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Shrinking projection method for proximal split feasibility and fixed point problems

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Abstract

In this paper, we consider and study proximal split feasibility and fixed point problem. For solving the problems, we introduce an iterative algorithm with shrinking projection technique. It is proven that the sequence generated by the proposed iterative algorithm converge strongly to the common solution of the proximal split feasibility and fixed point problems.

Keywords: shrinking projection method, proximal split feasibility problem, fixed point problem, k-strictly pseudo-contractive mapping, strong convergence.
2010 MSC: 47H10, 49M37, 49K35, 90C25.

1. Introduction

Throughout this paper, we assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces, $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint A^*, $f : \mathcal{H}_1 \to \mathcal{R} \bigcup \{+\infty\}$ and $g : \mathcal{H}_2 \to \mathcal{R} \bigcup \{+\infty\}$ are two proper, lower semi-continuous convex functions.

In the present manuscript, we try to solve the following minimization problem:

$$\min_{x \in \mathcal{H}_1} \{f(x) + g_\lambda(Ax)\}, \tag{1.1}$$

where g_λ stands for the Moreau-Yosida approximate of the function g of index $\lambda > 0$, that is,

$$g_\lambda(x) = \min_{y \in \mathcal{H}_2} \left\{ g(y) + \frac{1}{2\lambda} \|x - y\|^2 \right\}.$$

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The split feasibility problem [6] was introduced for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2], it is found that split feasibility problem can also be used to model the intensity modulated therapy [7]. It is worth pointing out that the split feasibility problem is a special case of problem (1.1). As a matter of fact, we choose f and g as the indicator functions of two nonempty closed convex sets $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$, that is,

$$f(x) = \delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$g(x) = \delta_Q(x) = \begin{cases} 0, & \text{if } x \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, the problem (1.1) collapses to

$$\min_{x \in \mathcal{H}_1} \left\{ \delta_C(x) + (\delta_Q)_\lambda(Ax) \right\},$$

which is equivalent to the following formulation

$$\min_{x \in C} \left\{ \frac{1}{2\lambda} \|(I - \mathrm{proj}_Q)Ax\|^2 \right\}. \tag{1.2}$$

Surely, solving (1.2) is to solve the following split feasibility problem of finding x such that

$$x \in C \quad \text{and} \quad Ax \in Q \tag{1.3}$$

provided $C \bigcap A^{-1}(Q) \neq \emptyset$.

To solve (1.3) that has been studied extensively by many authors; see, for instance, [1, 16, 17, 8, 3, 18, 4, 13, 5], one of the key points is to use the fixed point technique according to x^* which solves (1.3) if and only if

$$x^* = \mathrm{proj}_C(I - \gamma A^*(I - \mathrm{proj}_Q)A)x^*,$$

where $\gamma > 0$ is a constant and proj_C and proj_Q stand for the orthogonal projectional on the closed convex sets C and Q, respectively. According to the above fixed point formulation, A seemingly more popular algorithm that solves the split feasibility problem is the CQ algorithm presented by Byrne [2, 1]:

$$x_{n+1} = \mathrm{proj}_C(x_n - \tau_n A^*(I - \mathrm{proj}_Q)Ax_n),$$

where the step size $\tau_n \in (0, 2/\|A\|^2)$.

However, the step size τ_n depends on the operator norm $\|A\|$ which is not an easy work to calculate in practice. To overcome this difficulty, the so-called self-adaptive method was developed.

Self − adaptive algorithm [9] Let $x_0 \in \mathcal{H}_1$ be an initial arbitrarily point. Assume that a sequence $\{x_n\}$ in C has been constructed with $\nabla \bar{h}(x_n) \neq 0$ as follows: Compute x_{n+1} via the rule

$$x_{n+1} = \mathrm{proj}_C(x_n - \tau_n A^*(I - \mathrm{proj}_Q)Ax_n), \tag{1.4}$$

where $\tau_n = \rho_n \frac{\bar{h}(x_n)}{\|\nabla \bar{h}(x_n)\|^2}$ with $0 < \rho_n < 4$ and $\bar{h}(x) = \frac{1}{2}\|(I - \mathrm{proj}_Q)Ax\|^2$.

If $\nabla \bar{h}(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of the problem (1.3) and the iterative process stops. Otherwise, we set $n := n + 1$ and go to the sequence (1.4).

Our main purpose of the present manuscript is to solve the problem (1.1) by using the fixed point technique, the self-adaptive method and the shrinking projection technique. By the Fréchet differentiability of the Yosida approximate g_λ, we have

$$\begin{aligned} \partial(f(x) + g_\lambda(Ax)) &= \partial f(x) + A^*\nabla g_\lambda(Ax) \\ &= \partial f(x) + A^*\left(\frac{I - \text{prox}_{\lambda g}}{\lambda}\right)Ax, \end{aligned} \tag{1.5}$$

where $\partial f(x)$ denotes the subdifferential of f at x and $\text{prox}_{\lambda g}x$ is the proximal mapping of g, that is,

$$\partial f(x) = \{w \in \mathcal{H}_1 : f(y) \geq f(x) + \langle w, y - x\rangle, \forall y \in \mathcal{H}_1\}$$

and

$$\text{prox}_{\lambda g}x = \arg\min_{y\in\mathcal{H}_2}\left\{g(y) + \frac{1}{2\lambda}\|y - x\|^2\right\}.$$

Note that the optimality condition of (1.5) is as follows:

$$0 \in \partial f(x) + A^*\left(\frac{I - \text{prox}_{\lambda g}}{\lambda}\right)Ax,$$

which can be rewritten as

$$0 \in \mu\lambda\partial f(x) + \mu A^*\left(I - \text{prox}_{\lambda g}\right)Ax,$$

which is equivalent to the fixed point formalation:

$$x = \text{prox}_{\mu\lambda f}(I - \mu A^*(I - \text{prox}_{\lambda g})A)x$$

for all $\mu > 0$.

If $\arg\min f \bigcap A^{-1}(\arg\min g) \neq \emptyset$, then (1.1) is reduced to the following proximal split feasibility problem of finding x such that

$$x \in \arg\min f \quad \text{and} \quad Ax \in \arg\min g, \tag{1.6}$$

where $\arg\min f = \{x^* \in \mathcal{H}_1 : f(x^*) \leq f(x), \forall x \in \mathcal{H}_1\}$ and $\arg\min g = \{x^\dagger \in \mathcal{H}_2 : f(x^\dagger) \leq f(x), \forall x \in \mathcal{H}_2\}$. In the sequel, we use Γ to denote the solution set of the problem (1.6).

Recently, in order to solve the problem (1.6), Moudafi and Thakur [12] presented the following split proximal algorithm with a way of selecting the step sizes such that its implementation does not need any prior information as regards the operator norm.

Self − adaptive split proximal algorithm [12] Let $x_0 \in \mathcal{H}_1$ be an initial arbitrarily point. Assume that a sequence $\{x_n\}$ in $\mathcal{H}_1$ has been constructed with $\theta(x_n) \neq 0$ as follows: Compute x_{n+1} via the rule

$$x_{n+1} = \text{prox}_{\mu_n\lambda f}(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) \tag{1.7}$$

for all $n \geq 0$, where the step size $\mu_n = \rho_n\frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ in which $0 < \rho_n < 4$, $h(x_n) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax_n\|^2$, $l(x_n) = \frac{1}{2}\|(I - \text{prox}_{\mu_n\lambda f})x_n\|^2$ and $\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$.

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of the problem (1.6) and the iterative process stops. Otherwise, we set $n := n + 1$ and go to the sequence (1.7).

Consequently, they demonstrated the following weak convergence of the above split proximal algorithm.

Theorem 1. *Suppose that $\Gamma \neq \emptyset$. Assume the parameters satisfy the condition:*

$$\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n) + l(x_n)} - \epsilon$$

for some $\epsilon > 0$ small enough. Then the sequence $\{x_n\}$ generated by (1.7) weakly converges to a solutions of the problem (1.6).

They established the weak convergence of algorithm (1.7) under the condition that $\mathcal{H}_1$ is Hilbert space with Opial property. However, in some applied disciplines, the strong convergence is more desirable than the weak convergence. So, we need to adapt (1.7) such that the strong convergence is guaranteed. Our modification is mainly based on an idea in Takahashi *et al*. [15] for finding a fixed point of a nonexpansive mapping T in Hilbert space. Their algorithmic scheme is the following: For $u \in \mathcal{H}_1$ and $n \geq 1$, set

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = \text{proj}_{C_{n+1}} u. \end{cases}$$

In this paper, motivated by the recent works in this field, especially by Moudafi and Thakur [12] and Takahashi *et al*. [15], we introduce an iterative algorithm and prove its strong convergence for solving proximal split feasibility and fixed point problems involved in a k-strictly pseudo-contractive mapping.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

The notation $Fix(T)$ denotes the set of fixed points of the mapping T, that is, $Fix(T) = \{x \in \mathcal{H} : Tx = x\}$. Projections are an important tool for our work in this paper. Recall that the (nearest point or metric) projection from $\mathcal{H}$ onto C, denoted by proj_C, is defined in such a way that, for each $x \in \mathcal{H}$, $\text{proj}_C x$ is the unique point in C with the property

$$\|x - \text{proj}_C x\| = \min\{\|x - y\| : y \in C\}.$$

Some properties of projections are gathered in the following proposition.

Proposition 1. *Given $x \in \mathcal{H}$ and $z \in C$.*
(1) $z = proj_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0$ for all $y \in C$.
(2) $z = proj_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$ for all $y \in C$.
(3) $\langle x - y, proj_C x - proj_C y\rangle \geq \|proj_C x - proj_C y\|^2$ for all $y \in \mathcal{H}$, which hence implies that $proj_C$ is nonexpansive.

We also need other sorts of nonlinear operators which are introduced blow.

Definition 1. *A nonlinear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be*
(1) L-Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in \mathcal{H}$. If L=1, we call T nonexpansive.
(2) k-strictly pseudo-contractive if there exists $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$$

for all $x, y \in \mathcal{H}$, where I denotes the identity, which is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - T)x - (I - T)y\|^2$$

(3) Firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in \mathcal{H}$, which is equivalent to

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle$$

for all $x, y \in \mathcal{H}$. Also, the mapping $I - T$ is firmly nonexpansive.

Note that the proximal mapping of g is firmly nonexpansive, namely,

$$\left\|\text{prox}_{\lambda g}x - \text{prox}_{\lambda g}y\right\|^2 \le \langle \text{prox}_{\lambda g}x - \text{prox}_{\lambda g}y, x - y\rangle$$

for all x, $y \in \mathcal{H}_2$ and it is also the case for the complement $I - \text{prox}_{\lambda g}$.

For all x, $y \in \mathcal{H}$, the following conclusions hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \quad t \in [0, 1], \tag{2.1}$$

and

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

Usually, the convergence of iterative algorithms requires some additional smoothness properties of the mapping T such as demi-closedness.

Definition 2. *An operator T is said to be demi-closed if, for any sequence $\{x_n\}$ which weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to z, then $Tx = z$.*

Lemma 1. *[10] Let C be a nonempty closed convex subset of a real Hilbert $\mathcal{H}$ and $T : C \to C$ a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Then, $I - T$ is demi-closed at zero.*

Lemma 2. *[19] Let C be a nonempty closed convex subset of a real Hilbert $\mathcal{H}$ and $T : C \to C$ a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)k \le \gamma$. Then*

$$\|\gamma(x - y) + \delta(Tx - Ty)\| \le (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

Lemma 3. *[11] Let C be a nonempty closed convex subset of a real Hilbert $\mathcal{H}$ and let $\{x_n\}$ be a sequence in $\mathcal{H}$ and $x_0 \in \mathcal{H}$. Let $q = P_C x_0$. If $\{x_n\}$ satisfies the following conditions:*

(i) $\omega_w(x_n) \subset C$;

(ii) $\|x_n - x_0\| \le \|x_0 - q\|$ *for all* $n \ge 1$,

then one has $x_n \to q$.

3. Main result

In this section, we introduce an iterative algorithm and prove its strong convergence for solving proximal split feasibility and fixed point problems.

Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces, $f : \mathcal{H}_1 \to \mathcal{R}\bigcup\{+\infty\}$ and $g : \mathcal{H}_2 \to \mathcal{R}\bigcup\{+\infty\}$ are two proper, lower semi-continuous convex functions and that (1.6) is consistent. $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Let C be a nonempty closed convex subset of a real Hilbert $\mathcal{H}_1$. Assume that $T : C \to C$ is a k-strictly pseudo-contraction for some $0 \le k < 1$ such that $Fix(T) \neq \emptyset$. Set $\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$ with $h(x_n) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax_n\|^2$, $l(x_n) = \frac{1}{2}\|(I - \text{prox}_{\mu_n \lambda f})x_n\|^2$ for all $x \in \mathcal{H}_1$ and introduce the following algorithm.

Algorithm 1. *For an initialization $x_0, x_1 \in \mathcal{H}_1$, assume that a sequence $\{x_n\}$ generated by the rule with $\theta(x_n) \neq 0$*

$$\begin{cases} y_n = prox_{\mu_n \lambda f}(x_n - \mu_n A^*(I - prox_{\lambda g})Ax_n), \\ z_n = \beta_n x_n + \gamma_n y_n + \delta_n T y_n, \\ C_{n+1} = \{w \in C_n : \|z_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = proj_{C_{n+1}} x_0, \quad n \geq 1. \end{cases} \tag{3.1}$$

where $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ is a real number sequence and μ_n is the step size satisfying $\mu_n = \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$.

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of the problem (1.6) which is also a fixed point of a k-strictly pseudo-contractive mapping and the iterative process stops. Otherwise, we set $n := n + 1$ and go to the sequence (3.1).

Using (3.1),we prove the following strong convergence theorem for approximation of solution of problem (1.6) which is also a fixed point of a k-strictly pseudo-contractive mapping.

Theorem 2. *Suppose that $Fix(T) \bigcap \Gamma \neq \emptyset$. Assume the parameters satisfy the condition:*

(i) $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ *for some* $\epsilon > 0$ *small enough;*
(ii) $k \leq \alpha_n < 1$;
(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$;
(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$.

Then the sequence $\{x_n\}$ generated by (3.1) *strongly converges to $x^* \in Fix(T) \bigcap \Gamma$.*

Proof. Let $x^* \in Fix(T) \bigcap \Gamma$. Since minimizers of any function are exactly fixed points of its proximal mappings, we have $x^* = \text{prox}_{\mu_n \lambda f} x^*$ and $Ax^* = \text{prox}_{\lambda g} Ax^*$. Using the fact that $\text{prox}_{\mu_n \lambda f}$ is nonexpansive, we derive from (3.1) that

$$\begin{aligned} \|y_n - x^*\| &= \|\text{prox}_{\mu_n \lambda f}(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) - x^*\| \\ &\leq \|x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n - x^*\|. \end{aligned} \tag{3.2}$$

Note that $\nabla h(x_n) = A^*(I - \text{prox}_{\lambda g})Ax_n$ and $\nabla l(x_n) = (I - \text{prox}_{\mu_n \lambda f})x_n$. Since $I - \text{prox}_{\lambda g}$ is firmly nonexpansive. Hence, we obtain

$$\begin{aligned} &\|x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n - x^*\|^2 \\ &= \|x_n - x^*\|^2 - 2\mu_n \left\langle A^*(I - \text{prox}_{\lambda g})Ax_n, x_n - x^* \right\rangle + \mu_n^2 \|A^*(I - \text{prox}_{\lambda g})Ax_n\|^2 \\ &= \|x_n - x^*\|^2 - 2\mu_n \langle \nabla h(x_n), x_n - x^* \rangle + \mu_n^2 \|\nabla h(x_n)\|^2 \\ &\leq \|x_n - x^*\|^2 - 4\mu_n h(x_n) + \mu_n^2 \|\nabla h(x_n)\|^2 \\ &\leq \|x_n - x^*\|^2 - 4\rho_n \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \frac{h(x_n)}{h(x_n) + l(x_n)} + \rho_n^2 \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \\ &= \|x_n - x^*\|^2 - \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \rho_n \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)}. \end{aligned} \tag{3.3}$$

Since $(\gamma_n + \delta_n)k \le \gamma_n$, utilizing Lemma 2, from (2.1), we conclude that

$$\begin{aligned}\|z_n - x^*\|^2 &= \|\beta_n x_n + \gamma_n y_n + \delta_n T y_n - x^*\|^2\\ &= \left\|\beta_n(x_n - x^*) + (\gamma_n+\delta_n)\frac{1}{\gamma_n+\delta_n}\left(\gamma_n(y_n - x^*) + \delta_n(Ty_n - x^*)\right)\right\|^2\\ &= \beta_n\|x_n - x^*\|^2 + (\gamma_n+\delta_n)\left\|\frac{1}{\gamma_n+\delta_n}\left(\gamma_n(y_n - x^*) + \delta_n(Ty_n - x^*)\right)\right\|^2\\ &\quad - \beta_n(\gamma_n+\delta_n)\left\|\frac{1}{\gamma_n+\delta_n}\left(\gamma_n(y_n - x_n) + \delta_n(Ty_n - x_n)\right)\right\|^2\\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 - \frac{\beta_n}{1-\beta_n}\|z_n - x_n\|^2.\end{aligned} \tag{3.4}$$

Without loss of generality, by the control condition (i), we can assume that $(4h(x_n))/(h(x_n)+l(x_n)) - \rho_n \ge 0$ for all $n \ge 1$. Thus, from (3.2), (3.3), (3.4) and the conditions $(iii), (iv)$, we have

$$\begin{aligned}&\|z_n - x^*\|^2\\ &\le \|x_n - x^*\|^2 - (1-\beta_n)\left(\rho_n\left(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n\right)\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right)\\ &\le \|x_n - x^*\|^2.\end{aligned} \tag{3.5}$$

It follows that $x^* \in C_{n+1}$. Thus, we get $Fix(T)\bigcap\Gamma \in C_n$ for all $n \ge 1$. Next, we show that C_n is closed and convex for all $n \ge 1$. The set $C_1 = \mathcal{H}_1$ is obviously closed and convex. Suppose that C_k is closed and convex. We see that C_{k+1} is closed and convex since $\|z_n - w\| \le \|x_n - w\|$ is equivalent to

$$\langle x_n - z_n, w\rangle \le \frac{1}{2}\left(\|x_n\|^2 - \|z_n\|^2\right),$$

so that C_n is a halfspace, therefore C_n is closed and convex for all $n \ge 1$. Thus, we obtain that the sequence $\{x_n\}$ is well defined.

From Propsition 1(1) and $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \ge 0, \quad \forall y \in C_n.$$

Recalling that $Fix(T)\bigcap\Gamma \in C_n$, one has

$$\langle x_0 - x_n, x_n - x^*\rangle \ge 0, \quad \forall x^* \in Fix(T)\bigcap\Gamma.$$

Hence,

$$\begin{aligned}0 &\le \langle x_0 - x_n, x_n - x^*\rangle\\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x^*\rangle\\ &\le -\|x_n - x_0\|^2 + \|x_n - x_0\|\|x_0 - x^*\|.\end{aligned}$$

This implies that

$$\|x_n - x_0\| \le \|x_0 - x^*\|,$$

which yields that sequence $\{x_n\}$ is bounded. From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we get

$$\begin{aligned}0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle\\ &\le -\|x_n - x_0\|^2 + \|x_n - x_0\|\|x_0 - x_{n+1}\|,\end{aligned} \tag{3.6}$$

which gives that

$$\|x_n - x_0\| \leq \|x_0 - x_{n+1}\|.$$

Hence, the limit $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

It follows from (3.6) that

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2\\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2\\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.\end{aligned}$$

Thus, we get

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.7}$$

The fact that $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1}$ gives

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|. \tag{3.8}$$

The expressions (3.7) and (3.8) yield

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.9}$$

Returning to (3.5), we have

$$\begin{aligned}&(1-\beta_n)\left(\rho_n\left(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n\right)\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right)\\ &\leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2\\ &\leq (\|x_n - x^*\| + \|z_n - x^*\|)\|z_n - x_n\|.\end{aligned}$$

This together with (3.9), condition (iv) and $\rho_n\left(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n\right) \geq \epsilon^2$ implies that

$$\lim_{n\to\infty} \frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)} = 0.$$

Noting that $\theta^2(x_n) = \|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2$ is bounded, we deduce immediately that

$$\lim_{n\to\infty} (h(x_n) + l(x_n)) = 0.$$

Therefore, we have

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} l(x_n) = 0. \tag{3.10}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$. By the lower semi-continuity of h, we have

$$0 \leq h(p) \leq \liminf_{i\to\infty} h(x_{n_i}) = \lim_{n\to\infty} h(x_n) = 0,$$

so, we have

$$h(p) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g})Ap\| = 0,$$

that is, Ap is a fixed point of proximal mapping of g or, equivalently, $0 \in \partial g(Ap)$. In other words, Ap is a minimizer of g.

Similarly, from the lower semi-continuity of l, we have

$$0 \le l(p) \le \liminf_{i\to\infty} l(x_{n_i}) = \lim_{n\to\infty} l(x_n) = 0,$$

so, we have

$$l(p) = \frac{1}{2}\|(I - \text{prox}_{\mu_n\lambda f})p\| = 0,$$

that is, p is a fixed point of proximal mapping of f or, equivalently, $0 \in \partial f(p)$. In other words, p is a minimizer of f. Hence, $p \in \Gamma$.

We observe that $0 < \mu_n < \frac{4(h(x_n)+l(x_n))}{\theta^2(x_n)}$, which implies that $\lim_{n\to\infty} \mu_n = 0$. Hence, we have from (3.1) and the boundedness of $\theta(x_n)$ that

$$\|y_n - \text{prox}_{\mu_n\lambda f} x_n\| \le \mu_n \|A^*(I - \text{prox}_{\lambda g})Ax_n\| \le \mu_n M \tag{3.11}$$

for some $M > 0$.

From $l(x_n) = \frac{1}{2}\|(I - \text{prox}_{\mu_n\lambda f})x_n\|^2$, we have

$$\lim_{n\to\infty} \|(I - \text{prox}_{\mu_n\lambda f})x_n\| = 0.$$

This together with (3.11) implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.12}$$

Furthermore, we obtain

$$\begin{aligned}\|\delta_n(Ty_n - x_n)\| &= \|z_n - x_n - \gamma_n(y_n - x_n)\| \\ &\le \|z_n - x_n\| + \|y_n - x_n\|,\end{aligned}$$

from (3.9), (3.12) and the condition (iv) that

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0.$$

Again using (3.12), we have

$$\lim_{n\to\infty} \|Ty_n - y_n\| = 0.$$

Since $I - T$ is demi-closed at zero, from Lemma 1, we get $p \in Fix(T)$. So, $p \in Fix(T)\bigcap\Gamma$. Thus, we have obtained that $\omega_w(x_n) \in Fix(T)\bigcap\Gamma$. According to Lemma 3, we see that $x_n \to P_{Fix(T)\bigcap\Gamma} x_0$. □

Remark 1. *We make the following remark concerning our contributions in this paper.*

1. *Yao et al. [[20], Theorem 3.2] and Yao et al. [[21], Theorem 5] prove strong convergence theorems for proximal split feasibility problem by regularization method, respectively. Shehu and Ogbuisi [[14], Theorem 3.1] prove strong convergence theorem for approximation of a solution of proximal split feasibility problem which is also a fixed point of a k-strictly pseudo-contractive mapping by the damped-like algorithm. But now, shrinking projection method has been presented in this paper for solving proximal split feasibility problem which is also a fixed point of a k-strictly pseudo-contractive mapping.*
2. *The technique of proving strong convergence in Theorem 3.1 is different from that in [[14], Theorem 3.1] because our technique depends on Lemma* 2.
3. *To ensure the weak convergence of the algorithm proposed in [[12], Theorem 2.2], one has to assume that the Hilbert space satisfying Opial's property. The main advantage of our algorithm is that its convergence does not rely on the Opial's property. Furthermore, we establish the norm convergence of the proposed algorithm.*

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