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H1 = L2 (0,π;H) Uzayında İki Terimli Diferansiyel Operatörün Düzenli İzi

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ÖZET: Mevcut çalışmanın esas amacı Hilbert uzayında tanımlanmış bir kendine-eş diferansiyel operatör için bir iz formülü çıkarmaktır.

Anahtar Kelimeler: Hilbert Uzayı, Özdeğer, Spektrum, İz-sınıfı Operatör, Rezolvent Operatör.

The Regularized Trace of Two Terms Differential Operator in the Space

$H_1 = L_2(0,\pi; H)$.

ABSTRACT: The main purpose of this present paper is to derive a trace formula for a selfadjoint differential operator which is defined in Hilbert space.

Keywords: Hilbert Space, Eigenvalue, Spectrum, Trace-class Operator, Resolvent Operator.

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INTRODUCTION

The study of regularized trace of differential operators was started in the 20th century with the work of Gelfand and Levitan (Gelfand et al., 1953). They dealt with the Sturm-Liouville type of differential equation:

$$-y''+q(x)y=\mu y, \qquad y'(0)=y'(\pi)=0$$

and obtained the formula $\sum_{n=0}^{\infty}(\mu_n-\lambda_n)=\frac{1}{4}\left[q(0)+q(n)\right]$; here μ_n are the eigenvalues of this operator and $\lambda_n=n^2$ are the eigenvalues of the same operator with $q(x)=0$. This research provided the basis for new and important theory.

Many scientists focused on trace computation of various differential operators and obtained significant results. After the pioneering work by Gelfand and Levitan, Gelfand, Dikiy, Levitan, Gasymov, Sadovnichii (Dikiy, 1953; Gelfand et al., 1953; Dikiy, 1955; Gelfand, 1956; Gasymov, 1963; Levitan, 1964; Sadovnichii, 1966) investigated the regularized trace formulas. The list of these works on the subject is given by Sadovnichii and Podol'skii (Sadovnichii et al., 2009). The trace formulas of the abstract self-adjoint operators with continuous spectrum were investigated by some authors (Krein, 1953; Faddeev, 1957; Bayramoglu, 1986). Among the studies , regularized trace formulas for differential operators with operator coefficient play an important role (Adiguzel et al., 2004; Adiguzel et al., 2011; Baksi et al., 2017).

Let H be a separable Hilbert space. Let L be the operator in the space $H_1=L_2(0,\pi;H)$ defined by differential expression:

$$\ell(y)=-y''+Qy \text{ with boundary conditions } y'(0)=y(\pi)=0. \qquad (1)$$

Assume that the operator $Q(x)$ in the expression $\ell(y)$ satisfies the conditions:

$(Q1)$ For every $x\in[0,\pi]$, $Q(x)$ is a self-adjoint kernel operator from H to H, and $Q(x)$ has second order continuous derivative with respect to the norm $\sigma_1(H)$ in $[0,\pi]$,

$(Q2)$ $\|Q\|<3/2$,

$(Q3)$ There is an orthonormal basis in the space H such that $\sum_{f=1}^{\infty}\left\|Q(x)\varphi_f\right\|<\infty$.

Here, $\sigma_1(H):H\to H$ is the space of kernel operators. The norms in H_1 and H are denoted by $\|.\|_1$ and $\|.\|$. Furthermore, the sum of eigenvalues of a kernel operator Q is denoted by $\operatorname{tr}Q=\operatorname{trace}Q$. The spectrum and resolvent of the operator L are denoted by $\sigma(L)$ and $\rho(L)$, respectively.

Suppose that the operator L_0 formed by differential expression:

$$\ell_0(y)=-y'' \qquad \text{with the boundary conditions } y'(0)=y(\pi)=0. \qquad (2)$$

The spectrum of the operator L_0 is the set $\left\{\left(e+\frac{1}{2}\right)^2\right\}_{e=1}^{\infty}$ and every point of this set is an eigenvalue with infinite multiplicity. The orthonormal eigenvectors corresponding to eigenvalues $\left(e+\frac{1}{2}\right)^2$ are in the form $\psi_{ef}(x)=\sqrt{\frac{2}{\pi}}\cos\left(e+\frac{1}{2}\right)^2 x.\varphi_f \quad (f=1,2,...)$.

Our purpose in this paper is to find the trace equality

$$\sum_{e=0}^{\infty}\left\{\sum_{f=1}^{\infty}\left[\lambda_{ef}^{2}-\left(e+\frac{1}{2}\right)^{4}\right]-\frac{(2e+1)^{2}}{2\pi}\int_{0}^{\pi}trQ(x)dx-c\right\}$$

$$=\frac{1}{8}tr\left[Q''(0)-Q''(\pi)-2Q^{2}(0)+2Q^{2}(\pi)\right] \quad (3)$$

for the operator L. Here, $\left\{\lambda_{ef}\right\}_{f=1}^{\infty}$ is the set of the eigenvalues of the operator L, and belongs to the interval

$$I_e=\left[\left(e+\frac{1}{2}\right)^{2}-\|Q\|,\left(e+\frac{1}{2}\right)^{2}+\|Q\|\right] \quad (e=0,1,2,...) \quad (2)$$

and $c=\frac{1}{2\pi}\int_{0}^{\pi}trQ^{2}(x)dx+\frac{1}{2\pi^{2}}tr\left[\int_{0}^{\pi}Q(x)dx\right]^{2}-\frac{1}{2}\left[trQ'(0)+trQ'(\pi)\right].$

MATERIALS AND METHODS

Let R_{λ}^{0} and R_{λ} be resolvent operators of L_0 and L. One can prove that if $Q(x)$ satisfies the condition $(Q3)$, then $QR_{\lambda}^{0}: H_1 \to H_1$ is a kernel operator for every $\lambda \neq \left(e+\frac{1}{2}\right)^{2}$ $(e=0,1,2,...)$. Let $\left\{\lambda_{ef}\right\}_{f=1}^{\infty}$ be the eigenvalues on I_e of the operator L.

Theorem 2.1. If $Q(x)$ holds the conditions $(Q2)$ and $(Q3)$, the spectrum of the operator L is a subset of the intervals I_e which are pairwise disjoint and

(1) Every point different from $\left(e+\frac{1}{2}\right)^{2}$ on I_e is a discrete eigenvalue with finite multiplicity in $\sigma(L)$,

(2) $\left(e+\frac{1}{2}\right)^{2}$ can be an eigenvalue with finite or infinite multiplicity in $\sigma(L)$,

(3) $\lim_{f\to\infty}\lambda_{ef}=\left(e+\frac{1}{2}\right)^{2}$ $(e=0,1,2,...)$.

Moreover, one can show that the series $\sum_{f=1}^{\infty}\left[\lambda_{ef}-\left(e+\frac{1}{2}\right)^{2}\right]$ (e=0,1, 2, …) are absolutely convergent. Since $R_{\lambda}-R_{\lambda}^{0}$ is a kernel operator in the space $\sigma_1(H_1)$, the formula

$$tr(R_{\lambda}-R_{\lambda}^{0})=\sum_{e=0}^{\infty}\sum_{f=1}^{\infty}\left[\frac{1}{\lambda_{ef}-\lambda}-\frac{1}{\left(e+\frac{1}{2}\right)^{2}-\lambda}\right] \quad (3)$$

is true for every $\lambda \in \rho(L)$ (Levitan et al., 1991). If we multiply with $\frac{\lambda^2}{2\pi i}$ both sides of "Eq. 5." and integrate on the circle $|\lambda| = b_d = (d+1)^2$, we get the following equality

$$\frac{1}{2\pi i}\int_{|\lambda|=b_d} \lambda^2 tr(R_\lambda - R_\lambda^0)d\lambda = \sum_{e=0}^{d}\sum_{f=1}^{\infty}\left[\left(e+\frac{1}{2}\right)^4 - \lambda_{ef}^2\right]. \tag{4}$$

By equality $R_\lambda - R_\lambda^0 = -R_\lambda Q R_\lambda^0$ and "Eq. 6.", we have

$$\sum_{e=0}^{d}\sum_{f=1}^{\infty}\left[\left(e+\frac{1}{2}\right)^4 - \lambda_{ef}^2\right] = \sum_{s=1}^{N}\frac{(-1)^s}{2\pi i}\int_{|\lambda|=b_d}\lambda^2 tr(R_\lambda^0(QR_\lambda^0)^s)d\lambda + \frac{(-1)^{N+1}}{2\pi i}\int_{|\lambda|=b_d}\lambda^2 tr(R_\lambda^0(QR_\lambda^0)^{N+1})d\lambda \tag{7}$$

where N is a positive integer. Let

$$K_{ds} = \frac{(-1)^{s+1}}{2\pi i}\int_{|\lambda|=b_d}\lambda^2 tr\left[R_\lambda^0(QR_\lambda^0)^s\right]d\lambda\,, \tag{5}$$

$$K_d^{(N)} = \frac{(-1)^N}{2\pi i}\int_{|\lambda|=b_d} tr\left[R_\lambda(QR_\lambda^0)^{N+1}\right]d\lambda\ . \tag{6}$$

Then "Eq. 7." becomes

$$\sum_{e=0}^{d}\sum_{f=1}^{\infty}\left[\lambda_{ef}^2 - \left(e+\frac{1}{2}\right)^4\right] = \sum_{s=1}^{N}K_{ds} + K_d^{(N)}\ . \tag{7}$$

Since QR_λ^0 is a kernel operator for every $\lambda \neq \left(e+\frac{1}{2}\right)^2$ in the space $\sigma_1(H_1)$, one can prove that QR_λ^0 is analytic with respect to norm in $\sigma_1(H_1)$ in the domain $\mathbb{C} - \left\{\left(e+\frac{1}{2}\right)^2\right\}_{e=0}^{\infty}$ and the formula

$$K_{ds} = \frac{(-1)^s}{\pi i s}\int_{|\lambda|=b_d}\lambda tr\left[\left(QR_\lambda^0\right)^s\right]d\lambda\ , \tag{8}$$

is satisfied.

RESULTS AND DISCUSSION

In the last section, a formula for second regularized trace of the operator L will be found. By "Eq. 11."

$$K_{d1} = -\frac{1}{\pi i}\int_{|\lambda|=b_d}\lambda tr\left(R_\lambda\left(QR_\lambda^0\right)\right)d\lambda \tag{12}$$

$$= 2\sum_{e=0}^{\infty}\sum_{f=1}^{\infty}\left(Q\psi_{ef},\psi_{ef}\right)\frac{1}{2\pi i}\int_{|\lambda|=b_d}\frac{\lambda d\lambda}{\lambda - \left(e+\frac{1}{2}\right)^2} \tag{9}$$

$$= \frac{4}{\pi}\sum_{e=0}^{d}\sum_{f=1}^{\infty}\int_0^{\pi}\cos^2\left[\left(e+\frac{1}{2}\right)x\right]\left(Q(x)\varphi_f,\varphi_f\right)dx \tag{10}$$

$$= \frac{2}{\pi}\sum_{e=0}^{d}\left(e+\frac{1}{2}\right)^2 \int_0^{\pi} trQ(x)dx + \frac{2}{\pi}\sum_{e=0}^{d}\left(e+\frac{1}{2}\right)^2 \int_0^{\pi} trQ(x)\cos\left[(2e+1)x\right]dx \quad (11)$$

$$= \frac{1}{2\pi}\sum_{e=0}^{d}(2e+1)^2 \int_0^{\pi} trQ(x)dx - \frac{d+1}{2\pi}\left[trQ^{'}(0)+trQ^{'}(\pi)\right]$$

$$-\frac{1}{2\pi}\sum_{e=0}^{d}\int_0^{\pi} trQ^{''}(x)\cos\left[(2e+1)x\right]dx \ . \quad (12)$$

We now evaluate K_{d2}, by "Eq. 11."

$$K_{d2} = \frac{1}{2\pi i}\int_{|\lambda|=b_d} \lambda tr\left[(QR_\lambda^0)^2\right]d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=b_d} \lambda \sum_{e=0}^{\infty}\sum_{f=1}^{\infty}\left(\left(QR_\lambda^0\right)^2 \psi_{ef}, \psi_{ef}\right)d\lambda \ . \quad (13)$$

Moreover, we know that $(QR_\lambda^0)(\psi_{ef}) = Q\psi_{ef} \cdot \left(-\lambda + \left(e+\frac{1}{2}\right)^2\right)^{-1}$ and

$$(QR_\lambda^0)^2(\psi_{ef}) = QR_\lambda^0\left(QR_\lambda^0\psi_{ef}\right) \quad (14)$$

$$= \left(\left(e+\frac{1}{2}\right)^2 - \lambda\right)^{-1} QR_\lambda^0\left(\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}(Q\psi_{ef}, \psi_{rq})_1 \ \psi_{rq}\right) \quad (15)$$

$$= \left(\left(e+\frac{1}{2}\right)^2 - \lambda\right)^{-1}\left(\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\frac{(Q\psi_{ef}, \psi_{rq})_1 \ Q\psi_{rq}}{\left(\left(r+\frac{1}{2}\right)^2 - \lambda\right)}\right) \ . \quad (16)$$

If we substitute "Eq. 20." in "Eq. 17."

$$K_{d2} = \frac{1}{2\pi i}\int_{|\lambda|=b_d} \lambda\left[\sum_{e=0}^{\infty}\sum_{f=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\frac{\left(Q\psi_{ef}, \psi_{rq}\right)\left(Q\psi_{rq}, \psi_{ef}\right)}{\left(\left(e+\frac{1}{2}\right)^2-\lambda\right)\left(\left(r+\frac{1}{2}\right)^2-\lambda\right)}\right]d\lambda \quad (17)$$

$$= \sum_{e=0}^{\infty}\sum_{f=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\psi_{ef}, \psi_{rq}\right)\right|^2 \frac{1}{2\pi i}\int_{|\lambda|=b_d}\frac{\lambda d\lambda}{\left(\left(e+\frac{1}{2}\right)^2-\lambda\right)\left(\left(r+\frac{1}{2}\right)^2-\lambda\right)} \quad (18)$$

$$= \sum_{e=0}^{d}\sum_{f=1}^{\infty}\sum_{r=0}^{d}\sum_{q=1}^{\infty}\left|\left(Q\psi_{ef}, \psi_{rq}\right)\right|^2 + 2\sum_{e=0}^{d}\sum_{f=1}^{\infty}\sum_{r=d+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\psi_{ef}, \psi_{rq}\right)\right|^2 \frac{\left(e+\frac{1}{2}\right)^2}{\left(e+\frac{1}{2}\right)^2-\left(r+\frac{1}{2}\right)^2}$$

$$+2\sum_{e=0}^{d}\sum_{f=1}^{\infty}\sum_{r=d+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\psi_{ef}, \psi_{rq}\right)\right|^2 \frac{\left(e+\frac{1}{2}\right)^2}{\left(e+\frac{1}{2}\right)^2-\left(r+\frac{1}{2}\right)^2} \quad (19)$$

$$=\sum_{e=0}^{d}\sum_{f=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\psi_{ef},\psi_{rq}\right)\right|^2-\sum_{e=0}^{d}\sum_{f=1}^{\infty}\sum_{r=d+1}^{\infty}\sum_{q=1}^{\infty}\left(1+\frac{2\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}\right)\left|\left(Q\psi_{ef},\psi_{rq}\right)\right|$$

$$-\sum_{e=0}^{d}\sum_{f=1}^{\infty}\sum_{r=d+1}^{\infty}\sum_{q=1}^{\infty}\left(1+\frac{2\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}\right)\left|\left(Q\psi_{ef},\psi_{rq}\right)\right|^2 \qquad (20)$$

$$=\sum_{e=0}^{d}\sum_{f=1}^{\infty}\left\|Q\psi_{ef}\right\|_1^2-\sum_{e=0}^{d}\sum_{f=1}^{\infty}\sum_{r=d+1}^{\infty}\sum_{q=1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^2+\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}\left|\left(Q\psi_{ef},\Psi_{rq}\right)\right|^2 \qquad (21)$$

is found. Let

$$\beta_d(f,q)=\sum_{e=0}^{d}\sum_{r=d+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^2+\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}\left|\left(Q\psi_{ef},\psi_{rq}\right)\right|^2. \qquad (22)$$

Then we get

$$K_{d2}=\sum_{e=0}^{d}\sum_{f=1}^{\infty}\left\|Q\psi_{ef}\right\|_1^2-\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\beta_d(f,q). \qquad (23)$$

We now investigate $\beta_d(f,q)$. Since

$$\left|(Q\psi_{ef},\psi_{rq})\right|^2=\frac{1}{\pi^2}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e-r)x\,dx\right|^2$$

$$+\frac{2}{\pi^2}\mathrm{Re}\left[\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e-r)x\,dx\int_0^{\pi}\overline{(Q(x)\varphi_f,\varphi_q)}\cos(e+r+1)x\,dx\right]$$

$$+\frac{1}{\pi^2}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e+r+1)x\,dx\right|^2, \qquad (24)$$

then β_d is in the form:

$$\beta_d=\frac{1}{\pi^2}\sum_{e=0}^{d}\sum_{r=d+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^2+\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e-r)x\,dx\right|^2$$

$$+\frac{2}{\pi^2}\sum_{e=0}^{d}\sum_{r=d+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^2+\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}$$

$$\times\mathrm{Re}\left[\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e-r)xdx.\int_0^{\pi}\overline{(Q(x)\varphi_f,\varphi_q)}\cos(e+r+1)xdx\right]$$

$$+\frac{1}{\pi^2}\sum_{e=0}^{d}\sum_{r=d+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^2+\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e+r+1)xdx\right|^2 . \tag{25}$$

If we take

$$\beta_{d1}=\pi^{-2}\sum_{e=0}^{d}\sum_{r=d+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^2+\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e-r)xdx\right|^2 , \tag{26}$$

$$\beta_{d2}=\frac{2}{\pi^2}\sum_{e=0}^{d}\sum_{r=d+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^2+\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}$$

$$\times\mathrm{Re}\left[\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e-r)xdx\int_0^{\pi}\overline{(Q(x)\varphi_f,\varphi_q)}\cos(e+r+1)xdx\right] , \tag{27}$$

$$\beta_{d3}=\frac{1}{\pi^2}\sum_{e=0}^{d}\sum_{r=d+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^2+\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e+r+1)xdx\right|^2 , \tag{28}$$

and if we express β_d in terms of β_{d1}, β_{d2} and β_{d3} in “Eq. 29.”, we have $\beta_d(f,q)=\beta_{d1}+\beta_{d2}+\beta_{d3}$. Now, we calculate an asymptotic formula for the sum

$$\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\beta_{d1} . \tag{29}$$

For any integers $d\geq 1$ and $i\geq 1$, let $E_{di}=\{(r,e): r,e\in N;\ r-e=i;\ e\leq d;\ r>d\}$ then one can write “Eq. 30.” such that

$$\beta_{d1}=\pi^{-2}\sum_{i=1}^{\infty}\left(\sum_{e,r\in E_{di}}\frac{\left(r+\frac{1}{2}\right)^2+\left(e+\frac{1}{2}\right)^2}{\left(r+\frac{1}{2}\right)^2-\left(e+\frac{1}{2}\right)^2}\right)\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos(e-r)xdx\right|^2$$

$$=\pi^{-2}\sum_{i=1}^{\infty}\left[\sum_{e,r\in E_{di}}\left(1+\frac{2(2e+1)^2}{(2r+1)^2-(2e+1)^2}\right)\left|\int_0^{\pi}\left(Q(x)\varphi_f,\varphi_q\right)\cos ixdx\right|^2\right]. \tag{30}$$

Let us calculate the sum

$$\sum_{e,r\in E_{di}}\left(1+\frac{2\left(2e+1\right)^2}{\left(2r+1\right)^2-\left(2e+1\right)^2}\right). \tag{31}$$

If we take $i \le d+1$

$$\sum_{e,r\in E_{di}}\frac{\left(2e+1\right)^2}{\left(2r+1\right)^2-\left(2e+1\right)^2}=\frac{d}{2}+\frac{2-i}{2}+\sum_{s=0}^{i-1}\frac{i}{4(2d-2s+1)+4i} \quad , \tag{32}$$

and

$$\sum_{s=0}^{i-1}\frac{i}{2\left(2d-2s+i+1\right)}=i\sum_{s=0}^{i-1}\frac{1}{2\left(d+(d-s)+\left(i-s\right)+1\right)}<i\sum_{s=0}^{i-1}\frac{1}{2d}=\frac{i^2}{2d} \quad . \tag{33}$$

By using "Eq.36." and "Eq.37.", we rewrite the sum (35) for $i \le d+1$, $i \ge 1$, $d \ge 2$

$$\sum_{e,r\in E_{di}}\left(1+\frac{2\left(2e+1\right)^2}{\left(2r+1\right)^2-\left(2e+1\right)^2}\right)=d+2+i^2O(d^{-1}) \quad . \tag{34}$$

Here, $O(d^{-1})$ which satisfies inequality $0<O(d^{-1})<d^{-1}$, depends on d and i.

Similarly, for $i \ge d+1$, the sum (35) becomes

$$\sum_{e,r\in E_{di}}\left(1+\frac{2\left(2e+1\right)^2}{\left(2r+1\right)^2-\left(2e+1\right)^2}\right)=O(d) \quad \left(d\ge 2\right) \tag{35}$$

is obtained, where $O(d)$ which satisfies inequality $\left|O(d)\right|<4d$, depends on d and i.

Substituting "Eq.38." and "Eq.39." into "Eq. 33.", we get

$$\begin{aligned}\beta_{d1}&=\pi^{-2}\sum_{i=1}^{d+1}(d+2+i^2O(d^{-1}))\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos ixdx\right|^2\\&+\pi^{-2}\sum_{i=d+2}^{\infty}O(d)\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos ixdx\right|^2=\pi^{-2}(d+2)\sum_{i=1}^{\infty}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos ixdx\right|^2\\&+\pi^{-2}\sum_{i=1}^{d+1}i^2O(d^{-1})\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos ixdx\right|^2+\pi^{-2}\sum_{i=d+2}^{\infty}O(d)\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos ixdx\right|^2 .\end{aligned} \tag{36}$$

Since

$$\frac{1}{\pi}\sum_{i=1}^{\infty}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos ixdx\right|^2=\frac{1}{2}\int_0^{\pi}\left|(Q(x)\varphi_f,\varphi_q)\right|^2dx-\frac{1}{2\pi}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)dx\right|^2,$$

then we substitute last equality in "Eq. 40.":

$$\begin{aligned}\beta_{d1}&=\frac{d+2}{2\pi}\int_0^{\pi}\left|\left(Q(x)\varphi_f,\varphi_q\right)\right|^2dx-\frac{d+2}{2\pi^2}\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)dx\right|^2 \qquad +\pi^{-2}\sum_{i=1}^{d+1}i^2 0(d^{-1})\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos ixdx\right|^2\\&\quad+\pi^{-2}\sum_{i=1}^{d+1}i^2 0(d)\left|\int_0^{\pi}(Q(x)\varphi_f,\varphi_q)\cos ixdx\right|^2\end{aligned} \tag{37}$$

is obtained. Substituting "Eq. 41." into "Eq. 33.",

$$\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\beta_{d1} = \frac{d+2}{2\pi}\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\int_0^{\pi}\left|\left(Q(x)\varphi_f,\varphi_q\right)\right|^2 dx - \frac{d+2}{2\pi^2}\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\left|\int_0^{\pi}\left(Q(x)\varphi_f,\varphi_q\right)dx\right|^2$$

$$+\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=1}^{d+1} i^2 O(d^{-1})\left|\int_0^{\pi}\left(Q(x)\varphi_f,\varphi_q\right)\cos ixdx\right|^2$$

$$+\frac{1}{\pi^2}\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=d+2}^{\infty} O(d)\left|\int_0^{\pi}\left(Q(x)\varphi_f,\varphi_q\right)\cos ixdx\right|^2 . \quad (38)$$

Moreover,

$$\sum_{q=1}^{\infty}\sum_{i=1}^{d+1} i^2 O\left(d^{-1}\right)\left|\int_0^{\pi}\left(Q(x)\varphi_f,\varphi_q\right)\cos ixdx\right|^2 = O\left(d^{-1}\right) \quad , \quad (39)$$

and

$$\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=d+2}^{\infty} O(d)\left|\int_0^{\pi}\left(Q(x)\varphi_f,\varphi_q\right)\cos ixdx\right|^2 = O(d^{-1}) \ , \quad (40)$$

are obtained. If we substitute “Eq. 43.” and “Eq. 44.” into “Eq. 42.”, we have

$$\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\beta_{d1} = -\frac{d+2}{2\pi}\int_0^{\pi} trQ^2(x)dx + \frac{d+2}{2\pi^2} tr\left(\int_0^{\pi} Q(x)dx\right)^2 + O(d^{-1}) \ . \quad (41)$$

Since $Q(x)$ satisfies conditions $(Q1)-(Q3)$, then

$$\left|\sum_{f=1}^{\infty}\sum_{q=1}^{\infty}\beta_{dk}\right| \le cd^{-1} \ . \ (k = 2,3) \quad (42)$$

By using “Eq. 27.”, “Eq. 45.” and “Eq. 46.”

$$K_{d2} = \sum_{e=0}^{d}\sum_{f=1}^{\infty}\left\|Q\psi_{ef}\right\|^2 - \frac{d+2}{2\pi}\int_0^{\pi} trQ^2(x)dx + \frac{d+2}{2\pi^2} tr\left(\int_0^{\pi} Q(x)dx\right)^2 + O(d^{-1}) \quad (43)$$

is obtained. Now, we calculate the sum on the right side of “Eq. 47.”:

$$\sum_{e=0}^{d}\sum_{f=1}^{\infty}\left\|Q\psi_{ef}\right\|^2 = \frac{2}{\pi}\sum_{e=0}^{d}\sum_{f=1}^{\infty}\int_0^{\pi}\cos^2\left(e+\frac{1}{2}\right)x\left(Q^2(x)\varphi_f,\varphi_f\right)dx$$

$$+\frac{1}{\pi}\sum_{e=0}^{d}\sum_{f=1}^{\infty}\int_0^{\pi}(1+\cos(2e+1)x)\left(Q^2(x)\varphi_f,\varphi_f\right)dx = \frac{d+1}{\pi}\int_0^{\pi}\sum_{f=1}^{\infty}(Q^2(x)\varphi_f,\varphi_f)dx$$

$$+\frac{d+1}{\pi}\int_0^{\pi} trQ^2(x)\cos(2e+1)xdx = \frac{d+1}{\pi}\int_0^{\pi} trQ^2(x)dx + \frac{d+1}{\pi}\int_0^{\pi} trQ^2(x)\cos(2e+1)xdx \quad (44)$$

If we substitute “Eq. 48.” in “Eq. 47.”

$$K_{d2} = \frac{d}{2\pi}\int_0^{\pi} trQ^2(x)dx + \frac{d+1}{\pi}\int_0^{\pi} trQ^2(x)\cos(2e+1)xdx + \frac{d+2}{2\pi^2} tr\left(\int_0^{\pi} Q(x)dx\right)^2 + O(d^{-1}) \quad (49)$$

On the other hand, one can show there exists $c > 0$ such that

$$\| QR_{\lambda}^{0} \|_{\sigma_1(H_1)} < c \quad (45)$$

and

$$\| R_{\lambda}^{0} \| < cd^{-1}, \| R_{\lambda} \| < cd^{-1} \text{ for } |\lambda| = b_d = (d+1)^2 . \quad (46)$$

From “Eq.9.”, “Eq.11.”, “Eq.50.” and “Eq.51.”, we have

$$\left|K_{ds}\right| = \frac{1}{\pi i}\left|\int_{|\lambda|=b_d} \lambda tr(QR_\lambda^0)^s d\lambda\right| \le \frac{1}{\pi i}\int_{|\lambda|=b_d} |\lambda|\left|tr(QR_\lambda^0)^s\right| d\lambda \le \frac{b_d}{\pi s}\int_{|\lambda|=b_d} \left\|(QR_\lambda^0)^s\right\|_{\sigma_1(H_1)} d\lambda$$

$$\le \frac{b_d}{\pi s}\int_{|\lambda|=b_d} \| QR_\lambda^0 \|_{\sigma_1(H_1)} \| (QR_\lambda^0)^{s-1} \| d\lambda \le \frac{cb_d}{\pi s}\int_{|\lambda|=b_d} d^{1-s} d\lambda < cs^{-1}d^{5-s} \quad , \tag{47}$$

and

$$\left|K_d^{(N)}\right| = \frac{1}{2\pi}\left|\int_{|\lambda|=b_d} \lambda^2 tr\left[R_\lambda (QR_\lambda^0)^{N+1}\right] d\lambda\right| \le \frac{b_d^2}{2\pi}\int_{|\lambda|=b_d} \left\|R_\lambda (QR_\lambda^0)^{N+1}\right\|_{\sigma_1(H_1)} d\lambda$$

$$\le b_d^2 \int_{|\lambda|=b_d} \|R_\lambda\| \left\|(QR_\lambda^0)^{N+1}\right\|_{\sigma_1(H_1)} d\lambda \le cb_d^2 d^{-1} \int_{|\lambda|=b_d} \left\|\left(QR_\lambda^0\right)^N\right\| . \left\|QR_\lambda^0\right\|_{\sigma_1(H_1)} d\lambda \le cd^{5-N}. \tag{48}$$

From “Eq. 52.” and “Eq. 53.”

$$\lim_{d\to\infty} K_{ds} = 0 \quad \text{for} \quad s \ge 6 \tag{49}$$

and

$$\lim_{d\to\infty} K_d^{(N)} = 0 \quad \text{for} \quad N \ge 6 \tag{55}$$

are obtained.

Theorem 3.1. If $Q(x)$ satisfies the conditions $(Q1)-(Q3)$, then

$$\sum_{e=0}^{\infty}\left\{\sum_{f=1}^{\infty}\left[\lambda_{ef}^2 - \left(e+\frac{1}{2}\right)^4\right] - \frac{(2e+1)^2}{2\pi}\int_0^\pi trQ(x)dx - c\right\} = \frac{1}{8}tr\left[Q''(0) - Q''(\pi) - 2Q^2(0) + 2Q^2(\pi)\right] .$$

Here, $c = \frac{1}{2\pi}\int_0^\pi trQ^2(x)dx + \frac{1}{2\pi^2}tr\left[\int_0^\pi Q(x)dx\right]^2 - \frac{1}{2\pi}\left[trQ'(0) + trQ'(\pi)\right].$

Proof: By “Eq.10.”, “Eq.16.”, and “Eq.49.”, we can write for N=6

$$\sum_{e=0}^{d}\sum_{f=1}^{\infty}\left[\lambda_{ef}^2 - \left(e+\frac{1}{2}\right)^4\right] = \frac{1}{2\pi}\sum_{e=0}^{d}(2e+1)^2\int_0^\pi trQ(x)dx - \frac{1}{2\pi}\sum_{e=0}^{d}\left[trQ'(0) + trQ'(\pi)\right]$$

$$-\frac{1}{2\pi}\sum_{e=0}^{d}\int_0^\pi trQ''(x)\cos(2e+1)xdx + \frac{d}{2\pi}\int_0^\pi trQ^2(x)dx + \frac{d+2}{2\pi^2}tr\left(\int_0^\pi Q(x)dx\right)^2$$

$$+\frac{1}{\pi}\sum_{e=0}^{d}\int_0^\pi trQ^2(x)\cos(2e+1)xdx + O\left(d^{-1}\right) + \sum_{s=3}^{6}K_{ds} + K_d^{(6)} \tag{56}$$

is obtained. From “Eq. 56.”

$$\sum_{e=0}^{d}\left\{\sum_{f=1}^{\infty}\left[\lambda_{ef}^2 - \left(e+\frac{1}{2}\right)^4\right] - \frac{\left(2e+1\right)^2}{2\pi}\int_0^\pi trQ\left(x\right)dx - c\right\}$$

$$=\frac{1}{2\pi}\sum_{e=0}^{d}\int_0^{\pi}\left(2trQ^2(x)-trQ''(x)\right)\cos(2e+1)xdx$$

$$+\sum_{s=3}^{6}K_{ds}+K_d^{(6)}+\frac{1}{2\pi^2}tr\left[\int_0^{\pi}Q(x)dx\right]^2-\frac{1}{2\pi}\int_0^{\pi}trQ^2(x)dx+O(d^{-1}), \quad (57)$$

where $c=\frac{1}{2\pi}\int_0^{\pi}trQ^2(x)dx+\frac{1}{2\pi^2}tr\left[\int_0^{\pi}Q(x)dx\right]^2-\frac{1}{2\pi}\left[trQ'(0)+trQ'(\pi)\right]$.

Moreover, we can show that

$$\lim_{d\to\infty}K_{ds}=0 \qquad (s=3,4,5). \quad (58)$$

By using “Eq.54.”, “Eq.55.”, “Eq.56.” and “Eq.58.”, as $d\to\infty$

$$\sum_{e=0}^{\infty}\left\{\sum_{f=1}^{\infty}\left[\lambda_{ef}^2-\left(e+\frac{1}{2}\right)^4\right]-\frac{(2e+1)^2}{2\pi}\int_0^{\pi}trQ(x)dx-c\right\}$$

$$=\frac{1}{8}tr\left[Q''(\pi)-Q''(0)+2Q^2(0)-2Q^2(\pi)\right]+\frac{1}{2\pi^2}tr\left(\int_0^{\pi}Q(x)dx\right)^2-\frac{1}{2\pi}\int_0^{\pi}trQ^2(x)dx \quad (59)$$

is found. The theorem is proved. The last equality is called ‘’Second Regularized Trace Formula for Self- Adjoint Differential Operator’’.

CONCLUSION

In this work, we consider the self-adjoint operator with bounded operator coefficient in the infinite dimensional Hilbert space. In early studies on this subject, the coefficient of a self-adjoint operator has been considered as a scalar function. However, it is more important to have the operator coefficient for a self-adjoint operator in these type studies.

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