beta}$. We consider the difference

$$\begin{aligned}\boldsymbol{\Delta}_4 &= \mathrm{MSE}[\hat{\boldsymbol{\beta}}_r^{(1)}] - \mathrm{MSE}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}] \\ &= a(\phi)(\mathbf{M}_0 - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1) + \mathbf{S}^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}^{-1}\mathbf{R}^\top)^{-1}(\mathbf{R}\boldsymbol{\beta} - \mathbf{r})(\mathbf{R}\boldsymbol{\beta} - \mathbf{r})^\top \\ &\quad \times(\mathbf{R}\mathbf{S}^{-1}\mathbf{R}^\top)^{-1}\mathbf{R}^\top\mathbf{S}^{-1} - (d-1)^2[\mathbf{M}_1\boldsymbol{\beta} - \mathbf{S}_1^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}_1^{-1}\mathbf{R}^\top)^{-1}] \\ &\quad \times(\mathbf{R}\boldsymbol{\beta} - \mathbf{r})(\mathbf{R}\boldsymbol{\beta} - \mathbf{r})^\top[\mathbf{M}_1\boldsymbol{\beta} - \mathbf{S}_1^{-1}\mathbf{R}^\top(\mathbf{R}\mathbf{S}_1^{-1}\mathbf{R}^\top)^{-1}]^\top\end{aligned}$$

to compare the FOARML estimator and the FOARL estimator. Then, we give Theorem 3.4.

3.4. Theorem. *Under the condition that* $a(\phi)(\mathbf{M}_0 - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1)$ *is an nnd matrix, the FOARL estimator dominates the FOARML estimator in terms of the matrix MSE criterion if and only if*

$$\begin{aligned}&[\boldsymbol{\beta} - (\mathbf{R}\mathbf{R}^\top)^{-1}\mathbf{r}]^\top(\mathbf{M}_1\mathbf{S}_d - \mathbf{M}_0\mathbf{S})^\top(\mathbf{M}_0 - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1)^- \\ &\times(\mathbf{M}_1\mathbf{S}_d - \mathbf{M}_0\mathbf{S})[\boldsymbol{\beta} - (\mathbf{R}\mathbf{R}^\top)^{-1}\mathbf{r}] = 0\end{aligned}$$

when the restrictions are incorrect.

Proof. $\boldsymbol{\Delta}_4$ can be written as $\boldsymbol{\Delta}_4 = \mathbf{D} + \mathbf{d}_1\mathbf{d}_1^\top - \mathbf{d}_2\mathbf{d}_2^\top$ where $\mathbf{d}_1$ and $\mathbf{d}_2$ are the biases of the FOARML and FOARL estimators. To the end of the comparison, it is important whether $\mathbf{d}_i \in \mathbb{C}(\mathbf{D})$, $i = 1, 2$ or not. From Theorem B.3 in Appendix B $\mathbf{d}_1 \notin \mathbb{C}(\mathbf{D})$, $\mathbf{d}_2 \notin \mathbb{C}(\mathbf{D})$ and $\mathbf{d}_2 \in \mathbb{C}(\mathbf{D} : \mathbf{d}_1)$. $\mathbf{d}_1$ and $\mathbf{d}_2$ are linearly independent. Thus when $\mathbf{D}$ is nnd, $\boldsymbol{\Delta}_4$ is nnd. Because of similarity between the FOARL and FOARML estimators in GLMs and restricted Liu and restricted ML estimators in linear model, the proof can be followed from Özkale [16]. □

3.2.2. *The comparison between the FOARL estimator and the FOAML estimator.* In this case, $\boldsymbol{\Delta}_5$ equals to

$$\begin{aligned}\boldsymbol{\Delta}_5 &= \mathrm{MSE}[\hat{\boldsymbol{\beta}}^{(1)}] - \mathrm{MSE}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}] \\ &= a(\phi)\mathbf{S}^{-1} - a(\phi)\mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1 - (\mathbf{M}_1\mathbf{S}_d - \mathbf{I}) \\ &\quad \times[\boldsymbol{\beta} - \mathbf{R}^\top(\mathbf{R}\mathbf{R}^\top)^{-1}\mathbf{r}][\boldsymbol{\beta} - \mathbf{R}^\top(\mathbf{R}\mathbf{R}^\top)^{-1}\mathbf{r}]^\top(\mathbf{M}_1\mathbf{S}_d - \mathbf{I})^\top.\end{aligned}$$

to compare the FOARL estimator and the FOAML estimator. Then, we give Theorem 3.5.

3.5. Theorem. *Under the condition that* $a(\phi)(\mathbf{S}^{-1} - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1)$ *is nnd matrix, the FOARL estimator is better than the FOAML estimator in the sense of the matrix MSE criterion if and only if*

$$\begin{aligned}&[\boldsymbol{\beta} - \mathbf{R}^\top(\mathbf{R}\mathbf{R}^\top)^{-1}\mathbf{r}]^\top(\mathbf{M}_1\mathbf{S}_d - \mathbf{I})^\top(\mathbf{S}^{-1} - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d\mathbf{M}_1)^- \\ &\times(\mathbf{M}_1\mathbf{S}_d - \mathbf{I})[\boldsymbol{\beta} - \mathbf{R}^\top(\mathbf{R}\mathbf{R}^\top)^{-1}\mathbf{r}] \leq a(\phi)\end{aligned}$$

when the restrictions are not true.

Proof. If $(\mathbf{M}_1\mathbf{S}_d - \mathbf{I})[\boldsymbol{\beta} - \mathbf{R}^\top(\mathbf{R}\mathbf{R}^\top)^{-1}\mathbf{r}] \in \mathbb{C}(\mathbf{A})$ where $\mathbf{A} = a(\phi)[\mathbf{S}^{-1} - \mathbf{M}_1\mathbf{S}_d\mathbf{S}^{-1}\mathbf{S}_d \mathbf{M}_1]$, $\boldsymbol{\Delta}_5$ is nnd if and only if $\mathbf{A}\mathbf{A}^-(\mathbf{M}_1\mathbf{S}_d - \mathbf{I}) = \mathbf{M}_1\mathbf{S}_d - \mathbf{I}$. The proof of the equality $\mathbf{A}\mathbf{A}^-(\mathbf{M}_1\mathbf{S}_d - \mathbf{I}) = \mathbf{M}_1\mathbf{S}_d - \mathbf{I}$ can be followed from Özkale [16]. □

4. Numerical Examples

We illustrate the theoretical results of the proposed estimator on two different real life data sets.

4.1. Example 1: Mine Data Set. In this numerical example, the response has Poisson distribution with log link. The data set was originally given by Myers [14] and was also used by Marx [11]. Myers [14] presented 44 observations on mines in the coal fields of the Appalachian region of western Virginia. There are four continuous explanatory variables. These variables were analyzed for roles contributing to the number of injuries or fractures that occur in the upper seams of the mines.

In this study, as [14, 11] did, a generalized linear regression assuming the y_i observations are from Poisson distribution and the log link function is going to be considered. The regression model is

$$\log \hat{\mu}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i} + \hat{\beta}_3 x_{3i} + \hat{\beta}_4 x_{4i},$$

where $\hat{\mu}_i$ is the expected number of upper seam injuries or fractures in the ith coal mine area, x_{1i} is the corresponding inner burden thickness, which is the shortest distance between seam floor and lower seam, x_{2i} is the percent extraction of the lower previously mined seam, x_{3i} is the lower seam height and x_{4i} is the time that the mine has been opened.

Our computations here were performed by using R. We used 1×10^{-6} (sufficiently close to zero) as a convergence criterion. If the sum of absolute difference for the parameter estimates between the iterations is smaller than 1×10^{-6}, the iterations stop. The eigenvalues of $\mathbf{X}^\top \hat{\mathbf{W}} \mathbf{X}$ at final iteration of IRLS method are obtained as $\lambda_1 = 416468$, $\lambda_2 = 338966$, $\lambda_3 = 30607.8$, $\lambda_4 = 3824.86$, $\lambda_5 = 0.95040$. Thus, by following [11] the condition indices of the information matrix are 1.0000, 3.5052, 11.6647, 32.9976 and 2093.3225, indicating severe ill conditions and justifying an optional estimation technique.

To make comparisons, all the estimators are obtained by the first-order approximation. The ordinary least square (ls) estimator $\hat{\boldsymbol{\beta}}_{ls} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$ is used as an initial value of $\boldsymbol{\beta}$ in computing the estimators and calculated as $\hat{\boldsymbol{\beta}}^{(0)} = (-4.579885, -0.001896, 0.104574, -0.007993, -0.050250)^\top$. We calculate the weight matrix $\hat{\mathbf{W}}^{(0)}$ with the diagonal element $\hat{\mu}_i^{(0)}$, where $\hat{\mu}_i^{(0)}$ is $exp(x\beta^{(0)})$. The working response for Poisson distribution is defined as $\hat{z}_i^{(0)} = x_i^\top \beta^{(0)} + (y_i - \mu_i^{(0)})/\mu_i^{(0)}$, $i = 1, \ldots, n$.

The Liu-biasing parameter d is computed as $\hat{d}_h$ which is proposed by Kurtoğlu and Özkale [8]. Therefore, according to the conditions given by Kurtoğlu and Özkale [8], $\hat{d}_h$ value is arbitrarily chosen as 0.95.

The values of $\boldsymbol{\beta}$ affect the value of $\mathbf{R}\boldsymbol{\beta} - \mathbf{r}$ which measures the relative performances of the estimators. Therefore, to see how the restrictions affect the MSE values, four different types of restrictions are used corresponding to:

(i) $4\beta_1 - \beta_3 = 0$ where the effect of the third explanatory variable on the linear predictor is four times the first explanatory variable
($\mathbf{R}^\top = \begin{bmatrix} 0 & 4 & 0 & -1 & 0 \end{bmatrix}$, $r = 0$),

(ii) $4\beta_1 - \beta_3 = 0.01$ where the effect of the difference between four times the first explanatory variable and the third explanatory variable on the linear predictor is 0.01. Here, the number 0.01 is arbitrarily chosen to make difference from the restriction given by (i)
($\mathbf{R}^\top = \begin{bmatrix} 0 & 4 & 0 & -1 & 0 \end{bmatrix}$, $r = 0.01$),

(iii) $\beta_0 + \beta_1 + \beta_4 = -5$ which means that sum of the effects of the constant, first and fourth explanatory variables on the linear predictor is -5
($\mathbf{R}^\top = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \end{bmatrix}$, $r = -5$),

(iv) $\beta_0 + \beta_1 + \beta_4 = -5.5$ which means that sum of the effects of the constant, first and fourth explanatory variables on the linear predictor is -5.5 where the number -5.5 is arbitrarily chosen to make difference from the restriction given by (iii)
($\mathbf{R}^\top = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \end{bmatrix}$, $r = -5.5$).

Table 1. The parameter estimates and scalar MSE values of the estimators when $d = 0.95$.

Estimator	*Rest.*	β_0	β_1	β_2	β_3	β_4	Scalar MSE
$\hat{\boldsymbol{\beta}}^{(1)}$	-	-4.909674	-0.001789	0.097199	-0.007045	-0.047203	0.232122
$\hat{\boldsymbol{\beta}}(d)^{(1)}$	-	-4.863429	-0.001782	0.096665	-0.007101	-0.047090	0.229910
$\hat{\boldsymbol{\beta}}_r^{(1)}$	(i)	-4.903588	-0.001781	0.097159	-0.007124	-0.047166	0.209601
	(ii)	-4.366989	-0.001020	0.093567	-0.014083	-0.043899	0.504145
	(iii)	-4.950893	-0.001796	0.097676	-0.006995	-0.047310	0.001781
	(iv)	-5.449518	-0.001871	0.103437	-0.006389	-0.048609	0.291556
$\hat{\boldsymbol{\beta}}_r(d)^{(1)}$	(i)	-4.862113	-0.001780	0.096658	-0.007122	-0.047080	0.172544
	(ii)	-4.418463	-0.001021	0.094188	-0.014086	-0.044005	0.411639
	(iii)	-4.950882	-0.001796	0.097676	-0.006995	-0.047321	0.001780
	(iv)	-5.449489	-0.001871	0.103438	-0.006387	-0.048639	0.291524
Change %							
$\hat{\boldsymbol{\beta}}^{(1)}$ and $\hat{\boldsymbol{\beta}}(d)^{(1)}$	-	↓0.94%	↓0.38%	↓0.54%	↑0.79%	↓0.23%	
$\hat{\boldsymbol{\beta}}_r^{(1)}$	(i)-(ii)	↓10.94%	↓42.68%	↓3.69%	↑97.67%	↓6.92%	
	(iii)-(iv)	↑10.07%	↑4.19%	↑5.89%	↓8.66%	↑2.74%	
$\hat{\boldsymbol{\beta}}_r(d)^{(1)}$	(i)-(ii)	↓9.12%	↓42.62%	↓2.55%	↑97.78%	↓6.53%	
	(iii)-(iv)	↑10.07%	↑4.19%	↑5.89%	↓8.68%	↑2.78%	

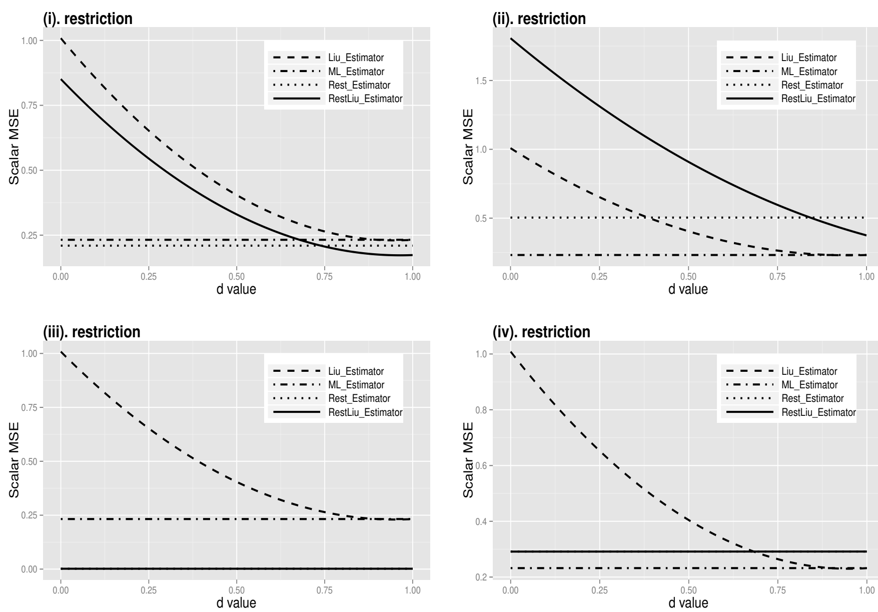

Figure 1. Scalar MSE values of the first-order approximated estimators versus d under different restrictions (Mine Data Set).

The estimated parameter values, scalar MSE values and the change percentage in coefficients are given in Table 1. The percentage change in $\boldsymbol{\beta}$s are calculated as $\hat{\beta}_i = [\{\hat{\beta}(new)_i - \hat{\beta}(old)_i\}/\hat{\beta}(old)_i] \times 100\%$ where $\hat{\boldsymbol{\beta}}$ shows any estimators. In computing the scalar MSE values, unknown $\boldsymbol{\beta}$ parameter vector is replaced by $\hat{\boldsymbol{\beta}}^{(1)}$ which is approximately unbiased (this idea is similar to those which are done the linear regression model). Table 1 shows that the estimators behave differently with respect to the model parameters and restrictions. For example, when the estimators are compared under restrictions (i)-(iii), it is observed that sMSE$[\hat{\boldsymbol{\beta}}_r(d)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}_r^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}(d)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}^{(1)}]$

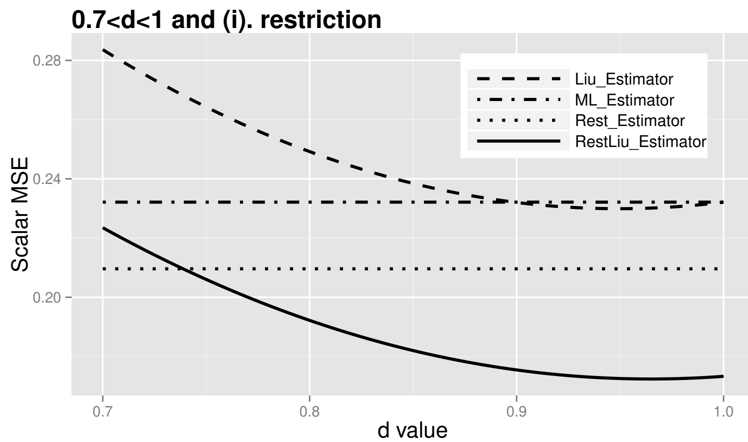

Figure 2. Scalar MSE values of the first-order approximated estimators versus d in interval $(0.7, 1)$ under restriction (i) (Mine Data Set).

under restriction (i), $\text{sMSE}[\hat{\boldsymbol{\beta}}(d)^{(1)}] < \text{sMSE}[\hat{\boldsymbol{\beta}}^{(1)}] < \text{sMSE}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}] < \text{sMSE}[\hat{\boldsymbol{\beta}}_r^{(1)}]$ under restriction (ii), $\text{sMSE}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}] < \text{sMSE}[\hat{\boldsymbol{\beta}}_r^{(1)}] < \text{sMSE}[\hat{\boldsymbol{\beta}}(d)^{(1)}] < \text{sMSE}[\hat{\boldsymbol{\beta}}^{(1)}]$ under restriction (iii) and $\text{sMSE}[\hat{\boldsymbol{\beta}}(d)^{(1)}] < \text{sMSE}[\hat{\boldsymbol{\beta}}^{(1)}] < \text{sMSE}[\hat{\boldsymbol{\beta}}_r^{(1)}] < \text{sMSE}[\hat{\boldsymbol{\beta}}_r(d)^{(1)}]$ under restriction (iv) where sMSE is the scalar MSE value.

There absolutely exists a biased estimator which outperforms $\hat{\boldsymbol{\beta}}^{(1)}$. It is observed that which estimator is superior than the other depends on the restriction and the Liu-biasing parameter used. Under the (iii) and (iv) restrictions, the FOARML and FOARL estimators give almost the same scalar MSE values. As seen from Table 1, under the (i) and (iii) restrictions, the FOARML and FOARL estimators have smaller scalar MSE value than the FOAML and FOAL estimators.

The change in the percentage of coefficients between (i) and (ii) restrictions is maximum for the FOARML and FOARL estimators (approximately 97%). The change in the percentage of coefficients between (i) and (ii) restrictions is minimum for the FOAML and FOAL estimators (approximately 0.2%). In both the FOARML and FOARL estimators, the changes in percentage of the coefficients for (iii)-(iv) restrictions give the same amount of change in percentage.

To see the scalar MSE behavior of the estimators in GLMs for values of d in the interval $(0, 1)$, Figure 1 is given. To see the changes of the FOAL estimator more clearly in the scalar MSE values under (i) restriction, we have narrowed the interval of d to $(0.7 < d < 1)$ in Figure 2. Figure 2 clearly indicates that as d moves away from zero, scalar MSE of the FOAL is decreasing up to a d value and then slightly increasing as d approaches to one. Figure 2 suggests that the FOAL estimator has smaller scalar MSE value than the FOAML estimator when d is larger than approximately 0.90. As d moves away from 0.90 to 0, the difference between the scalar MSE values of the FOAL and FOAML estimators increases where FOAML estimator gets better than FOAL estimator. Figure 1 also supports the results of Table 1. In other words, since $d = 0.95$ is larger than 0.90, the FOARL estimator has less scalar MSE value than the others as Figure 2 shows. As seen from Figure 1 that under restrictions (iii) and (iv), scalar MSE values of the FOARML and FOARL estimators are almost same (see Table 1 for $d = 0.95$). As seen from Table 1 and Figure 1 that FOARML estimator performs better than FOAML estimator in the sense of scalar MSE criterion under the restrictions (i) and (iii). But it seen that FOARML estimator for the restrictions (ii) and (iv) is not superior over FOAML estimator in terms of scalar MSE. Figure 1 displays the behaviour of the scalar MSE value of FOAL estimator that d gets larger (approaches 0.95), the scalar MSE value of FOAL estimator decreases more rapidly until $d = 0.95$. After this value, the scalar MSE value of FOAL estimator increases slightly. As a consequence, we can say that the restrictions and the Liu-biasing parameter affect the performance of the estimators.

4.2. Example 2: Weather Data Set. In this numerical example, the response has gamma distribution with reciprocal link. The data set was first analyzed by Chatterjee and Hadi [3]. The data correspond to the weather factors and nitrogen dioxide concentrations (y), in parts per hundred million (p.p.h.m.), for 26 days in September 1984 as measured at a monitoring station in the San Francisco Bay area. There are four explanatory variables. The variables considered in the study are: mean wind speed in miles per hour (x_1) in m.p.h., maximum temperature (x_2) in $^\circ F$, insolation (x_3) in langleys per day and stability factor (x_4) in $^\circ F$. Our computations here were performed by using R. Chatterjee and Hadi [3] have shown that this data set has a gamma distribution.

The gamma model with reciprocal link is $\hat{\mu}_i = (\sum_{j=1}^{q} \hat{\beta}_j x_{ij})^{-1}$. The Pearson estimate of $a(\phi)$ is obtained as $\hat{\varphi}^2 = 0.07572852$.

The constant term is added to the $\mathbf{X}$ matrix. The eigenvalues of the $\mathbf{X}^\top \hat{\mathbf{W}} \mathbf{X}$ matrix where $\hat{\mathbf{W}}$ is obtained at the final iteration of the IRLS method are obtained as $\lambda_1 = 1240.428467$, $\lambda_2 = 72.912574$, $\lambda_3 = 45.938929$, $\lambda_4 = 19.375420$, and $\lambda_5 = 5.801554$. Thus, the condition number (see [10]) which is computed as $\lambda_{\max}/\lambda_{\min} = 213.8097$ indicates that multicollinearity exists.

To compute the IRLS estimator, we assigned with Hardin and Hilbe's [5] initial fitted values $\hat{\mu}_i^{(0)} = (y + \overline{y})/2$. Then we define the weight matrix $\mathbf{W}$ from $(\hat{\mathbf{W}}^{(0)})^{-1} = (d\boldsymbol{\eta}/d\boldsymbol{\mu})^2\, V^{(0)}$ and the working response z with the ith observation $\hat{z}_i^{(0)} = 1/\hat{\mu}_i^{(0)} - (y_i - \hat{\mu}_i^{(0)})/(\hat{\mu}_i^{(0)})^2$ for the gamma distribution with reciprocal link.

All the estimators are obtained by the first-order approximation. The ordinary ls estimator $\hat{\boldsymbol{\beta}}_{ls} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$ is used as an initial value of $\boldsymbol{\beta}$ and calculated as $\hat{\boldsymbol{\beta}}^{(0)} = (6.269231,\ -5.622560,\ 7.502692,\ -2.000586,\ -0.356589)^\top$.

Liu-biasing parameter d is computed as $\hat{d}_h$ which is proposed by [8]. Therefore, according to the conditions given by [8], two d_h values are arbitrarily chosen as 0.25 and 0.90.

The restrictions are chosen as follows:

- (i) $\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$ which means that the sum of the effects of explanatory variables and the constant on the linear predictor is zero ($\mathbf{R}^\top = [1\ 1\ 1\ 1\ 1]$, $r = 0$),
- (ii) $\beta_1 + \beta_2 + \beta_3 = 0$ which means that sum of the effects of the explanatory variables on the linear predictor is 0 ($\mathbf{R}^\top = [0\ 1\ 1\ 1\ 0]$, $r = 0$),
- (iii) $\beta_1 + \beta_2 + \beta_3 = 0.01$ which means that sum of the effects of the explanatory variables on the linear predictor is 0.01 where the number 0.01[§] is arbitrarily chosen to make difference from the restriction given by (ii) ($\mathbf{R}^\top = [0\ 1\ 1\ 1\ 0]$, $r = 0.01$).

Table 2 shows that the restrictions and the $\hat{d}_h$ values affect the MSE performance of the estimators. In computing the scalar MSE values, unknown $\boldsymbol{\beta}$ parameter vector is replaced by $\hat{\boldsymbol{\beta}}^{(1)}$ which is approximately unbiased. When the results in Table 2 are considered, it

[§]We recognized that as the deviation from 0 increases, scalar MSE values of the FOAL and FOARL estimators inflate. Therefore, 0.01 is arbitrarily choosen.

Table 2. The parameter estimates and scalar MSE values of the estimators for different d values and restrictions.

Estimator	d	Rest.	β_0	β_1	β_2	β_3	β_4	Scalar MSE
$\hat{\boldsymbol{\beta}}^{(1)}$	-	-	0.3089166	0.0132961	-0.0037102	0.0000512	0.0000507	0.0417548
$\hat{\boldsymbol{\beta}}(d)^{(1)}$	0.25	-	0.2267008	0.0150655	-0.0029761	0.0000668	-0.0003206	0.0292329
	0.90	-	0.2979545	0.0135320	-0.0036123	0.0005332	0.0000012	0.0389611
$\hat{\boldsymbol{\beta}}_r^{(1)}$	-	(i)	-0.0181439	0.0202662	-0.0007861	0.0001155	-0.0014516	0.1070521
	-	(ii)	0.5720477	0.0059614	-0.0060111	0.0000497	0.0010802	0.0761032
	-	(iii)	0.2990107	0.0135723	-0.0036235	0.0000513	0.0000120	0.0069032
$\hat{\boldsymbol{\beta}}_r(d)^{(1)}$	0.25	(i)	-0.0181264	0.0202520	-0.0007855	0.0001159	-0.0014558	0.1070405
	0.25	(ii)	0.5312684	0.0055127	-0.0056088	0.0000960	0.0007696	0.0555099
	0.25	(iii)	0.2807248	0.0133711	-0.0034431	0.0000720	-0.0001272	0.0067997
	0.90	(i)	-0.0180952	0.0202273	-0.0007844	0.0001166	-0.0014642	0.1070197
	0.90	(ii)	0.5478234	0.0056949	-0.0057722	0.0000772	0.0008957	0.0640246
	0.90	(iii)	0.2972799	0.0135532	-0.0036065	0.0000532	-0.0000011	0.0070209

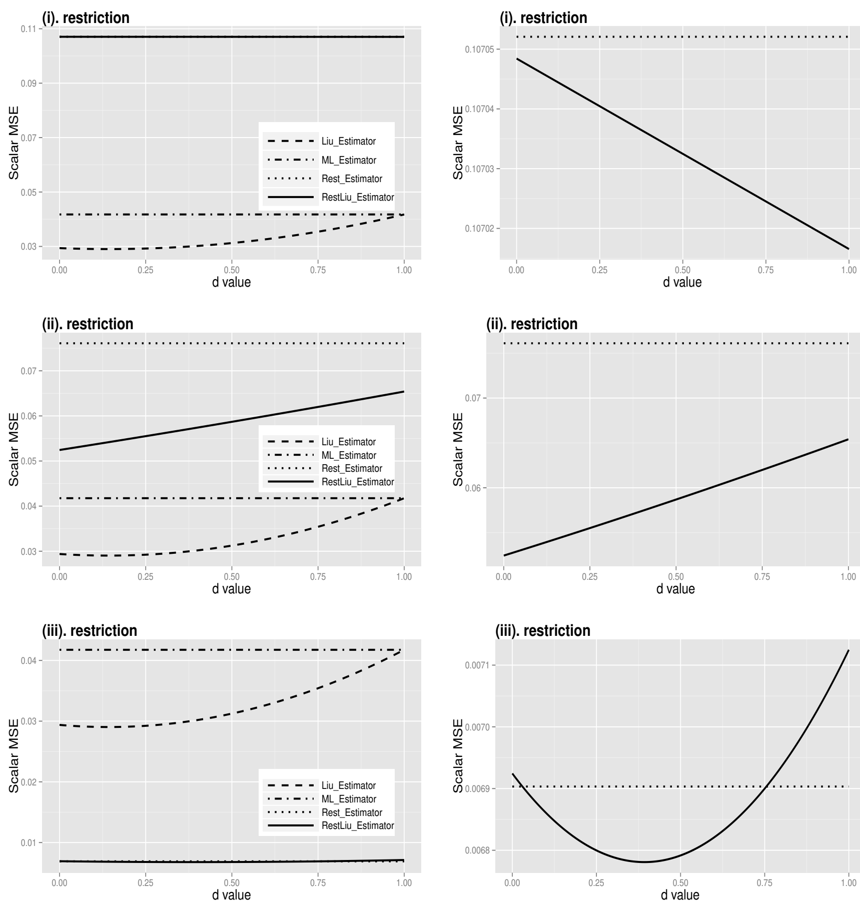

Figure 3. Scalar MSE values of the first-order approximated estimators versus d under different restrictions (Weather Data Set).

is observed that under restriction (i), the ordering sMSE$[\hat{\boldsymbol{\beta}}(d=0.25)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}(d=0.90)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}_r(d=0.90)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}_r(d=0.25)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}_r^{(1)}]$ exists. This ordering changes to sMSE$[\hat{\boldsymbol{\beta}}(d=0.25)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}(d=0.90)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}_r(d=0.25)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}_r(d=0.90)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}_r^{(1)}]$ under restriction (ii), and sMSE$[\hat{\boldsymbol{\beta}}_r(d=0.25)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}_r^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}_r(d=0.90)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}(d=0.25)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}(d=0.90)^{(1)}] <$ sMSE$[\hat{\boldsymbol{\beta}}^{(1)}]$ under restriction (iii).

In general, there absolutely exists a biased estimator which outperforms $\hat{\boldsymbol{\beta}}^{(1)}$. It is observed that which estimator is superior than the other depends on the restriction and the Liu-biasing parameter used.

To consider the effect of the restrictions and the Liu-biasing parameter on the estimators, the scalar MSE values of the estimators corresponding to d values in the interval $(0,1)$ are given in Figure 3. Because of scaling, the performance of the FOARML and FOARL estimators are not obviously seen from the left side of Figure 3. Therefore, the scalar MSE values of the FOARML and FOARL estimators versus d under all restrictions are plotted separately in the right side of Figure 3. As seen from Figure 3, as d goes from 0 to 1, the scalar MSE value of the FOAL estimator increases where as the scalar MSE value of the FOARL estimator decreases in linear manner under restriction (i), increase in linear manner under restriction (ii) and decrease and increase in a parabolic manner under restriction (iii). Figure 3 suggests that under restriction (iii), the FOARML and restricted Liu estimators perform better than the others. At restrictions (i) and (ii), the FOAL estimator shows better performance than the other estimators which is also supported by Table 2.

5. Monte Carlo simulation studies

In this section, we examine the performance of the FOARL to the FOARML, FOAL and FOAML estimators in GLMs via simulation studies. Since the distribution of response, degree of multicollinearity, sample size, restriction types, and Liu-biasing parameter impact the performance of the estimators, simulation studies are conducted under various choices of these parameters. Our Monte Carlo simulation studies are performed by using R and the seed value of the generator of R is taken as 12345. We design two simulation studies where one considers gamma response and the other Poisson response.

5.1. Experiment 1: Simulation study with gamma response. In this experiment, the response has gamma distribution with log link. The setting is as follows.

(1) To see the effect of the number of observations, the sample sizes are chosen as $n = 100$, 200, 300, 400, 500, 600, 700, 800, 900, 1000 and the number of explanatory variables is chosen as $q = 4$.

(2) Following McDonald and Galarneau [13], the explanatory variables are generated by

$$x_{ij} = (1-\gamma^2)^{1/2} w_{ij} + \gamma w_{iq+1},\ i = 1,\dots,n,\ j = 1,\dots,q,$$

where w_{ij} are independent standard normal pseudo-random numbers and γ is specified so that the correlation between any two explanatory variables is given by γ^2. The explanatory variables are then standardized by using unit length scaling so that $\mathbf{X}^\top\mathbf{X}$ is a matrix of correlations.

(3) To see the correlation effect in the simulation, we considered $\gamma^2 = 0.90$, 0.95, 0.99.

(4) For each set of explanatory variables, parameter vector $\boldsymbol{\beta}$ is chosen as $\boldsymbol{\beta} = (1/2) \times j_{q\times 1}$ where $j_{q\times 1}$ is a $q \times 1$ vector of ones.

(5) The values of $\boldsymbol{\beta}$ affect the value of $\mathbf{R}\boldsymbol{\beta} = \mathbf{r}$ which measures the correctness of the restrictions and the performances of the estimators depend on the magnitude of $\mathbf{R}\boldsymbol{\beta} = \mathbf{r}$. Therefore, following Özkale [16] four different scenarios are chosen:
   (i) $\beta_1 - 2\beta_2 = 0$, $\beta_3 - 2\beta_4 = 0$ where the effect of the first explanatory variable on the linear predictor is twice the second explanatory and the effect of the third explanatory variable on the linear predictor is the fourth explanatory variable
   $\left(\mathbf{R}^\top = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \mathbf{r} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$,
   (ii) $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$ where the sum of the effects of the explanatory variables on the linear predictor is 0 ($\mathbf{R}^\top = [1\ 1\ 1\ 1]$, $r = 0$),
   (iii) $\beta_1 - \beta_2 = 0$, $\beta_3 - \beta_4 = 0$ where the first and second explanatory variables have same effect on the linear predictor, the third and fourth explanatory variables have the same effect on the linear predictor
   $\left(\mathbf{R}^\top = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \mathbf{r} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$,
   (iv) $\beta_1 - \beta_2 = 0$, $\beta_3 - \beta_4 = 0$, $\beta_1 - \beta_3 = 0$ where each explanatory variable affect the linear predictor in the same way
   $\left(\mathbf{R}^\top = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$.

   Note that in (i)-(ii), the restrictions are not true that is $\mathbf{R}\boldsymbol{\beta} \neq \mathbf{r}$ and in (iii)-(iv) the restrictions are indeed true that is $\mathbf{R}\boldsymbol{\beta} = \mathbf{r}$.

(6) The response of the gamma model is generated using pseudo-random numbers from $Gamma(u = \hat{\mu}_i^2/\text{var}_i,\ v = \text{var}_i/\hat{\mu}_i)$ distribution with log link function where $\text{var}_i = \hat{\mu}_i^2$ and

$$\hat{\mu}_i = \exp(\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_q x_{iq}), \quad i = 1, \ldots, n,\ j = 1, \ldots, q.$$

(7) Model parameters are estimated using the FOAML, FOARML and FOAL methods. We set the initial fitted values $\hat{\mu}_i^{(0)} = \exp(\mathbf{X}\hat{\boldsymbol{\beta}}^{(0)})$ where $\hat{\boldsymbol{\beta}}^{(0)}$ is the ordinary ls estimator. We calculate the weight matrix $\hat{\mathbf{W}}^{(0)}$ with the diagonal element $(\hat{\mu}_i^{(0)})^2$.

(8) We assign the working response z with the ith observation $\hat{z}_i^{(0)} = \log(\hat{\mu}_i^{(0)}) + (y_i - \hat{\mu}_i^{(0)})/\hat{\mu}_i^{(0)}$ for the gamma distribution.

(9) In the simulation procedure, the dispersion parameter is estimated to be the Pearson method $\hat{\varphi}_p^2 = (n-q)^{-1}\sum_{i=1}^n (y_i - \hat{\mu}_i^{(0)})^2/(\hat{\mu}_i^{(0)})^2$ for each sample size and explanatory variables.

(10) The Liu-biasing parameters are chosen as $d = 0.1$, 0.3, 0.5, 0.7, and 0.9.

(11) The number of replications in Monte Carlo simulation study is taken as 2000. The performance of the estimators is calculated in terms of the estimated MSE (EMSE):

$$\text{EMSE}(\tilde{\boldsymbol{\beta}}) = \frac{1}{2000}\sum_{r=1}^{2000}(\tilde{\boldsymbol{\beta}}_{(r)} - \boldsymbol{\beta})^\top(\tilde{\boldsymbol{\beta}}_{(r)} - \boldsymbol{\beta}),$$

   where the subscript (r) refers to the rth replication and $\tilde{\boldsymbol{\beta}}_{(r)}$ is the estimate of $\boldsymbol{\beta}$ in the rth replication of the experiment.

The simulation results are reported in Tables C1-C10. The main conclusions obtained from the simulation results are as follows:

(a) While the degree of collinearity becomes larger at fixed Liu-biasing parameter except for restriction (iv), the EMSE values of the estimators increase.

(b) When the Liu-biasing parameter becomes larger at fixed degree of collinearity and sample size, the EMSE values of the FOAL and FOARL estimators increase.
(c) The sample sizes do not have an obvious effect on the EMSE values of the estimators at fixed degree of collinearity and Liu-biasing parameter.
(d) Taking account of restrictions, the difference between the EMSE values of the FOARL and FOARML estimators is least at the restrictions (iv). The largest difference of these estimators is at the restriction (ii).
(e) The Liu-biasing parameter, degree of collinearity and sample size do not have an obvious effect on these results.

5.2. Experiment 2: Simulation study with Poisson response. This experiment is designed mainly for Poisson distributed response. The explanatory variables are generated as in Experiment 1 and n values, degree of correlation, Liu-biasing parameter d, parameter vector $\boldsymbol{\beta}$ and restrictions are chosen as in Experiment 1.

The y_i observations are generated from a Poisson distribution $y_i \sim \mathcal{P}(\mu_i)$, $i = 1, \ldots, n$ where μ_i is determined by

$$\hat{\mu}_i = \exp(\hat{\beta}_1 x_{i1} + \ldots + \hat{\beta}_q x_{iq}).$$

We calculate the weight matrix $\hat{\mathbf{W}}^{(0)}$ with the diagonal element $\hat{\mu}_i^{(0)}$ where $\hat{\mu}_i^{(0)} = \exp(\mathbf{X}\hat{\boldsymbol{\beta}}^{(0)})$. The working response for Poisson distribution is defined as $\hat{z}_i^{(0)} = x_i^\top \hat{\boldsymbol{\beta}}^{(0)} + (y_i - \hat{\mu}_i^{(0)})/\hat{\mu}_i^{(0)}$, $i = 1, \ldots, n$.

The simulation results are reported in Tables C11-C20. The main conclusions obtained from the simulation results are as follows:

(a) The EMSE values of the FOAML and FOARML (except restriction (iv)) increase as the degree of correlation increases. This case is valid for the FOAL and FOARL estimators for fixed d value; however, the increasement is not as severe as compared to the FOAL and FOARL estimators, respectively.
(b) As d increases from 0.1 to 0.9 when correlation is fixed, the EMSE values of the FOAL and FOARL estimators increase.
(c) The sample sizes do not have an obvious effect on the EMSE values of all the estimators considered when correlation is fixed.

6. Conclusions

In this paper, to cope with the effects of multicollinearity in GLMs, a new estimator, called restricted Liu estimator in GLMs is introduced by unifying Liu and restricted ML estimators in GLMs. The superiority of the FOARL estimator over the FOAL, FOARML and FOAML estimators is examined with respect to the scalar MSE criterion. The numerical examples and simulation studies are conducted on the superiority of the FOARL estimator over the FOAML, FOARML, and FOAL estimators.

Based on the analyses, we conclude that the FOARL estimator outperforms the other estimators under certain restrictions. The Liu-biasing parameter affects the performance of the FOARL and FOAL estimators in GLMs.

Appendix A. Derivation of the restricted Liu estimator

Assuming that $\boldsymbol{\Lambda}^*$ and $\mathbf{X}^\top\mathbf{W}\mathbf{X} + \mathbf{I}$ are invertible, application of inverse theorem (see p.428 [6]), we obtain

$$\begin{aligned}&\left\{\mathrm{E}[\mathrm{H}_l(\boldsymbol{\beta}, d, \boldsymbol{\lambda})]_{\beta=\hat{\beta}_r(d)^{(m)}}\right\}^{-1} = a(\phi)\{(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1} - (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\\ &\times\mathbf{R}^\top[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}\mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\}. \qquad \text{(A.1)}\end{aligned}$$

By using (2.2), (2.4) and (A.1), we get

$$\mathrm{E}[\mathrm{H}_l(\boldsymbol{\beta}, d, \boldsymbol{\lambda})]_{\beta=\hat{\beta}_r(d)^{(m)}} = \hat{\boldsymbol{\beta}}_r(d)^{(m)} + \left[\frac{\partial\psi(\boldsymbol{\beta}, d, \boldsymbol{\lambda})}{\partial\boldsymbol{\beta}}\right]_{\beta=\hat{\beta}_r(d)^{(m)}}$$

$$= \frac{1}{a(\phi)}[(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})\hat{\boldsymbol{\beta}}_r(d)^{(m)}$$

$$+\mathbf{X}^\top\mathbf{W}\mathbf{D}(\mathbf{y}-\boldsymbol{\mu}) - \hat{\boldsymbol{\beta}}_r(d)^{(m)} + d\hat{\boldsymbol{\beta}} + \mathbf{R}^\top\boldsymbol{\Lambda}^*\mathbf{r}]. \tag{A.2}$$

Following (A.1) and (A.2), $\hat{\boldsymbol{\beta}}_r(d)^{(m+1)}$ simplificies to

$$\begin{aligned}\hat{\boldsymbol{\beta}}_r(d)^{(m+1)} \quad = \quad & \{(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1} - (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top \\ & \times[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}\mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\} \\ & \times[(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})\hat{\boldsymbol{\beta}}_r(d)^{(m)} + \mathbf{X}^\top\mathbf{W}\mathbf{D}(\mathbf{y}-\boldsymbol{\mu}) - \hat{\boldsymbol{\beta}}_r(d)^{(m)} \\ & +d\hat{\boldsymbol{\beta}} + \mathbf{R}^\top\boldsymbol{\Lambda}^*\mathbf{r}][(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top\boldsymbol{\Lambda}^*\mathbf{r}].\end{aligned}$$

After multiplying the terms and by using the following equation $\hat{\boldsymbol{\beta}}(d)^{(m+1)} = (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}[(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})\hat{\boldsymbol{\beta}}_r(d)^{(m)} + \mathbf{X}^\top\mathbf{W}\mathbf{D}(\mathbf{y}-\boldsymbol{\mu}) - \hat{\boldsymbol{\beta}}_r(d)^{(m)} + d\hat{\boldsymbol{\beta}}]$, we get, after algebric simplifications,

$$\begin{aligned}\hat{\boldsymbol{\beta}}_r(d)^{(m+1)} \quad = \quad & \hat{\boldsymbol{\beta}}(d)^{(m+1)} - (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1} \\ & \times\mathbf{R}^\top]^{-1}\mathbf{R}\hat{\boldsymbol{\beta}}(d)^{(m+1)} + [\mathbf{I} - (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}[(\boldsymbol{\Lambda}^*)^{-1} \\ & +\mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}\mathbf{R}] \\ = \quad & \hat{\boldsymbol{\beta}}(d)^{(m+1)} - (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1} \\ & \times\mathbf{R}^\top]^{-1}\mathbf{R}\hat{\boldsymbol{\beta}}(d)^{(m+1)} + [(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I}) + \mathbf{R}^\top\boldsymbol{\Lambda}^*\mathbf{r}]^{-1}\mathbf{R}^\top\boldsymbol{\Lambda}^*\mathbf{r}.\end{aligned}$$

For further simplifications and from (A.1), we obtain

$$\begin{aligned} & [(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I}) + \mathbf{R}^\top\boldsymbol{\Lambda}^*\mathbf{r}]^{-1}\mathbf{R}^\top\boldsymbol{\Lambda}^* \\ = \quad & (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top\boldsymbol{\Lambda}^* - (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top \\ & \times[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}\mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top\boldsymbol{\Lambda}^* \\ = \quad & (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1} \\ & \times\{[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top] - \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top\}\boldsymbol{\Lambda}^* \\ = \quad & (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}(\boldsymbol{\Lambda}^*)^{-1}\boldsymbol{\Lambda}^*.\end{aligned}$$

Finally, $\hat{\boldsymbol{\beta}}_r(d)^{(m+1)}$ results in

$$\begin{aligned}\hat{\boldsymbol{\beta}}_r(d)^{(m+1)} \quad = \quad & \hat{\boldsymbol{\beta}}(d)^{(m+1)} - (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1} \\ & \times\mathbf{R}^\top]^{-1}\mathbf{R}\hat{\boldsymbol{\beta}}(d)^{(m+1)} + (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top[(\boldsymbol{\Lambda}^*)^{-1} \\ & +\mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}\mathbf{r} \\ = \quad & \hat{\boldsymbol{\beta}}(d)^{(m+1)} - (\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1} \\ & \times\mathbf{R}^\top[(\boldsymbol{\Lambda}^*)^{-1} + \mathbf{R}(\mathbf{X}^\top\mathbf{W}\mathbf{X}+\mathbf{I})^{-1}\mathbf{R}^\top]^{-1}[\mathbf{R}\hat{\boldsymbol{\beta}}(d)^{(m+1)} - \mathbf{r}].\end{aligned}$$

Appendix B. Theorems

B.1. Theorem. *[[6]] Let* $\mathbf{R}$ *represent an* $n \times q$ *matrix,* $\mathbf{S}$ *an* $n \times m$ *matrix,* $\mathbf{T}$ *an* $m \times p$ *matrix, and* $\mathbf{U}$ *a* $p \times q$ *matrix. If* $\Re(\mathbf{STU}) \subset \Re(\mathbf{R})$ *and* $\mathbb{C}(\mathbf{STU}) \subset \mathbb{C}(\mathbf{R})$*, then the matrix* $\mathbf{R}^- - \mathbf{R}^-\mathbf{STT}^-(\mathbf{T}^- + \mathbf{T}^-\mathbf{TUR}^-\mathbf{STT}^-)^-\mathbf{T}^-\mathbf{TUR}^-$ *is a g-inverse of the matrix* $\mathbf{R} + \mathbf{STU}$.

B.2. Theorem. *([4]) Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *symmetric matrices*

(1) If $\mathbf{A}$ *is pd, there exists a nonsingular matrix* $\mathbf{Q}$ *such that* $\mathbf{Q}^\top\mathbf{A}\mathbf{Q} = \mathbf{I}$ *and* $\mathbf{Q}^\top\mathbf{B}\mathbf{Q} = \mathbf{D}$*, where* $\mathbf{D}$ *is a diagonal matrix and the diagonal elements of* $\mathbf{D}$ *are roots* $\boldsymbol{\lambda}$ *of the polynomial equation* $|\mathbf{B} - \boldsymbol{\lambda}\mathbf{A}| = 0$*.*

(2) If $\mathbf{A}$ *and* $\mathbf{B}$ *are both nonnegative (neither has to be positive definite), there exists a nonsingular matrix* $\mathbf{Q}$ *such that* $\mathbf{Q}^\top\mathbf{A}\mathbf{Q}$ *and* $\mathbf{Q}^\top\mathbf{B}\mathbf{Q}$ *are each diagonal.*

B.3. Theorem. *([2]) Let* $\varsigma_{n\times p}$ *be the set of* $n \times p$ *complex matrices and let* $\mathbf{H}_n$ *be the subset of* $\varsigma_{n\times n}$ *consisting of Hermitian matrices. Further, given* $\mathbf{L} \in \varsigma_{n\times p}$*, the symbols* $\mathbf{L}^*$*,* $\mathbf{R}(\mathbf{L})$ *and* $\boldsymbol{\varsigma}(\mathbf{L})$ *stand for the conjugate transpose, the range and set of all g-inverses, respectively of* $\mathbf{L}$*. Now, let* $\mathbf{A} \in \mathbf{H}_n$*,* $\mathbf{a}_1$ *and* $\mathbf{a}_2 \in \varsigma_{n\times 1}$ *be linearly independent,* $\mathbf{f}_{ij} = \mathbf{a}_i^*\mathbf{A}^-\mathbf{a}_j$*,* $i, j = 1, 2$ *for* $\mathbf{A}^- \in \boldsymbol{\varsigma}(\mathbf{A})$*, and* $\mathbf{s} = [\mathbf{a}_1^*(\mathbf{I}_n - \mathbf{A}\mathbf{A}^-)^*(\mathbf{I}_n - \mathbf{A}\mathbf{A}^-)\mathbf{a}_2]/[\mathbf{a}_1^*(\mathbf{I}_n - \mathbf{A}\mathbf{A}^-)^*(\mathbf{I}_n - \mathbf{A}\mathbf{A}^-)\mathbf{a}_1]$*, provided that* $\mathbf{a}_1 \notin \mathbf{R}(\mathbf{A})$*. Then* $\mathbf{A} + \mathbf{a}_1\mathbf{a}_1^* - \mathbf{a}_2\mathbf{a}_2^*$ *is nnd if and only if any one of the following sets of conditions hold.*

(a) $\mathbf{A}$ *is nnd,* $\mathbf{a}_1 \in \mathbf{R}(\mathbf{A})$*,* $\mathbf{a}_2 \in \mathbf{R}(\mathbf{A})$ *and* $(\mathbf{f}_{11} + 1)(\mathbf{f}_{22} - 1) \le |\mathbf{f}_{12}|^2$*;*

(b) $\mathbf{A}$ *is nnd,* $\mathbf{a}_1 \notin \mathbf{R}(\mathbf{A})$*,* $\mathbf{a}_2 \in \mathbf{R}(\mathbf{A} : \mathbf{a}_1)$ *and* $(\mathbf{a}_2 - \mathbf{s}\mathbf{a}_1)^*\mathbf{A}^-(\mathbf{a}_2 - \mathbf{s}\mathbf{a}_1) \le 1 - |\mathbf{s}|^2$*;*

(c) $\mathbf{A} = \mathbf{U}\Delta\mathbf{U}^* - \delta\mathbf{v}\mathbf{v}^*$*,* $\mathbf{a}_1 \in \mathbf{R}(\mathbf{A})$*,* $\mathbf{a}_2 \in \mathbf{R}(\mathbf{A})$*,* $\mathbf{v}^*a_1 \ne 0$ *and* $(\mathbf{f}_{11} + 1) \le 0$*,* $(\mathbf{f}_{22} - 1) \le 0$*,* $(\mathbf{f}_{11} + 1)(\mathbf{f}_{22} - 1) \ge |\mathbf{f}_{12}|^2$ *where* $(\mathbf{U} : \mathbf{v})$ *(with* $\mathbf{U}$ *possibly absent) is a subunitary matrix,* $\boldsymbol{\Delta}$ *is a pd diagonal matrix (occuring when* $\mathbf{U}$ *is present) and* $\boldsymbol{\delta}$ *is a positive scalar. Further, the conditions (a)-(c) are all independent of the choice of* $\mathbf{A}^- \in \boldsymbol{\varsigma}(\mathbf{A})$*.*

Appendix C. Tables of simulation studies

Table C1: EMSEs of estimators for GLMs with gamma response when n = 100.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	29.5422	11.0529	30.3281	11.5347	0.2743	1.1915	0.6463	1.9655	0.5626	0.2267
0.1	0.95	59.4653	21.6902	60.2694	22.9911	0.2647	1.4184	0.7308	2.1978	0.6528	0.2208
	0.99	301.0209	106.8740	301.8662	115.3358	0.2582	3.7647	1.5997	4.5479	1.5706	0.2169
	0.90	29.5422	11.0529	30.3281	11.5347	0.2743	4.0886	1.7324	4.8596	1.6718	0.2297
0.3	0.95	59.4653	21.6902	60.2694	22.9911	0.2647	6.7766	2.7276	7.5534	2.7145	0.2235
	0.99	301.0209	106.8740	301.8662	115.3358	0.2582	28.5083	10.7601	29.2891	11.1787	0.2194
	0.90	29.5422	11.0529	30.3281	11.5347	0.2743	8.9303	3.5675	9.6937	3.5591	0.2370
0.5	0.95	59.4653	21.6902	60.2694	22.9911	0.2647	16.4440	6.3548	17.2141	6.4784	0.2302
	0.99	301.0209	106.8740	301.8662	115.3358	0.2582	76.8595	28.6882	77.6340	30.0105	0.2257
	0.90	29.5422	11.0529	30.3281	11.5347	0.2743	15.7166	6.1515	16.4678	6.2244	0.2487
0.7	0.95	59.4653	21.6902	60.2694	22.9911	0.2647	30.4206	11.6123	31.1798	11.9444	0.2410
	0.99	301.0209	106.8740	301.8662	115.3358	0.2582	148.8183	55.3839	149.5828	58.0658	0.2358
	0.90	29.5422	11.0529	30.3281	11.5347	0.2743	24.4475	9.4846	25.1821	9.6677	0.2648
0.9	0.95	59.4653	21.6902	60.2694	22.9911	0.2647	48.7065	18.5001	49.4506	19.1125	0.2558
	0.99	301.0209	106.8740	301.8662	115.3358	0.2582	244,3848	90,8474	245.1353	95.3447	0.2498

Table C2: EMSEs of estimators for GLMs with gamma response when n = 200.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	30.1531	11.6998	30.8828	11.4509	0.2650	1.1771	0.6343	1.9630	0.5419	0.2140
0.1	0.95	59.7589	22.9170	60.4954	22.5290	0.2552	1.4040	0.7221	2.1958	0.6283	0.2082
	0.99	295.7997	112.4884	296.5275	110.9072	0.2479	3.6976	1.5997	4.4937	1.4941	0.2039
	0.90	30.1531	11.6998	30.8828	11.4509	0.2650	4.1285	1.7485	4.9106	1.6345	0.2175
0.3	0.95	59.7589	22.9170	60.4954	22.5290	0.2552	6.7874	2.7575	7.5758	2.6311	0.2113
	0.99	295.7997	112.4884	296.5275	110.9072	0.2479	28.0228	10.8483	28.8158	10.6216	0.2068
	0.90	30.1531	11.6998	30.8828	11.4509	0.2650	9.0729	3.6384	9.8467	3.4950	0.2255
0.5	0.95	59.7589	22.9170	60.4954	22.5290	0.2552	16.5047	6.4616	17.2856	6.2850	0.2187
	0.99	295.7997	112.4884	296.5275	110.9072	0.2479	75.5401	28.9512	76.3263	28.5005	0.2136
	0.90	30.1531	11.6998	30.8828	11.4509	0.2650	16.0103	6.3038	16.7712	6.1235	0.2380
0.7	0.95	59.7589	22.9170	60.4954	22.5290	0.2552	30.5560	11.8342	31.3251	11.5899	0.2302
	0.99	295.7997	112.4884	296.5275	110.9072	0.2479	146.2498	55.9084	147.0251	55.1307	0.2244
	0.90	30.1531	11.6998	30.8828	11.4509	0.2650	24.9406	9.7449	25.6842	9.5199	0.2549
0.9	0.95	59.7589	22.9170	60.4954	22.5290	0.2552	48.9411	18.8754	49.6943	18.5458	0.2458
	0.99	295.7997	112.4884	296.5275	110.9072	0.2479	240.1517	91.7198	240.9122	90.5122	0.2391

Table C3: EMSEs of estimators for GLMs with gamma response when n = 300.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	29.2874	9.1568	30.0099	9.2613	0.2631	1.1714	0.6014	1.9570	0.5148	0.2142
0.1	0.95	58.7009	18.0131	59.4287	18.3725	0.2517	1.3928	0.6622	2.1859	0.5788	0.2067
	0.99	295.3786	89.3077	296.0891	91.7421	0.2430	3.6899	1.3523	4.4890	1.2934	0.2009
	0.90	29.2874	9.1568	30.0099	9.2613	0.2631	4.0491	1.4915	4.8314	1.4170	0.2173
0.3	0.95	58.7009	18.0131	59.4287	18.3725	0.2517	6.6896	2.2856	7.4798	2.2377	0.2095
	0.99	295.3786	89.3077	296.0891	91.7421	0.2430	27.9808	8.7243	28.7773	8.8954	0.2034
	0.90	29.2874	9.1568	30.0099	9.2613	0.2631	8.8528	2.9642	9.6271	2.9228	0.2248
0.5	0.95	58.7009	18.0131	59.4287	18.3725	0.2517	16.2369	5.1944	17.0199	5.2271	0.2164
	0.99	295.3786	89.3077	296.0891	91.7421	0.2430	75.4310	23.1019	76.2210	23.7442	0.2098
	0.90	29.2874	9.1568	30.0099	9.2613	0.2631	15.5822	5.0195	16.3441	5.0321	0.2368
0.7	0.95	58.7009	18.0131	59.4287	18.3725	0.2517	30.0347	9.3885	30.8063	9.5470	0.2274
	0.99	295.3786	89.3077	296.0891	91.7421	0.2430	146.0406	44.4849	146.8201	45.8398	0.2201
	0.90	29.2874	9.1568	30.0099	9.2613	0.2631	24.2375	7.6575	24.9824	7.7450	0.2533
0.9	0.95	58.7009	18.0131	59.4287	18.3725	0.2517	48.0829	14.8680	48.8390	15.1974	0.2426
	0.99	295.3786	89.3077	296.0891	91.7421	0.2430	239.8095	72.8734	240.5746	75.1823	0.2344

Table C4: EMSEs of estimators for GLMs with gamma response when n = 400.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	30.4989	10.1679	31.2269	10.1423	0.2692	1.1982	0.6309	1.9880	0.5447	0.2102
0.1	0.95	60.7168	19.9452	61.4530	20.0150	0.2591	1.4280	0.7012	2.2239	0.6146	0.2040
	0.99	302.5332	98.3480	303.2697	99.1015	0.2516	3.7765	1.4669	4.5772	1.3820	0.1993
	0.90	30.4989	10.1679	31.2269	10.1423	0.2692	4.1940	1.6256	4.9787	1.5367	0.2152
0.3	0.95	60.7168	19.9452	61.4530	20.0150	0.2591	6.9077	2.5100	7.6989	2.4243	0.2087
	0.99	302.5332	98.3480	303.2697	99.1015	0.2516	28.6622	9.6375	29.4584	9.5716	0.2038
	0.90	30.4989	10.1679	31.2269	10.1423	0.2692	9.1987	3.2757	9.9735	3.1866	0.2248
0.5	0.95	60.7168	19.9452	61.4530	20.0150	0.2591	16.7849	5.7552	17.5669	5.6759	0.2177
	0.99	302.5332	98.3480	303.2697	99.1015	0.2516	77.2648	25.5768	78.0524	25.5526	0.2123
	0.90	30.4989	10.1679	31.2269	10.1423	0.2692	16.2122	5.5810	16.9725	5.4944	0.2391
0.7	0.95	60.7168	19.9452	61.4530	20.0150	0.2591	31.0595	10.4370	31.8281	10.3695	0.2310
	0.99	302.5332	98.3480	303.2697	99.1015	0.2516	149.5844	49.2849	150.3594	49.3249	0.2249
	0.90	30.4989	10.1679	31.2269	10.1423	0.2692	25.2345	8.5417	25.9756	8.4600	0.2580
0.9	0.95	60.7168	19.9452	61.4530	20.0150	0.2591	49.7317	16.5552	50.4824	16.5050	0.2487
	0.99	302.5332	98.3480	303.2697	99.1015	0.2516	245.6210	80.7618	246.3792	80.8885	0.2417

Table C5: EMSEs of estimators for GLMs with gamma response when n = 500.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	29.6030	10.4164	30.3499	10.0334	0.2697	1.1968	0.6330	1.9752	0.5455	0.2218
0.1	0.95	59.1960	20.5059	30.3499	19.8935	0.2588	1.4185	0.7065	2.2041	0.6151	0.2145
	0.99	296.8833	101.5791	297.6659	99.1856	0.2506	3.7243	1.4973	4.5155	1.3830	0.2088
	0.90	29.6030	10.4164	30.3499	10.0334	0.2697	4.1111	1.6422	4.8867	1.5207	0.2248
0.3	0.95	59.1960	20.5059	59.9570	19.8935	0.2588	6.7674	2.5513	7.5505	2.4043	0.2172
	0.99	296.8833	101.5791	297.6659	99.1856	0.2506	28.1492	9.8955	28.9380	9.5523	0.2113
	0.90	29.6030	10.4164	30.3499	10.0334	0.2697	8.9672	3.3284	9.7358	3.1490	0.2321
0.5	0.95	59.1960	20.5059	59.9570	19.8935	0.2588	16.3966	5.8776	17.1731	5.6288	0.2240
	0.99	296.8833	101.5791	297.6659	99.1856	0.2506	75.8436	26.2998	76.6260	25.5085	0.2177
	0.90	29.6030	10.4164	30.3499	10.0334	0.2697	15.7651	5.6915	16.5222	5.4306	0.2438
0.7	0.95	59.1960	20.5059	59.9570	19.8935	0.2588	30.3062	10.6853	31.0719	10.2887	0.2348
	0.99	296.8833	101.5791	297.6659	99.1856	0.2506	146.8074	50.7102	147.5796	49.2516	0.2280
	0.90	29.6030	10.4164	30.3499	10.0334	0.2697	24.5049	8.7316	25.2461	8.3652	0.2600
0.9	0.95	59.1960	20.5059	59.9570	19.8935	0.2588	48.4960	16.9745	49.2470	16.3840	0.2498
	0.99	296.8833	101.5791	297.6659	99.1856	0.2506	241.0406	83.1267	241.7988	80.7817	0.2421

Table C6: EMSEs of estimators for GLMs with gamma response when n = 600.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	31.0492	9.4549	31.7829	9.7342	0.2644	1.2202	0.6271	2.0056	0.5519	0.2145
0.1	0.95	61.6927	18.4864	62.4344	19.1613	0.2527	1.4518	0.6873	2.2449	0.6155	0.2068
	0.99	306.5041	90.7966	307.2401	94.5599	0.2433	3.8274	1.3867	4.6269	1.3474	0.2005
	0.90	31.0492	9.4549	31.7829	9.7342	0.2644	4.2684	1.5579	5.0504	1.5158	0.2178
0.3	0.95	61.6927	18.4864	62.4344	19.1613	0.2527	7.0191	2.3710	7.8092	2.3675	0.2097
	0.99	306.5041	90.7966	307.2401	94.5599	0.2433	29.0418	8.9219	29.8386	9.2231	0.2031
	0.90	31.0492	9.4549	31.7829	9.7342	0.2644	9.3625	3.0874	10.1367	3.1082	0.2255
0.5	0.95	61.6927	18.4864	62.4344	19.1613	0.2527	17.0548	5.3713	17.8377	5.5007	0.2168
	0.99	306.5041	90.7966	307.2401	94.5599	0.2433	78.2831	23.5922	79.0733	24.5711	0.2097
	0.90	31.0492	9.4549	31.7829	9.7342	0.2644	16.5027	5.2156	17.2645	5.3291	0.2377
0.7	0.95	61.6927	18.4864	62.4344	19.1613	0.2527	31.5587	9.6885	32.3303	10.0153	0.2281
	0.99	306.5041	90.7966	307.2401	94.5599	0.2433	151.5513	45.3974	152.3309	47.3913	0.2202
	0.90	31.0492	9.4549	31.7829	9.7342	0.2644	25.6889	7.9426	26.4337	8.1786	0.2544
0.9	0.95	61.6927	18.4864	62.4344	19.1613	0.2527	50.5310	15.3225	51.2870	15.9110	0.2435
	0.99	306.5041	90.7966	307.2401	94.5599	0.2433	248.8465	74.3378	249.6116	77.6837	0.2346

Table C7: EMSEs of estimators for GLMs with gamma response when n = 700.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	30.6508	10.6061	31.4007	10.6001	0.2682	1.2114	0.6376	1.9914	0.5519	0.2202
0.1	0.95	61.1418	20.8514	61.9077	20.9660	0.2581	1.4429	0.7122	2.2298	0.6270	0.2132
	0.99	305.5251	103.0444	306.3232	104.1052	0.2507	3.8167	1.5126	4.6087	1.4331	0.2080
	0.90	30.6508	10.6061	31.4007	10.6001	0.2682	4.2199	1.6633	4.9972	1.5751	0.2232
0.3	0.95	61.1418	20.8514	61.9077	20.9660	0.2581	6.9573	2.5850	7.7416	2.5029	0.2160
	0.99	305.5251	103.0444	306.3232	104.1052	0.2507	28.9423	10.0209	29.7317	9.9892	0.2107
	0.90	30.6508	10.6061	31.4007	10.6001	0.2682	9.2476	3.3764	10.0177	3.2915	0.2305
0.5	0.95	61.1418	20.8514	61.9077	20.9660	0.2581	16.9015	5.9606	17.6790	5.8935	0.2229
	0.99	305.5251	103.0444	306.3232	104.1052	0.2507	78.0226	26.6400	78.8055	26.7119	0.2173
	0.90	30.6508	10.6061	31.4007	10.6001	0.2682	16.2945	5.7769	17.0531	5.7012	0.2423
0.7	0.95	61.1418	20.8514	61.9077	20.9660	0.2581	31.2753	10.8391	32.0420	10.7987	0.2339
	0.99	305.5251	103.0444	306.3232	104.1052	0.2507	151.0576	51.3701	151.8301	51.6011	0.2277
	0.90	30.6508	10.6061	31.4007	10.6001	0.2682	25.3605	8.8647	26.1033	8.8041	0.2585
0.9	0.95	61.1418	20.8514	61.9077	20.9660	0.2581	50.0788	17.2204	50.8307	17.2186	0.2490
	0.99	305.5251	103.0444	306.3232	104.1052	0.2507	248.0473	84.2112	248.8056	84.6569	0.2421

Table C8: EMSEs of estimators for GLMs with gamma response when n = 800.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	28.7676	9.2882	29.4890	8.9809	0.2787	1.1544	0.6029	1.9432	0.5108	0.2112
0.1	0.95	57.2782	18.1590	58.0090	17.6806	0.2683	1.3691	0.6659	2.1643	0.5704	0.2048
	0.99	285.5347	89.2477	286.2683	87.3721	0.2607	3.5840	1.3566	4.3840	1.2416	0.2000
	0.90	28.7676	9.2882	29.4890	8.9809	0.2787	3.9868	1.5065	4.7690	1.3868	0.2177
0.3	0.95	57.2782	18.1590	58.0090	17.6806	0.2683	6.5468	2.3050	7.3357	2.1641	0.2111
	0.99	285.5347	89.2477	286.2683	87.3721	0.2607	27.0810	8.7428	27.8750	8.4313	0.2060
	0.90	28.7676	9.2882	29.4890	8.9809	0.2787	8.7072	3.0032	9.4780	2.8377	0.2292
0.5	0.95	57.2782	18.1590	58.0090	17.6806	0.2683	15.8654	5.2434	16.6435	5.0196	0.2218
	0.99	285.5347	89.2477	286.2683	87.3721	0.2607	72.9545	23.1494	73.7381	22.4508	0.2163
	0.90	28.7676	9.2882	29.4890	8.9809	0.2787	15.3154	5.0931	16.0700	4.8634	0.2454
0.7	0.95	57.2782	18.1590	58.0090	17.6806	0.2683	29.3249	9.4810	30.0877	9.1370	0.2370
	0.99	285.5347	89.2477	286.2683	87.3721	0.2607	141.2043	44.5766	141.9734	43.3001	0.2309
	0.90	28.7676	9.2882	29.4890	8.9809	0.2787	23.8116	7.7763	24.5451	7.4638	0.2664
0.9	0.95	57.2782	18.1590	58.0090	17.6806	0.2683	46.9252	15.0178	47.6683	14.5163	0.2568
	0.99	285.5347	89.2477	286.2683	87.3721	0.2607	231.8305	73.0242	232.5807	70.9791	0.2497

Table C9: EMSEs of estimators for GLMs with gamma response when n = 900.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	30.8776	10.1850	31.6047	10.4930	0.2745	1.2322	0.6455	2.0123	0.5616	0.2199
0.1	0.95	61.5356	19.9866	62.2734	20.7300	0.2637	1.4632	0.7147	2.2502	0.6343	0.2130
	0.99	307.0856	98.6128	307.8316	102.7954	0.2557	3.8459	1.4826	4.6382	1.4311	0.2078
	0.90	30.8776	10.1850	31.6047	10.4930	0.2745	4.2710	1.6455	5.0469	1.5851	0.2241
0.3	0.95	61.5356	19.9866	62.2734	20.7300	0.2637	7.0238	2.5329	7.8069	2.5050	0.2169
	0.99	307.0856	98.6128	307.8316	102.7954	0.2557	29.1138	9.6898	29.9023	9.9218	0.2114
	0.90	30.8776	10.1850	31.6047	10.4930	0.2745	9.3379	3.2999	10.1050	3.2911	0.2328
0.5	0.95	61.5356	19.9866	62.2734	20.7300	0.2637	17.0352	5.7889	17.8102	5.8715	0.2250
	0.99	307.0856	98.6128	307.8316	102.7954	0.2557	78.4495	25.6918	79.2304	26.4979	0.2191
	0.90	30.8776	10.1850	31.6047	10.4930	0.2745	16.4327	5.6087	17.1865	5.6796	0.2461
0.7	0.95	61.5356	19.9866	62.2734	20.7300	0.2637	31.4973	10.4830	32.2599	10.7339	0.2373
	0.99	307.0856	98.6128	307.8316	102.7954	0.2557	151.8530	49.4887	152.6223	51.1593	0.2308
	0.90	30.8776	10.1850	31.6047	10.4930	0.2745	25.5556	8.5719	26.2915	8.7505	0.2639
0.9	0.95	61.5356	19.9866	62.2734	20.7300	0.2637	50.4101	16.6150	51.1562	17.0921	0.2539
	0.99	307.0856	98.6128	307.8316	102.7954	0.2557	249.3244	81.0805	250.0780	83.9060	0.2464

Table C10: EMSEs of estimators for GLMs with gamma response when n = 1000.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	29.0214	9.3668	29.7541	9.2088	0.2720	1.1730	0.6068	1.9594	0.5176	0.2137
0.1	0.95	57.9086	18.3709	58.6554	18.1790	0.2600	1.3884	0.6688	2.1824	0.5785	0.2060
	0.99	289.5640	90.6743	290.3296	90.1641	0.2506	3.6342	1.3676	4.4345	1.2717	0.1998
	0.90	29.0214	9.3668	29.7541	9.2088	0.2720	4.0338	1.5161	4.8154	1.4136	0.2185
0.3	0.95	57.9086	18.3709	58.6554	18.1790	0.2600	6.6265	2.3231	7.4161	2.2142	0.2105
	0.99	289.5640	90.6743	290.3296	90.1641	0.2506	27.4651	8.8525	28.2612	8.6892	0.2039
	0.90	29.0214	9.3668	29.7541	9.2088	0.2720	8.7962	3.0220	9.5684	2.9015	0.2280
0.5	0.95	57.9086	18.3709	58.6554	18.1790	0.2600	16.0485	5.2890	16.8294	5.1510	0.2193
	0.99	289.5640	90.6743	290.3296	90.1641	0.2506	73.9868	23.4544	74.7747	23.1618	0.2122
	0.90	29.0214	9.3668	29.7541	9.2088	0.2720	15.4601	5.1245	16.2183	4.9815	0.2421
0.7	0.95	57.9086	18.3709	58.6554	18.1790	0.2600	29.6545	9.5664	30.4224	9.3889	0.2323
	0.99	289.5640	90.6743	290.3296	90.1641	0.2506	143.1995	45.1733	143.9751	44.6895	0.2245
	0.90	29.0214	9.3668	29.7541	9.2088	0.2720	24.0256	7.8236	24.7651	7.6534	0.2609
0.9	0.95	57.9086	18.3709	58.6554	18.1790	0.2600	47.4446	15.1552	48.1952	14.9279	0.2497
	0.99	289.5640	90.6743	290.3296	90.1641	0.2506	235.1031	74.0092	235.8623	73.2722	0.2409

Table C11: EMSEs of estimators for GLMs with Poisson response when n = 100.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	28.5934	11.1609	31.2301	11.4054	0.2844	1.2374	0.6653	2.0402	0.5681	0.2295
0.1	0.95	57.4946	21.9826	62.2156	22.7807	0.2741	1.4450	0.7474	2.2483	0.6511	0.2236
	0.99	290.8867	108.3680	312.8578	114.2393	0.2661	3.6979	1.5955	4.5010	1.5155	0.2180
	0.90	28.5934	11.1609	31.2301	11.4054	0.2844	4.0791	1.7422	4.9075	1.6391	0.2344
0.3	0.95	57.4946	21.9826	62.2156	22.7807	0.2741	6.6750	2.7243	7.5087	2.6312	0.2281
	0.99	290.8867	108.3680	312.8578	114.2393	0.2661	27.6698	10.5985	28.5078	10.6483	0.2224
	0.90	28.5934	11.1609	31.2301	11.4054	0.2844	8.7707	3.5469	9.6233	3.4438	0.2431
0.5	0.95	57.4946	21.9826	62.2156	22.7807	0.2741	16.0338	6.2959	16.8969	6.2235	0.2362
	0.99	290.8867	108.3680	312.8578	114.2393	0.2661	74.4118	28.1940	75.2842	28.5201	0.2301
	0.90	28.5934	11.1609	31.2301	11.4054	0.2844	15.3123	6.0792	16.1877	5.9822	0.2558
0.7	0.95	57.4946	21.9826	62.2156	22.7807	0.2741	29.5215	11.4622	30.4131	11.4278	0.2479
	0.99	290.8867	108.3680	312.8578	114.2393	0.2661	143.9241	54.3819	144.8302	55.1308	0.2412
	0.90	28.5934	11.1609	31.2301	11.4054	0.2844	23.7039	9.3393	24.6007	9.2544	0.2723
0.9	0.95	57.4946	21.9826	62.2156	22.7807	0.2741	47.1380	18.2231	48.0572	18.2443	0.2631
	0.99	290.8867	108.3680	312.8578	114.2393	0.2661	236.2066	89.1623	237.1457	90.4804	0.2557

Table C12: EMSEs of estimators for GLMs with Poisson response when n = 200.

d	corr	FOAML	FOARML Restrictions (i)	FOARML Restrictions (ii)	FOARML Restrictions (iii)	FOARML Restrictions (iv)	FOAL	FOARL Restrictions (i)	FOARL Restrictions (ii)	FOARL Restrictions (iii)	FOARL Restrictions (iv)
	0.90	30.7105	12.2138	32.2011	11.8859	0.2752	1.2254	0.6582	2.0225	0.5619	0.2139
0.1	0.95	60.7527	24.0159	63.0798	23.4800	0.2640	1.4497	0.7482	2.2511	0.6499	0.2077
	0.99	301.0278	118.7868	310.0946	116.2995	0.2526	3.7785	1.6578	4.5842	1.5424	0.2015
	0.90	30.7105	12.2138	32.2011	11.8859	0.2752	4.2499	1.8099	5.0521	1.6878	0.2193
0.3	0.95	60.7527	24.0159	63.0798	23.4800	0.2640	6.9454	2.8543	7.7539	2.7156	0.2126
	0.99	301.0278	118.7868	310.0946	116.2995	0.2526	28.5607	11.2675	29.3755	10.9749	0.2059
	0.90	30.7105	12.2138	32.2011	11.8859	0.2752	9.2902	3.7526	10.0940	3.5923	0.2292
0.5	0.95	60.7527	24.0159	63.0798	23.4800	0.2640	16.8313	6.6741	17.6435	6.4677	0.2217
	0.99	301.0278	118.7868	310.0946	116.2995	0.2526	76.9276	30.0649	77.7486	29.4308	0.2141
	0.90	30.7105	12.2138	32.2011	11.8859	0.2752	16.3464	6.4863	17.1481	6.2752	0.2436
0.7	0.95	60.7527	24.0159	63.0798	23.4800	0.2640	31.1072	12.2077	31.9200	11.9062	0.2349
	0.99	301.0278	118.7868	310.0946	116.2995	0.2526	148.8792	58.0501	149.7035	56.9103	0.2261
	0.90	30.7105	12.2138	32.2011	11.8859	0.2752	25.4185	10.0110	26.2145	9.7366	0.2624
0.9	0.95	60.7527	24.0159	63.0798	23.4800	0.2640	49.7733	19.4551	50.5834	19.0312	0.2523
	0.99	301.0278	118.7868	310.0946	116.2995	0.2526	244.4154	95.2229	245.2401	93.4134	0.2419

Table C13: EMSEs of estimators for GLMs with Poisson response when n = 300.

d	corr	FOAML	FOARML Restrictions (i)	FOARML Restrictions (ii)	FOARML Restrictions (iii)	FOARML Restrictions (iv)	FOAL	FOARL Restrictions (i)	FOARL Restrictions (ii)	FOARL Restrictions (iii)	FOARL Restrictions (iv)
	0.90	29.6153	10.0287	31.7611	10.0938	0.2783	1.2322	0.6549	2.0277	0.5615	0.2240
0.1	0.95	59.3247	19.6938	62.9979	19.9774	0.2664	1.4498	0.7180	2.2499	0.6266	0.2162
	0.99	297.9529	97.6277	313.8923	99.6760	0.2581	3.7596	1.4641	4.5615	1.3865	0.2112
	0.90	29.6153	10.0287	31.7611	10.0938	0.2783	4.1650	1.6319	4.9763	1.5379	0.2285
0.3	0.95	59.3247	19.6938	62.9979	19.9774	0.2664	6.8319	2.4889	7.6509	2.4094	0.2203
	0.99	297.9529	97.6277	313.8923	99.6760	0.2581	28.2971	9.4620	29.1206	9.5031	0.2149
	0.90	29.6153	10.0287	31.7611	10.0938	0.2783	9.0261	3.2366	9.8511	3.1538	0.2372
0.5	0.95	59.3247	19.6938	62.9979	19.9774	0.2664	16.4877	5.6463	17.3237	5.6044	0.2283
	0.99	297.9529	97.6277	313.8923	99.6760	0.2581	76.1710	25.0394	77.0148	25.3345	0.2224
	0.90	29.6153	10.0287	31.7611	10.0938	0.2783	15.8155	5.4689	16.6519	5.4092	0.2501
0.7	0.95	59.3247	19.6938	62.9979	19.9774	0.2664	30.4172	10.1900	31.2685	10.2115	0.2402
	0.99	297.9529	97.6277	313.8923	99.6760	0.2581	147.3814	48.1965	148.2440	48.8806	0.2335
	0.90	29.6153	10.0287	31.7611	10.0938	0.2783	24.5333	8.3289	25.3788	8.3041	0.2671
0.9	0.95	59.3247	19.6938	62.9979	19.9774	0.2664	48.6204	16.1202	49.4851	16.2308	0.2560
	0.99	297.9529	97.6277	313.8923	99.6760	0.2581	241.9283	78.9330	242.8081	80.1415	0.2484

Table C14: EMSEs of estimators for GLMs with Poisson response when n = 400.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	30.7848	10.1645	32.3040	10.0751	0.2847	1.2492	0.6552	2.0313	0.5654	0.2292
0.1	0.95	61.3707	19.9339	63.7764	19.8870	0.2734	1.4794	0.7232	2.2669	0.6329	0.2222
	0.99	305.0298	97.5975	314.5162	97.9331	0.2663	3.8416	1.4750	4.6314	1.3826	0.2181
	0.90	30.7848	10.1645	32.3040	10.0751	0.2847	4.2842	1.6473	5.0739	1.5452	0.2338
0.3	0.95	61.3707	19.9339	63.7764	19.8870	0.2734	7.0330	2.5256	7.8301	2.4207	0.2264
	0.99	305.0298	97.5975	314.5162	97.9331	0.2663	28.9513	9.5400	29.7526	9.4166	0.2221
	0.90	30.7848	10.1645	32.3040	10.0751	0.2847	9.3355	3.2860	10.1296	3.1692	0.2427
0.5	0.95	61.3707	19.9339	63.7764	19.8870	0.2734	17.0183	5.7497	17.8224	5.6255	0.2346
	0.99	305.0298	97.5975	314.5162	97.9331	0.2663	77.9587	25.2596	78.7692	25.0840	0.2298
	0.90	30.7848	10.1645	32.3040	10.0751	0.2847	16.4030	5.5711	17.1985	5.4374	0.2559
0.7	0.95	61.3707	19.9339	63.7764	19.8870	0.2734	31.4355	10.3957	32.2439	10.2473	0.2468
	0.99	305.0298	97.5975	314.5162	97.9331	0.2663	150.8639	48.6337	151.6813	48.3848	0.2413
	0.90	30.7848	10.1645	32.3040	10.0751	0.2847	25.4868	8.5028	26.2807	8.3499	0.2734
0.9	0.95	61.3707	19.9339	63.7764	19.8870	0.2734	50.2843	16.4634	51.0945	16.2863	0.2630
	0.99	305.0298	97.5975	314.5162	97.9331	0.2663	247.6667	79.6623	248.4888	79.3191	0.2566

Table C15: EMSEs of estimators for GLMs with Poisson response when n = 500.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	29.1453	10.4823	30.2451	10.2481	0.2666	1.1867	0.6286	1.9790	0.5464	0.2130
0.1	0.95	58.3401	20.6028	59.8329	20.2990	0.2566	1.4050	0.7008	2.2027	0.6162	0.2067
	0.99	292.0949	101.8702	296.6694	101.1129	0.2499	3.6703	1.4824	4.4718	1.3897	0.2021
	0.90	29.1453	10.4823	30.2451	10.2481	0.2666	4.0644	1.6324	4.8586	1.5351	0.2169
0.3	0.95	58.3401	20.6028	59.8329	20.2990	0.2566	6.6869	2.5312	7.4875	2.4249	0.2104
	0.99	292.0949	101.8702	296.6694	101.1129	0.2499	27.7135	9.7979	28.5185	9.6280	0.2057
	0.90	29.1453	10.4823	30.2451	10.2481	0.2666	8.8480	3.3050	9.6401	3.1798	0.2254
0.5	0.95	58.3401	20.6028	59.8329	20.2990	0.2566	16.1805	5.8267	16.9803	5.6770	0.2183
	0.99	292.0949	101.8702	296.6694	101.1129	0.2499	74.6430	26.0374	75.4483	25.7097	0.2132
	0.90	29.1453	10.4823	30.2451	10.2481	0.2666	15.5375	5.6465	16.3237	5.4806	0.2383
0.7	0.95	58.3401	20.6028	59.8329	20.2990	0.2566	29.8856	10.5873	30.6812	10.3724	0.2303
	0.99	292.0949	101.8702	296.6694	101.1129	0.2499	144.4590	50.2010	145.2612	49.6347	0.2247
	0.90	29.1453	10.4823	30.2451	10.2481	0.2666	24.1329	8.6568	24.9093	8.4374	0.2557
0.9	0.95	58.3401	20.6028	59.8329	20.2990	0.2566	47.8024	16.8129	48.5902	16.5112	0.2465
	0.99	292.0949	101.8702	296.6694	101.1129	0.2499	237.1613	82.2886	237.9574	81.4031	0.2402

Table C16: EMSEs of estimators for GLMs with Poisson response when n = 600.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
0.1	0.90	29.7498	9.1261	31.2383	9.1824	0.2779	1.1963	0.6252	1.9901	0.5368	0.2166
	0.95	59.0678	17.7827	61.3714	18.0406	0.2659	1.4155	0.6817	2.2151	0.5954	0.2091
	0.99	293.6343	86.8990	302.4162	88.5403	0.2560	3.6892	1.3474	4.4937	1.2754	0.2026
0.3	0.90	29.7498	9.1261	31.2383	9.1824	0.2779	4.1267	1.5238	4.9256	1.4446	0.2223
	0.95	59.0678	17.7827	61.3714	18.0406	0.2659	6.7585	2.3003	7.5651	2.2411	0.2144
	0.99	293.6343	86.8990	302.4162	88.5403	0.2560	27.8600	8.5508	28.6733	8.6270	0.2076
0.5	0.90	29.7498	9.1261	31.2383	9.1824	0.2779	9.0084	2.9950	9.8090	2.9396	0.2325
	0.95	59.0678	17.7827	61.3714	18.0406	0.2659	16.3693	5.1780	17.1800	5.1781	0.2238
	0.99	293.6343	86.8990	302.4162	88.5403	0.2560	75.0374	22.5665	75.8570	22.9457	0.2164
0.7	0.90	29.7498	9.1261	31.2383	9.1824	0.2779	15.8414	5.0389	16.6405	5.0220	0.2471
	0.95	59.0678	17.7827	61.3714	18.0406	0.2659	30.2478	9.3149	31.0597	9.4064	0.2373
	0.99	293.6343	86.8990	302.4162	88.5403	0.2560	145.2213	43.3944	146.0445	44.2314	0.2291
0.9	0.90	29.7498	9.1261	31.2383	9.1824	0.2779	24.6258	7.6554	25.4200	7.6916	0.2661
	0.95	59.0678	17.7827	61.3714	18.0406	0.2659	48.3942	14.7109	49.2044	14.9259	0.2549
	0.99	293.6343	86.8990	302.4162	88.5403	0.2560	238.4117	71.0346	239.2361	72.4843	0.2457

Table C17: EMSEs of estimators for GLMs with Poisson response when n = 700.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
0.1	0.90	30.4922	10.1295	31.5624	10.3847	0.2690	1.2144	0.6259	2.0009	0.5451	0.2183
	0.95	60.7094	19.9315	62.1660	20.5798	0.2590	1.4416	0.6975	2.2339	0.6194	0.2119
	0.99	303.3710	97.9897	307.8700	101.8113	0.2507	3.7972	1.4554	4.5940	1.4042	0.2066
0.3	0.90	30.4922	10.1295	31.5624	10.3847	0.2690	4.2134	1.6067	5.0014	1.5479	0.2219
	0.95	60.7094	19.9315	62.1660	20.5798	0.2590	6.9246	2.4889	7.7194	2.4602	0.2152
	0.99	303.3710	97.9897	307.8700	101.8113	0.2507	28.7549	9.5415	29.5553	9.7594	0.2097
0.5	0.90	30.4922	10.1295	31.5624	10.3847	0.2690	9.2165	3.2409	10.0023	3.2276	0.2299
	0.95	60.7094	19.9315	62.1660	20.5798	0.2590	16.8005	5.7125	17.5945	5.7842	0.2226
	0.99	303.3710	97.9897	307.8700	101.8113	0.2507	77.4921	25.3281	78.2930	26.0856	0.2166
0.7	0.90	30.4922	10.1295	31.5624	10.3847	0.2690	16.2237	5.5283	17.0035	5.5841	0.2422
	0.95	60.7094	19.9315	62.1660	20.5798	0.2590	31.0694	10.3684	31.8592	10.5912	0.2340
	0.99	303.3710	97.9897	307.8700	101.8113	0.2507	150.0090	48.8153	150.8072	50.3828	0.2273
0.9	0.90	30.4922	10.1295	31.5624	10.3847	0.2690	25.2350	8.4690	26.0050	8.6176	0.2587
	0.95	60.7094	19.9315	62.1660	20.5798	0.2590	49.7312	16.4565	50.5134	16.8814	0.2494
	0.99	303.3710	97.9897	307.8700	101.8113	0.2507	246.3054	80.0030	247.0979	82.6510	0.2417

Table C18: EMSEs of estimators for GLMs with Poisson response when n = 800.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	30.1563	10.2279	30.9995	9.9267	0.2666	1.2082	0.6329	1.9987	0.5441	0.2113
0.1	0.95	60.0867	20.0413	61.0733	19.5458	0.2575	1.4335	0.7025	2.2299	0.6101	0.2052
	0.99	299.6464	98.7883	301.7316	96.7418	0.2499	3.7574	1.4645	4.5581	1.3524	0.2006
	0.90	30.1563	10.2279	30.9995	9.9267	0.2666	4.1814	1.6256	4.9695	1.5113	0.2157
0.3	0.95	60.0867	20.0413	61.0733	19.5458	0.2575	6.8701	2.5048	7.6645	2.3682	0.2095
	0.99	299.6464	98.7883	301.7316	96.7418	0.2499	28.4215	9.6027	29.2209	9.2942	0.2046
	0.90	30.1563	10.2279	30.9995	9.9267	0.2666	9.1316	3.2706	9.9129	3.1131	0.2245
0.5	0.95	60.0867	20.0413	61.0733	19.5458	0.2575	16.6480	5.7361	17.4366	5.5184	0.2179
	0.99	299.6464	98.7883	301.7316	96.7418	0.2499	76.5650	25.4768	77.3595	24.7803	0.2125
	0.90	30.1563	10.2279	30.9995	9.9267	0.2666	16.0588	5.5680	16.8290	5.3495	0.2379
0.7	0.95	60.0867	20.0413	61.0733	19.5458	0.2575	30.7674	10.3964	31.5461	10.0606	0.2304
	0.99	299.6464	98.7883	301.7316	96.7418	0.2499	148.1880	49.0869	148.9738	47.8107	0.2244
	0.90	30.1563	10.2279	30.9995	9.9267	0.2666	24.9629	8.5178	25.7177	8.2207	0.2556
0.9	0.95	60.0867	20.0413	61.0733	19.5458	0.2575	49.2282	16.4859	49.9931	15.9949	0.2472
	0.99	299.6464	98.7883	301.7316	96.7418	0.2499	243.2904	80.4329	244.0638	78.3853	0.2402

Table C19: EMSEs of estimators for GLMs with Poisson response when n = 900.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	29.3950	9.3950	30.4309	9.5818	0.2676	1.1864	0.6134	1.9774	0.5261	0.2133
0.1	0.95	58.6737	18.3986	60.0636	18.8864	0.2570	1.4065	0.6754	2.2041	0.5905	0.2062
	0.99	292.5257	90.3224	296.7109	93.5075	0.2492	3.6737	1.3723	4.4757	1.3122	0.2012
	0.90	29.3950	9.3950	30.4309	9.5818	0.2676	4.0840	1.5300	4.8756	1.4577	0.2175
0.3	0.95	58.6737	18.3986	60.0636	18.8864	0.2570	6.7150	2.3391	7.5142	2.2896	0.2101
	0.99	292.5257	90.3224	296.7109	93.5075	0.2492	27.7514	8.8481	28.5559	9.0142	0.2049
	0.90	29.3950	9.3950	30.4309	9.5818	0.2676	8.9080	3.0442	9.6960	3.0086	0.2261
0.5	0.95	58.6737	18.3986	60.0636	18.8864	0.2570	16.2621	5.3155	17.0592	5.3449	0.2182
	0.99	292.5257	90.3224	296.7109	93.5075	0.2492	74.7500	23.4210	75.5537	24.0473	0.2125
	0.90	29.3950	9.3950	30.4309	9.5818	0.2676	15.6581	5.1560	16.4387	5.1791	0.2392
0.7	0.95	58.6737	18.3986	60.0636	18.8864	0.2570	30.0478	9.6047	30.8392	9.7564	0.2304
	0.99	292.5257	90.3224	296.7109	93.5075	0.2492	144.6696	45.0912	145.4692	46.4113	0.2241
	0.90	29.3950	9.3950	30.4309	9.5818	0.2676	24.3345	7.8655	25.1037	7.9689	0.2568
0.9	0.95	58.6737	18.3986	60.0636	18.8864	0.2570	48.0721	15.2065	48.8542	15.5242	0.2468
	0.99	292.5257	90.3224	296.7109	93.5075	0.2492	237.5101	73.8585	238.3024	76.1064	0.2396

Table C20: EMSEs of estimators for GLMs with Poisson response when n = 1000.

		FOAML	FOARML				FOAL	FOARL			
			Restrictions					Restrictions			
d	corr		(i)	(ii)	(iii)	(iv)		(i)	(ii)	(iii)	(iv)
	0.90	30.0800	10.1238	31.0495	10.1334	0.2765	1.2220	0.6406	2.0077	0.5543	0.2178
0.1	0.95	59.9603	19.9813	61.2093	20.1167	0.2667	1.4447	0.7114	2.2360	0.6250	0.2116
	0.99	299.5451	98.5618	302.9818	99.6335	0.2589	3.7670	1.4701	4.5626	1.3887	0.2068
	0.90	30.0800	10.1238	31.0495	10.1334	0.2765	4.1932	1.6263	4.9776	1.5408	0.2229
0.3	0.95	59.9603	19.9813	61.2093	20.1167	0.2667	6.8758	2.5116	7.6666	2.4326	0.2164
	0.99	299.5451	98.5618	302.9818	99.6335	0.2589	28.4283	9.5852	29.2242	9.5561	0.2114
	0.90	30.0800	10.1238	31.0495	10.1334	0.2765	9.1310	3.2556	9.9101	3.1767	0.2324
0.5	0.95	59.9603	19.9813	61.2093	20.1167	0.2667	16.6339	5.7342	17.4206	5.6747	0.2255
	0.99	299.5451	98.5618	302.9818	99.6335	0.2589	76.5565	25.4075	77.3493	25.4878	0.2199
	0.90	30.0800	10.1238	31.0495	10.1334	0.2765	16.0356	5.5285	16.8051	5.4621	0.2465
0.7	0.95	59.9603	19.9813	61.2093	20.1167	0.2667	30.7192	10.3795	31.4978	10.3515	0.2387
	0.99	299.5451	98.5618	302.9818	99.6335	0.2589	148.1518	48.9371	148.9378	49.1836	0.2324
	0.90	30.0800	10.1238	31.0495	10.1334	0.2765	24.9068	8.4450	25.6628	8.3969	0.2651
0.9	0.95	59.9603	19.9813	61.2093	20.1167	0.2667	49.1315	16.4472	49.8984	16.4629	0.2561
	0.99	299.5451	98.5618	302.9818	99.6335	0.2589	243.2139	80.1738	243.9898	80.6436	0.2489

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