Advances in the Theory of Nonlinear Analysis and its Applications **3** (2019) No. 3, 111–120.
https://doi.org/10.31197/atnaa.604962
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Some variants of contraction principle in the case of operators with Volterra property: step by step contraction principle

Ioan A. Rus[a]

[a] *Department of Mathematics, Babeş-Bolyai University, 1, Kogălniceanu Street, 400084, Cluj-Napoca, Romania.*

Abstract

Following the idea of T.A. Burton, of progressive contractions, presented in some examples (T.A. Burton, *A note on existence and uniqueness for integral equations with sum of two operators: progressive contractions*, Fixed Point Theory, 20 (2019), No. 1, 107-113) and the forward step method (I.A. Rus, *Abstract models of step method which imply the convergence of successive approximations*, Fixed Point Theory, 9 (2008), No. 1, 293-307), in this paper we give some variants of contraction principle in the case of operators with Volterra property. The basic ingredient in the theory of step by step contraction is G-contraction (I.A. Rus, *Cyclic representations and fixed points*, Ann. T. Popoviciu Seminar of Functional Eq. Approxim. Convexity, 3 (2005), 171-178). The relevance of step by step contraction principle is illustrated by applications in the theory of differential and integral equations.

Keywords: Space of continuous function, operator with Volterra property, max-norm, Bielecki norm, contraction, G-contraction, fiber contraction, progressive contraction, step by step contraction, fixed point, Picard operator, weakly Picard operator, differential equation, integral equation, conjecture.
2010 MSC: 47H10, 47H09, 34K05, 34K12, 45D05, 45G10, 54H25.

1. Introduction

Following an idea of T.A. Burton ([7], [8], [9], ...) of progressive contractions, and the forward step method ([21]), in this paper we give some variants of contraction principle in the case of operators with

Email address: iarus@math.ubbcluj.ro (Ioan A. Rus)

Received July 12, 2019, Accepted: August 08, 2019, Online: August 20, 2019.

Volterra property. The basic ingredient in our variant, step by step contraction principle, is G-contraction ([20]). Some applications to differential and integral equations are also given. In connection with our abstract results, a conjecture is formulated.

2. Preliminaries

2.1. G-contractions

Let (X, d) be a metric space and $G \subset X \times X$ be a nonempty subset. An operator $f : X \to X$ is a G-contraction if there exists $l \in]0, 1[$ such that,

$$d(f(x), f(y)) \leq l d(x, y), \forall (x, y) \in G.$$

Here are some examples of subsets $G \subset X \times X$:

(1) $G := G(f)$, the graphic of the operator f. In this case, a G-contraction is a graphic contraction ([17], [24], ...).

(2) Let $A_i \subset X$, $i = \overline{1, p}$, be nonempty closed subsets such that:

(i) $X = \bigcup_{i=1}^{p} A_i$;

(ii) $f(A_i) \subset A_{i+1}$, $i = \overline{1, p}$, $(A_{p+1} = A_1)$.

For, $G := \bigcup_{i=1}^{p} (A_i \times A_{i+1})$, a G-contraction is a cyclic contraction of Kirk-Srinivasan-Veeramani (see the references in [20]).

(3) Let $a, b, c \in \mathbb{R}$, $a < c < b$ and $X := C[a, b]$ with $d(x, y) := \max_{t \in [a,b]} |x(t) - y(t)|$. For $K, H \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$, we consider the operator, $f : C[a, b] \to C[a, b]$, defined by,

$$f(x)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, \ t \in [a, b].$$

We suppose that there exists $L_H > 0$ such that

$$|H(t, s, u) - H(t, s, v)| \leq L_H |u - v|, \ \forall \ t, s \in [a, b], \ \forall \ u, v \in \mathbb{R}.$$

If, $L_H(b - c) < 1$ and if we take

$$G := \{(x, y) \in C[a, b] \times C[a, b] \mid x\big|_{[a,c]} = y\big|_{[a,c]}\},$$

then f is a G-contraction.

For other examples of G-contractions see [20] and [24], pp. 282-284.

2.2. Weakly Picard operators

Let $(X, \to)$ be an L-space $((X, d), \xrightarrow{d}; (X, \tau), \xrightarrow{\tau}; (X, \|\cdot\|), \xrightarrow{\|\cdot\|}, \rightharpoonup; \ldots)$. An operator $f : X \to X$ is weakly Picard operator (WPO) if the sequence, $(f^n(x))_{n \in \mathbb{N}}$, converges for all $x \in X$ and the limit (which generally depend on x) is a fixed point of f.

If an operator f is WPO and the fixed point set of f, $F_f = \{x^*\}$, then by definition f is Picard operator (PO).

For a WPO, $f : X \to X$, we define the operator $f^\infty : X \to X$, by $f^\infty(x) := \lim_{n \to \infty} f^n(x)$.

We remark that, $f^\infty(X) = F_f$, i.e., f^∞ is a set retraction of X on F_f.

For the case of ordered L-spaces, we have some properties of WPO and PO.

Abstract Gronwall Lemma. Let $(X, \to, \leq)$ be an ordered L-space and $f : X \to X$ be an operator. We suppose that:

(1) f is increasing;

(2) f is WPO.

Then:

(i) $x \leq f(x) \Rightarrow x \leq f^\infty(x)$;

(ii) $x \geq f(x) \Rightarrow x \geq f^\infty(x)$.

Abstract Comparison Lemma. Let $(X, \to, \leq)$ be an ordered L-space and $f, g, h : X \to X$ be such that:

(1) $f \leq g \leq h$;

(2) the operators f, g, h are WPO;

(3) the operator g is increasing.

Then:

$$x \leq y \leq z \Rightarrow f^\infty(x) \leq g^\infty(y) \leq h^\infty(z).$$

Regarding the theory of WPO and PO see [18], [19], [22], [23], [26], [17], [24], [2], ...

2.3. Fiber Contraction Principle

In order to present our results, we need the following theorems (see [22], [25], [26], [27], ...).

Fiber Contraction Theorem. Let $(X, \to)$ be an L-space, (Y, ρ) be a metric space, $g : X \to X$, $h : X \times Y \to Y$ and $f : X \times Y \to X \times Y$, $f(x, y) := (g(x), h(x, y))$. We suppose that:

(1) (Y, ρ) is a complete metric space;

(2) g is WPO;

(3) $h(x, \cdot) : Y \to Y$ is l-contraction, $\forall\, x \in X$;

(4) $h : X \times Y \to Y$ is continuous.

Then, f is WPO. Moreover, if g is a PO, then f is a PO.

Generalized Fiber Contraction Theorem. Let $(X, \to)$ be an L-space, (X_i, d_i), $i = \overline{1, m}$, $m \geq 1$ be metric spaces. Let, $f_i : X_0 \times \ldots \times X_i \to X_i$, $i = \overline{0, m}$, be some operators. We suppose that:

(1) (X_i, d_i), $i = \overline{1, m}$, are complete metric spaces;

(2) f_0 is a WPO;

(3) $f_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \to X_i$, $i = \overline{1, m}$, are l_i-contractions;

(4) f_i, $i = \overline{1, m}$, are continuous.

Then, the operator $f : X_0 \times \ldots \times X_m \to X_0 \times \ldots \times X_m$, defined by,

$$f(x_0, \ldots, x_m) := (f_0(x_0), f_1(x_0, x_1), \ldots, f_m(x_0, \ldots, x_m))$$

is a WPO.

If f_0 is a PO, then f is a PO.

3. Operators with Volterra property with respect to a subinterval

Let $(\mathbb{B}, +, \mathbb{R}, |\cdot|)$ be a Banach space, $a, b, c \in \mathbb{R}$, $a < c < b$. In what follows, we consider on $C([a, b], \mathbb{B})$, $C([a, c], \mathbb{B})$ norms of uniform convergence (max-norm, $\|\cdot\|$, Bielecki norm, $\|\cdot\|_\tau$). In, $C([a, b], \mathbb{B}) \times C([a, b], \mathbb{B})$, we consider a subset defined by,

$$G := \{(x, y) \mid x, y \in C([a, b], \mathbb{B}),\ x\big|_{[a,c]} = y\big|_{[a,c]}\},$$

and in, $C([a, b], \mathbb{B})$, for each $x \in C([a, c], \mathbb{B})$ we consider the subset,

$$X_x := \{y \in C([a, b], \mathbb{B}) \mid y\big|_{[a,c]} = x\}.$$

Definition 3.1. *An operator, $V : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$, has the Volterra property with respect to the subinterval, $[a, c]$, if the following implication holds,*

$$x, y \in C([a, b], \mathbb{B}),\ x\big|_{[a,c]} = y\big|_{[a,c]} \Rightarrow V(x)\big|_{[a,c]} = V(y)\big|_{[a,c]}.$$

Definition 3.2. *An operator, $V : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$, has the Volterra property if it has the Volterra property with respect to each subinterval, $[a, t]$, for $a < t < b$.*

For example, let $K, H \in C([a, b] \times [a, b] \times \mathbb{B}, \mathbb{B})$ and $V : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ be defined by,

$$V(x)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds,\ t \in [a, b].$$

This operator has the Volterra property with respect to the subinterval $[a, c]$, but V has not the Volterra property.

If, $V : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$, is an operator with Volterra property with respect to $[a, c]$, then the operator V induces an operator, V_1, on $C([a, c], \mathbb{B})$, defined by

$$V_1(x) := V(\tilde{x})\big|_{[a,c]},\ \text{where } \tilde{x} \in C([a, b], \mathbb{B}) \text{ with, } \tilde{x}\big|_{[a,c]} = x.$$

Remark 3.3. *If V has the Volterra property with respect to $[a, c]$ and V is a G-contraction (see section 2.1.), then the operator*

$$V\big|_{X_x} : X_x \to X_{V_1(x)},$$

is a contraction for all $x \in C([a, c], \mathbb{B})$. If $x^* \in F_{V_1}$, then, $V(X_{x^*}) \subset X_{x^*}$.

The first abstract result of our paper is the following.

Theorem 3.4. *In terms of the above notations, we suppose that:*

(1) *V has the Volterra property with respect to $[a, c]$;*

(2) *V_1 is a contraction;*

(3) *V is a G-contraction.*

Then:

(*i*) *$F_V = \{x^*\}$;*

(*ii*) *$x^*\big|_{[a,c]} = V_1^\infty(x),\ \forall\ x \in C([a, c], \mathbb{B})$;*

(*iii*) *$x^* = V^\infty(x),\ \forall\ x \in X_{x^*|_{[a,c]}}$.*

Proof. From (1) we have that, $F_{V_1} = \{x_1^*\}$, $x_1^* \in C([a,c],\mathbb{B})$. From (3) and Remark 3.3, $V\big|_{X_{x_1^*}} : X_{x_1^*} \to X_{x_1^*}$, is a contraction, i.e., it has a unique fixed point, x^*, and $x^*\big|_{[a,c]} = x_1^*$. From these we have (*i*), (*ii*) and (*iii*). □

Conjecture 3.5. *In the conditions of Theorem 3.4, the operator V is PO, i.e., $x^* = V^\infty(x)$, $\forall\ x \in C([a,b],\mathbb{B})$.*

For a better understanding of Theorem 3.4 and Conjecture 3.5, in what follows, we present some examples.

Example 3.6. *Let a,b,c be as above and $\mathbb{B} := \mathbb{R}$. For $K,H \in C([a,b]\times[a,b]\times\mathbb{R},\mathbb{R})$ we consider the following functional integral equation,*

$$x(t) = \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,\max_{\theta\in[a,s]} x(\theta))ds,\ t \in [a,b]. \tag{3.1}$$

We are looking for the solution of this equation in $C[a,b]$. In addition, we suppose that:

(2′) *there exists $L_K > 0$ such that:*

$$|K(t,s,u) - K(t,s,v)| \le L_K|u-v|,\ \forall\ t \in [a,b],\ \forall\ s \in [a,c],\ \forall\ u,v \in \mathbb{R};$$

(3′) *there exists $L_H > 0$ such that,*

$$|H(t,s,u) - H(t,s,v)| \le L_H|u-v|,\ \forall\ t,s \in [a,b],\ \forall\ u,v \in \mathbb{R}.$$

In this case:

$$V(x)(t) = \text{ the second part of } (3.1);$$
$$V_1(x)(t) = \text{ the second part of } (3.1), \text{ for } t \in [a,c].$$

It is clear that V has the Volterra property with respect to the subinterval $[a,c]$.

We consider on $C[a,c]$ and $C[a,b]$ max-norms and if, $(L_K + L_H)(c-a) < 1$, the operator V_1 is a contraction and if, $L_H(b-c) < 1$, the operator V is a G-contraction.

So, by Theorem 3.4, in the above conditions, equation (3.1) has in $C[a,b]$ a unique solution, x^*. Moreover, for $t \in [a,c]$, $x^*(t) = \lim_{n\to\infty} x_n(t)$, for each $x_0 \in C[a,c]$, where $\{x_n\}_{n\in\mathbb{N}}$ is defined by,

$$x_{n+1}(t) = \int_a^c K(t,s,x_n(s))ds + \int_a^t H(t,s,\max_{\theta\in[a,s]} x_n(\theta))ds,$$

and for $t \in [a,b]$, $x^*(t) = \lim_{n\to\infty} y_n(t)$, where $\{y_n\}_{n\in\mathbb{N}}$, is defined by

$y_0 \in C[a,b]$, with $y_0\big|_{[a,c]} = x^*\big|_{[a,c]}$, and

$$y_{n+1}(t) = \int_a^c K(t,s,x^*(s))ds + \int_a^t H(t,s,\max_{\theta\in[a,s]} y_n(\theta))ds.$$

Remark 3.7. *In the case of operator V, in this example, Conjecture 3.5 is a theorem. Indeed, let $X_0 := C[a,c]$, $X_1 := C[c,b]$ and $C[a,b]$ be endowed with max-norms. We take, $f_0 := V_1$ and $f_1(x,y) : C[a,c] \times C[c,b] \to C[c,b]$ be defined by*

$$f_1(x,y)(t) := \int_a^c K(t,s,x(s))ds + \int_a^c H(t,s,\max_{\theta\in[a,s]} x(\theta))ds +$$
$$+ \int_c^t H(t,s,\max(\max_{\theta\in[a,c]} x(\theta), \max_{\theta\in[c,s]} y(\theta)))ds.$$

We remark that, f_0 is a PO, and $f_1(x,\cdot): C[c,b] \to C[c,b]$ is $L_H(b-c)$-contraction. By Fiber Contraction Theorem, in the conditions, $(L_K + L_H)(c-a) < 1$ and $L_H(b-c) < 1$, the operator f is a Picard operator. Let,

$$x_0 \in C[a,c],\ x_{n+1} = f_0(x_n),\ n \in \mathbb{N},$$

and

$$y_0 \in C[c,b],\ y_{n+1} = f_1(x_n, y_n),\ n \in \mathbb{N}.$$

Then, $x_n \to x^*\big|_{[a,c]}$ as $n \to \infty$, $y_n \to x^*\big|_{[c,b]}$ as $n \to \infty$.

We denote,

$$u_n(t) = \begin{cases} x_n(t), & t \in [a,c], \\ y_n(t), & t \in [c,b]. \end{cases}$$

Then, $u_n \in C[a,b]$, for $n \in \mathbb{N}^*$, and, $u_{n+1} = V(u_n)$ with $u_n \to x^*$ as $n \to \infty$, i.e., V is a PO.

This result is very important because we can apply for V, the Abstract Gronwall Lemma. So we have:

Theorem 3.8. *Let us consider the equation (3.1) in the following conditions: $(L_K + L_H)(c-a) < 1$, $L_H(b-c) < 1$ and $K(t,s,\cdot)$, $H(t,s,\cdot): \mathbb{R} \to \mathbb{R}$ are increasing functions, for all $t,s \in [a,b]$. Let us denote by x^* the unique solution of (3.1). Then the following implications hold:*

(i) $x \in C[a,b]$, $x(t) \le \displaystyle\int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,\max_{\theta\in[a,s]} x(\theta))ds$, $t \in [a,b]$, $\Rightarrow x \le x^*$;

(ii) $x \in C[a,b]$, $x(t) \ge \displaystyle\int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,\max_{\theta\in[a,s]} x(\theta))ds$, $t \in [a,b]$, $\Rightarrow x \ge x^*$.

Also, from the Abstract Comparison Lemma we have a comparison result for equation (3.1).

Remark 3.9. *For the functional integral equations with maxima, see [1], [11], [16], [22], [13], ...*

Example 3.10. *Let $a,b,c \in \mathbb{R}$, $a < b < c$, and $(\mathbb{B}, +, \mathbb{R}, |\cdot|)$ be a Banach space. For $K, H \in C([a,b] \times [a,b] \times \mathbb{B}, \mathbb{B})$ we consider the following integral equation,*

$$x(t) = \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds,\ t \in [a,b]. \tag{3.2}$$

We are looking for solutions of these equations in $C([a,b],\mathbb{B})$. To do this, in addition, we suppose that:

$(2'')$ *there exists $L_K > 0$ such that,*

$$|K(t,s,u) - K(t,s,v)| \le L_K|u-v|,\ \forall\ t \in [a,b],\ \forall\ s \in [a,c],\ \forall\ u,v \in \mathbb{B}\ ;$$

$(3'')$ *there exists $L_H > 0$ such that,*

$$|H(t,s,u) - H(t,s,v)| \le L_H|u-v|,\ \forall\ t,s \in [a,b],\ \forall\ u,v \in \mathbb{B}\ .$$

In the case of equation (3.2) we have:

$$\begin{aligned} V(x)(t) = &\ \text{the second part of (3.2)}; \\ V_1(x)(t) = &\ \text{the second part of (3.2), for } t \in [a,c]. \end{aligned}$$

First, we remark that V has the Volterra property with respect to the subinterval $[a,c]$.

If we consider on $(C[a,c],\mathbb{B})$ and $C[a,b]$ max-norms, then if, $(L_K + L_H)(c-a) < 1$, the operator V_1 is a contraction (i.e., PO) and if, $L_H(b-c) < 1$, the operator V is a G-contraction. By Theorem

3.4, in these conditions, equation (3.2) *has in* $C([a,b],\mathbb{B})$ *a unique solution,* x^*. *Moreover, for* $t\in[a,c]$, $x^*(t)=\lim_{n\to\infty}x_n(t)$, *where* $x_0\in C[a,c]$,

$$x_{n+1}(t)=\int_a^c K(t,s,x_n(s))ds+\int_a^t H(t,s,x_n(s))ds,\ n\in\mathbb{N}$$

and for $t\in[a,b]$, $x^*(t)=\lim_{n\to\infty}y_n(t)$, *where* $y_0\in C([a,b],\mathbb{B})$, *with* $y_0\big|_{[a,c]}=x^*$, *and*

$$y_{n+1}(t)=\int_a^c K(t,s,x^*(s))ds+\int_a^t H(t,s,y_n(s))ds,\ n\in\mathbb{N}.$$

Remark 3.11. *In a similar way, as in the case of Example 3.6, the Conjecture 3.5 is a theorem for the operator* V *in Example 3.10.*

Remark 3.12. *We can work, in the case of Example 3.10 with max-norm on* $C([a,c],\mathbb{B})$ *and with a Bielecki norm on* $C[c,b]$, *i.e., on* $C([a,b],\mathbb{B})$ *with the norm,* $\|x\|=\max\big(\max_{t\in[a,c]}|x(t)|,\max_{t\in[c,b]}e^{-\tau(t-c)}|x(t)|\big)$.

If $\mathbb{B}:=\mathbb{R}^m$, *then we can work with vectorial max-norms and with vectorial Bielecki norms.*

Remark 3.13. *For example of integral operator like* V *in Example 3.10, which appear in differential equations, see: [5], [14], [4], [3] and the references in [3].*

4. Operators with Volterra property

Let, $V: C([a,b],\mathbb{B})\to C([a,b],\mathbb{B})$, be an operator with Volterra property. Let $m\in\mathbb{N}$, $m\geq 2$, $t_0:=a$, $t_1:=t_0+\frac{b-a}{m}$, $\dots$, $t_k:=t_0+\frac{k(b-a)}{m}$, $\dots$, $t_m:=b$. We denote by $V_k: C([t_0,t_k],\mathbb{B})\to C([t_0,t_k],\mathbb{B})$, $k=\overline{1,m-1}$, the operators induced by V on $[t_0,t_k]$ (see the definition of V_1 in section 3). We also consider the following sets,

$$G_k:=\{(x,y)\mid x,y\in C([t_0,t_{k+1}],\mathbb{B}),\ x\big|_{[t_0,t_k]}=y\big|_{[t_0,t_k]}\},\ k=\overline{1,m-1}.$$

For, $x_k\in C([t_0,t_k],\mathbb{B})$, $k=\overline{1,m-1}$, we denote,

$$X_{x_k}:=\{y\in C([t_0,t_{k+1}],\mathbb{B})\mid y\big|_{[t_0,t_k]}=x_k\}.$$

The second basic result of this paper is the following.

Theorem 4.1 (Theorem of step by step contraction)**.** *We suppose that:*

(1) V *has the Volterra property;*

(2) V_1 *is a contraction;*

(3) V_k *is a* G_{k-1}*-contraction, for* $k=\overline{2,m}$.

Then:

(*i*) $F_V=\{x^*\}$;

(*ii*)

$$x^*\big|_{[t_0,t_1]}=V_1^\infty(x),\ \forall\ x\in C([t_0,t_1],\mathbb{B}),$$
$$x^*\big|_{[t_0,t_2]}=V_2^\infty(x),\ \forall\ x\in X_{x^*|_{[t_0,t_1]}},$$
$$\vdots$$
$$x^*\big|_{[t_0,t_{m-1}]}=V_{m-1}^\infty(x),\ \forall\ x\in X_{x^*|_{[t_0,t_{m-2}]}}.$$

(iii) $x^* = V^\infty(x),\ \forall\ x \in X_{x^*|_{[t_0,t_{m-1}]}}$.

Proof. It follows from successive (step by step !) application of Theorem 3.4, for the pairs, (V_{k+1}, V_k), $k = \overline{1, m-1}$, with V_{k+1} as V and V_k as V_1. □

Conjecture 4.2. *In the condition of Theorem 4.1 the operator V is PO, with respect to uniform convergence on $C([a,b],\mathbb{B})$.*

Example 4.3. *For $K \in C([a,b]\times[a,b]\times\mathbb{R})$ we consider the following functional integral equation with maxima,*

$$x(t) = \int_a^t K(t, s, \max_{\theta\in[a,s]} x(\theta))ds,\ \ t \in [a,b] \tag{4.1}$$

By step by step contraction principle we shall prove that, if there exists $L_K > 0$ such that,

$$|K(t,s,u) - K(t,s,v)| \le L_K|u-v|,\ \forall\ t,s \in [a,b],\ \forall\ u,v \in \mathbb{R},$$

then the equation (4.1) has in $C[a,b]$ a unique solution.

Indeed, let $m \in \mathbb{N}^*$ be such that, $\frac{L_K(b-a)}{m} < 1$. Let, $V : C[a,b] \to C[a,b]$ be defined by,

$$V(x)(t) := \text{ the second part of } (4.1).$$

First, we remark that V has the Volterra property. In this case:

$$V_1 : C[t_0,t_1] \to C[t_0,t_1],\ V_1(x)(t) = \int_{t_0}^{t_1} K(t,s,\max_{\theta\in[t_0,s]} x(\theta))ds,\ \ t\in[t_0,t_1].$$

A Lipschitz constant for V_1 is, $\frac{L_K(b-a)}{m}$. So, V_1 is a contraction with respect to max-norm.

In a similar way, V_2 is a G_1-contraction, V_k is a G_{k-1}-contraction and V is G_{m-1}-contraction.

So, we are in the conditions of Theorem 4.1. From this theorem we have that:

The equation (4.1) has in $C[a,b]$ a unique solution, x^*. Moreover,

- *for $t \in [t_0,t_1]$, $x^*(t) = \lim_{n\to\infty} x_n(t)$, where $x_0 \in C[t_0,t_1]$, $x_{n+1}(t) = \int_{t_0}^t K(t,s,\max_{\theta\in[t_0,s]} x_n(\theta))ds$;*
- *for $t \in [t_0,t_2]$, $x^*(t) = \lim_{n\to\infty} x_n(t)$, where $x_0 \in C[t_0,t_2]$ with $x_0|_{[t_0,t_1]} = x^*|_{[t_0,t_1]}$, and $x_{n+1}(t) = \int_{t_0}^t K(t,s,\max_{\theta\in[t_0,s]} x_n(\theta))ds$, $n \in \mathbb{N}$;*

$\vdots$

- *for $t \in [t_0,t_m]$, $x^*(t) = \lim_{n\to\infty} x_n(t)$, where $x_0 \in C[t_0,t_m]$ with $x_0|_{[t_0,t_{m-1}]} = x^*|_{[t_0,t_{m-1}]}$, and $x_{n+1}(t) = \int_{t_0}^t K(t,s,\max_{\theta\in[t_0,s]} x_n(\theta))ds$.*

Remark 4.4. *In a similar way as in the Example 3.6, by Generalized fiber contraction theorem, we have that, for V in Example 4.3, the Conjecture 4.2 is a theorem.*

Example 4.5. *For $f \in C([a,b]\times\mathbb{R})$, we consider the following Cauchy problem*

$$\begin{aligned} x'(t) &= f(t, \max_{\theta\in[a,t]} x(\theta)),\ \ t\in[a,b] \\ x(a) &= 0 \end{aligned} \tag{4.2}$$

This problem with $x \in C^1[a,b]$ is equivalent with the following functional integral equation with maxima, in $C[a,b]$,

$$x(t) = \int_a^t f(s, \max_{\theta\in[a,s]} x(\theta))ds, \tag{4.3}$$

From the result, in Example 4.3, we have that, if there exists $L_f > 0$ *such that,*

$$|f(t,u) - f(t,v)| \leq L_f |u - v|, \ \forall \ t \in [a,b], \ \forall \ u, v \in \mathbb{R},$$

then the equation (4.3) *has in* $C[a,b]$ *a unique solution, i.e., the Cauchy problem* (4.2) *has in* $C^1[a,b]$ *a unique solution.*

Remark 4.6. *For functional differential equations see: [1], [6], [11], [12], [16], [22], ...*

Remark 4.7. *For operators with Volterra property see: [10], [21], [15] and the references therein.*

5. Step by step generalized contraction principles

There is a large class of generalized contraction principle (see, for example, [24], [2], [17]). As an example in what follows, we consider the case of φ-contractions.

Let (X,d) be a metric space, $G \subset X \times X$ a nonempty subset and $f : X \to X$ be an operator.

Definition 5.1. *Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a comparison function. By definition,* f *is a* (G,φ)*-contraction if,*

$$d(f(x), f(y)) \leq \varphi(d(x,y)), \ \forall \ x, y \in G.$$

In the terms of notations in section 4, in a similar way as in the case of Theorem 4.1, we have:

Theorem 5.2 (Theorem of step by step φ-contraction)**.** *We suppose that:*

(1) V *has the Volterra property;*

(2) V_1 *is a* φ*-contraction;*

(3) V_k *is a* (G_{k-1},φ)*-contraction, for* $k = \overline{2,m}$.

Then:

(*i*) $F_V = \{x^*\}$;

(*ii*)

$$\begin{aligned} x^*\big|_{[t_0,t_1]} &= V_1^\infty(x), \ \forall \ x \in C([t_0,t_1],\mathbb{B}), \\ x^*\big|_{[t_0,t_2]} &= V_2^\infty(x), \ \forall \ x \in X_{x^*|_{[t_0,t_1]}}, \\ &\vdots \\ x^*\big|_{[t_0,t_{m-1}]} &= V_{m-1}^\infty(x), \ \forall \ x \in X_{x^*|_{[t_0,t_{m-2}]}}. \end{aligned}$$

(*iii*) $x^* = V^\infty(x)$, $\forall \ x \in X_{x^*|_{[t_0,t_{m-1}]}}$.

Problem 5.3. *For which generalized contractions we have step by step corresponding result ? If such generalized contractions are found, then the problem is to give relevant applications of such result.*

References

[1] D.D. Bainov, S.G. Hristova, Differential Equations with Maxima, CRC Press, 2011.

[2] V. Berinde, Iterative Approximation of Fixed Points, Springer, 2007.

[3] O.-M. Bolojan, Fixed Point Methods for Nonlinear Differential Systems with Nonlocal Conditions, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2013.

[4] A. Boucherif, First-order differential inclusions with nonlocal initial conditions, Appl. Math. Letters, 15 (2002), 409-414.

[5] A. Boucherif, R. Precup, On the nonlocal initial value problem for first order differential equations, Fixed Point Theory, 4 (2003), 205-212.

[6] T.A. Burton, Stability by Fixed Point Theory for Functional Differential Equations, Dover Publ., New York, 2008.

[7] T.A. Burton, A note on existence and uniqueness for integral equations with sum of two operators: progressive contractions, Fixed Point Theory, 20 (2019), No. 1, 107-113.

[8] T.A. Burton, I.K. Purnaras, The shrinking fixed point map, Caputo and integral equations: progressive contraction, J. Fractional Calculus Appl., 9 (2018), No. 1, 188-194.

[9] T.A. Burton, I.K. Purnaras, Progressive contractions, product contractions, quadratic integro-differential equations, AIMS Math., 4 (2019), No. 3, 482-496.

[10] C. Corduneanu, Abstract Volterra equations: a survey, Mathematical and Computer Modeling, 32 (2000), 1503-1528.

[11] A. Halanay, Differential Equations: Stability, Oscillations, Time Lags, Acad. Press, New York, 1966.

[12] V. Kolmanovskii, A. Myshkis, Applied theory of functional-differential equations, Kluwer, 1992.

[13] V. Lakshmikantham, L. Wen, B. Zhang, Theory of Differential Equations with Unbounded Delay, Kluwer, 1994.

[14] O. Nica, R.Precup, On the nonlocal initial value problem for first order differential systems, Studia Univ. Babeş-Bolyai Math., 56 (2011), No. 3, 125-137.

[15] D. Otrocol, Ulam stabilities of differential equation with abstract Volterra operator in a Banach space, Nonlinear Funct. Anal. Appl., 15 (2010), No. 4, 613-619.

[16] D. Otrocol, I.A. Rus, Functional-differential equations with "maxima" via weakly Picard operator theory, Bull. Math. Soc. Sci. Math. Roumanie, 51 (2008), No. 3, 253-261.

[17] I.A. Rus, Generalized Contractions and Applications, Cluj Univ. Press, Cluj-Napoca, 2001.

[18] I.A. Rus, Picard operators and applications, Sc. Math. Jpn., 58 (2003), no. 1, 191-219.

[19] I.A. Rus, Fixed points, upper and lower fixed points: abstract Gronwall lemmas, Carpathian J. Math., 20 (2004), No. 1, 125-134.

[20] I.A. Rus, Cyclic reprezentations and fixed points, Ann. T. Popoviciu Seminar of Functional Eq. Approximation and Convexity, 3 (2005), 171-178.

[21] I.A. Rus, Abstract models of step method which imply the convergence of successive approximations, Fixed Point Theory, 9 (2008), No. 1, 293-307.

[22] I.A. Rus, Some nonlinear functional differential and integral equations, via weakly Picard operator theory: a survey, Carpathian J. Math., 26 (2010), No. 2, 230-258.

[23] I.A. Rus, Some problems in the fixed point theory, Adv. Theory of Nonlinear Analysis Appl., 2 (2018), No. 1, 1-10.

[24] I.A. Rus, A. Petruşel, G. Petruşel, Fixed Point Theory, Cluj Univ. Press, Cluj-Napoca, 2008.

[25] I.A. Rus, M.A. Şerban, Operators on infinite dimensional cartesian product, Analele Univ. de Vest Timişoara, 48 (2010), 253-263.

[26] I.A. Rus, M.A. Şerban, Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem, Carpathian J. Math., 29 (2013), No. 2, 239-258.

[27] M.A. Şerban, Teoria punctului fix pentru operatori definiţi pe produs cartezian, Presa Univ. Clujeană, Cluj-Napoca, 2002.