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Research Article

On λ-biminimal conformal immersions

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Abstract

In the present paper, we consider λ-biminimal conformal immersions. We find the Euler-Lagrange equation of λ-biminimal immersions under conformal change of metrics. We also consider λ-biminimal immersions from a surface (M^2, g) to a Riemannian manifold (N^3, h) under homothetic change of metric and give an example.

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1. Introduction

Harmonic maps have been analyzed widely over the last 50 years as a generalization of essential topics such as geodesics, minimal surfaces and harmonic functions. Moreover, harmonic maps have involved significant applications in mathematics and theoretical physics. Biharmonic maps are natural generalizations of harmonic maps, and they include essential objects like harmonic functions, geodesics, minimal submanifolds, and Riemannian submersions with minimal fibers as special cases [12].

A map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is called a *harmonic map* and *biharmonic map* , if it is a critical point of the *energy functional* and *bienergy functional*

$$E(\varphi) = \frac{1}{2}\int_{\Omega} \|d\varphi\|^2 \, d\nu_g,$$

$$E_2(\varphi) = \frac{1}{2}\int_{\Omega} \|\tau(\varphi)\|^2 \, d\nu_g,$$

for every compact domain Ω of M, respectively. The Euler-Lagrange equation for the harmonic maps is given by

$$\tau(\varphi) = tr(\nabla d\varphi) = 0,$$

where $\tau(\varphi) = tr(\nabla d\varphi)$ is called the *tension field* of the map φ [2]. The Euler-Lagrange equation of $E_2(\varphi)$ can be written as

$$\tau_2(\varphi) = tr(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla})\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi) = 0, \tag{1.1}$$

which is the *bitension field* of φ [6] and R^N is the curvature tensor of N. Clearly, every harmonic map is biharmonic.

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An isometric immersion φ is called λ-*biminimal* ([4], [8]), if it is a critical point of the functional $E_2(\varphi)$ for variations normal to the image $\varphi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that φ is a critical point of the λ-bienergy

$$E_{2,\lambda}(\varphi) = E_2(\varphi) + \lambda E(\varphi), \tag{1.2}$$

for any smooth variation of the map $\varphi_t :]-\varepsilon, +\varepsilon[\,, \varphi_0 = \varphi$ such that $V = \frac{d\varphi_t}{dt} |_{t=0} = 0$ is normal to $\varphi(M)$.

The Euler-Lagrange equation for λ-biminimal immersions is given by

$$[\tau_{2,\lambda}(\varphi)]^{\perp} = [\tau_2(\varphi)]^{\perp} - \lambda[\tau(\varphi)]^{\perp} = 0, \tag{1.3}$$

for some value of $\lambda \in \mathbb{R}$, where $[\cdot]^{\perp}$ denotes the normal component of $[\cdot]$ [8]. In addition, a biharmonic immersion is λ-biminimal such that $\lambda = 0$ (it means that the immersion is biminimal), but the converse is not always true. Moreover, if we take $\lambda = 0$ into equation (1.3), we obtain $[\tau_2(\varphi)]^{\perp} = 0$, which is not the biharmonic equation. λ-biminimal submanifolds are a natural extension of minimal submanifolds from the perspective of variational calculus. In general, λ-biminimal submanifolds are minimal, but the converse is not true.

The fascinating relationship between harmonicity and conformality has a long history. Let $\varphi : M^2 \to \mathbb{R}^3$ be a conformal immersion. Weierstrass showed that φ is harmonic if and only if $\varphi(M)$ is a minimal submanifold in $\mathbb{R}^3$. Harmonic conformal immersions of surfaces are exactly conformal minimal immersions of surfaces. A very nice generalization of the theory on conformal minimal immersions to conformal biharmonic immersions was given in [10]. Therefore, generalizing conformal minimal immersions to λ-biminimal conformal immersions would be an interesting problem for us. Moreover, since the tension field is normal to the submanifold, to study on λ-biminimality condition on the submanifold will be one of the most effective deformations in the normal direction.

In [6], Jiang defined biharmonic maps between Riemannian manifolds. In [10], Ou studied conformal biharmonic immersions. In [11], Ou obtained a classification of biharmonic conformal immersions of complete constant mean curvature surfaces into $\mathbb{R}^3$ and hyperbolic 3-spaces. In [8], Loubeau and Montaldo defined biminimal immersions and investigated biminimal curves under conformal changes of the metric. In [4], Inoguchi studied λ-biminimal curves and surfaces in contact 3-manifolds. For other developments about biminimal curves and biminimal submanifolds see [1, 3, 5, 7, 9, 13, 14]. Motivated by the above studies, in this paper, we consider the notion of a λ-biminimal conformal immersion. We investigate the Euler-Lagrange equation of λ-biminimal immersions under conformal and homothetic changes of the metrics. Finally, we find the necessary and sufficient conditions for conformal immersions to be λ-biminimal. We also give an example of λ-biminimal surfaces under a homothetic change of the metrics.

2. λ-Biminimal immersions under conformal change of metrics

In this section, we consider λ-biminimal immersions under conformal change of metrics.

Let $\varphi : (M^m, g) \to (N^{m+1}, h)$ be an isometric immersion. By B, ξ, A and $\mathbf{H} = H\xi$, we denote the second fundamental form, the unit normal vector field, the shape operator and the mean curvature vector field of φ, respectively, where H is the mean curvature function.

Firstly, we have the following theorem for the Euler-Lagrange equation for λ-biminimal immersions under conformal change of metrics.

Theorem 2.1. *Let $\varphi : (M^m, g) \to (N^{m+1}, h)$ be an isometric immersion. Then under the conformal change of metrics $\overline{g} = F^{-2}g$, the transformation of $[\tau_{2,\lambda}(\varphi, \overline{g})]^{\perp}$ and $[\tau_2(\varphi, g)]^{\perp}$ is given by*

$$[\tau_{2,\lambda}(\varphi, \overline{g})]^{\perp} = F^4 \left\{ \left[[\tau_2(\varphi, g)]^{\perp} - \frac{\lambda}{F^2}[\tau(\varphi, g)]^{\perp} \right] \right.$$

$$-(m-2)traceB(\cdot,\nabla.d\varphi(grad\ln F))-(m-2)trace\nabla^{\perp}_{\cdot}B(\cdot,d\varphi(grad\ln F))$$
$$+2\left(\Delta\ln F-(m-4)\left|grad\ln F\right|^{2}\right)[\tau(\varphi,g)]^{\perp}-(m-6)\left(\nabla^{\varphi}_{grad\ln F}[\tau(\varphi,g]\right)^{\perp}$$
$$+(m-2)(m-6)B\left(grad\ln F,d\varphi(grad\ln F)\right)\},$$

where grad and Δ denote the gradient and the Laplacian with respect to the metric g, respectively.

Proof. Let $\{e_i\}$ with respect to g and $\{\overline{e_i}=Fe_i\}$ with respect to $\overline{g}$ be local geodesic orthonormal frames. Using [10], we can write the normal part of the tension field under the conformal change of a metric as

$$[\tau(\varphi,\overline{g})]^{\perp}=F^{2}[\tau(\varphi,g)]^{\perp}. \tag{2.1}$$

From [10], the bitension field under the conformal change of a metric is known that

$$\tau_2(\varphi,\overline{g})=F^{4}\Big\{[\tau_2(\varphi,g)]+(m-2)J^{\varphi}_{g}(d\varphi(grad\ln F))$$
$$+2\left(\Delta\ln F-(m-4)\left|grad\ln F\right|^{2}\right)[\tau(\varphi,g)]-(m-6)\nabla^{\varphi}_{grad\ln F}[\tau(\varphi,g)]$$
$$-2(m-2)\left(\Delta\ln F-(m-4)\left|grad\ln F\right|^{2}\right)d\varphi(grad\ln F)$$
$$+(m-2)(m-6)\nabla^{\varphi}_{grad\ln F}d\varphi(grad\ln F)\Big\}, \tag{2.2}$$

where $J^{\varphi}_{g}(X)$ is the Jacobi operator of φ. By the use of definition of Jacobi operator, the Gauss equation and the Weingarten formula, we find

$$\left[J^{\varphi}_{g}(d\varphi(grad\ln F))\right]^{\perp}=\Big[-trace_{g}\left(\nabla^{\varphi}\nabla^{\varphi}-\nabla^{\varphi}_{\nabla^{M}}\right)d\varphi(grad\ln F)$$
$$+trace_{g}R^{N}\left(d\varphi,d\varphi(grad\ln F)\right)d\varphi\Big]^{\perp}$$
$$=-\sum_{i=1}^{m}\{\nabla_{e_i}\nabla_{e_i}d\varphi(grad\ln F)+B\left(e_i,\nabla_{e_i}d\varphi(grad\ln F)\right)$$
$$-A_{B\left(e_i,\nabla_{e_i}d\varphi(grad\ln F)\right)}e_i+\nabla^{\perp}_{e_i}B\left(e_i,\nabla_{e_i}d\varphi(grad\ln F)\right)\Big\}$$
$$+\sum_{i=1}^{m}R^{N}\left(d\varphi\left(e_i\right),d\varphi(grad\ln F)\right)d\varphi\left(e_i\right)$$
$$=-traceB(\cdot,\nabla.d\varphi(grad\ln F))-trace\nabla^{\perp}_{\cdot}B(\cdot,d\varphi(grad\ln F)). \tag{2.3}$$

Moreover, using the Gauss equation, we have

$$\left[\nabla^{\varphi}_{grad\ln F}d\varphi(grad\ln F)\right]^{\perp}=B\left(grad\ln F,d\varphi(grad\ln F)\right). \tag{2.4}$$

Substituting the equations (2.3) and (2.4) into (2.2), we obtain the normal part of the bitension field under the conformal change of a metric as

$$[\tau_2(\varphi,\overline{g})]^{\perp}=F^{4}\Big\{[\tau_2(\varphi,g)]^{\perp}-(m-2)traceB(\cdot,\nabla.d\varphi(grad\ln F))$$
$$-(m-2)trace\nabla^{\perp}_{\cdot}B(\cdot,d\varphi(grad\ln F))$$
$$+2\left(\Delta\ln F-(m-4)\left|grad\ln F\right|^{2}\right)[\tau(\varphi,g)]^{\perp}$$
$$-(m-6)\left(\nabla^{\varphi}_{grad\ln F}[\tau(\varphi,g)]\right)^{\perp}$$
$$+(m-2)(m-6)B\left(grad\ln F,d\varphi(grad\ln F)\right)\}. \tag{2.5}$$

By the use of the equations (2.1) and (2.5) into (1.3), we obtain the desired result. □

Substituting $m=2$ into Theorem 2.1, we have the following corollary:

Corollary 2.2. *Let $\varphi : (M^2, g) \to (N^3, h)$ be an isometric immersion. Then under the conformal change of metrics $\overline{g} = F^{-2}g$, the transformation of $[\tau_{2,\lambda}(\varphi, \overline{g})]^{\perp}$ and $[\tau_2(\varphi, g)]^{\perp}$ is given by*

$$[\tau_{2,\lambda}(\varphi, \overline{g})]^{\perp} = F^4 \left\{ \left[[\tau_2(\varphi, g)]^{\perp} - \frac{\lambda}{F^2} [\tau(\varphi, g)]^{\perp} \right] \right.$$
$$\left. + \left(2\Delta \ln F + 4 \left| grad \ln F \right|^2 \right) [\tau(\varphi, g)]^{\perp} + 4 \left(\nabla^{\varphi}_{grad \ln F} [\tau(\varphi, g)] \right)^{\perp} \right\},$$

where grad and Δ denote the gradient and the Laplacian with respect to the metric g, respectively.

3. λ-Biminimal conformal immersions

Now we give the following definition from [11] and [15]:

Definition 3.1. An isometric immersion $\varphi : (M^m, \overline{g}) \to (N^{m+1}, h)$ is said to admit a λ-*biminimal conformal immersion*, if there exists a function $\mu : (M^m, \overline{g}) \to \mathbb{R}^+$ such that the conformal immersion $\varphi : (M^m, \mu^{-2}\overline{g}) \to (N^{m+1}, h)$ with conformal factor μ is a λ-biminimal immersion. In particular, if $\mu =$ constant $\neq 1$, then the conformal immersion is called a *homothetic immersion*.

Let $\varphi : (M^m, \overline{g}) \to (N^{m+1}, h)$ be the associated isometric immersion, where $\overline{g} = \varphi^* h = \mu^2 g$.

Theorem 3.2. *The homothetic immersion $\varphi : (M^m, g) \to (N^{m+1}, h)$ is $\lambda\mu^2$-biminimal such that $\lambda \neq 0$ if and only if $\varphi : (M^m, \mu^2 g) \to (N^{m+1}, h)$ is λ-biminimal.*

Proof. Let $\varphi : (M^m, g) \to (N^{m+1}, h)$ be a homothetic immersion with $\varphi^* h = \mu^2 g$. Let us take $F = \mu^{-1}$ and $lnF = -\ln \mu$. From Theorem 2.1, we can write

$$\begin{aligned}
[\tau_{2,\lambda}(\varphi, \overline{g})]^{\perp} = \mu^{-4} \Big\{ & [\tau_2(\varphi, g)]^{\perp} - \lambda\mu^2 [\tau(\varphi, g)]^{\perp} \\
& + (m-2) trace B(\cdot, \nabla_{\cdot} d\varphi(grad \ln \mu)) \\
& + (m-2) trace \nabla^{\perp}_{\cdot} B(\cdot, d\varphi(grad \ln \mu)) \\
& - 2 \left(\Delta \ln \mu + (m-4) \left| grad \ln \mu \right|^2 \right) [\tau(\varphi, g)]^{\perp} \\
& + (m-6) \left(\nabla^{\varphi}_{grad \ln \mu} [\tau(\varphi, g)] \right)^{\perp} \\
& + (m-2)(m-6) B \left(grad \ln \mu, d\varphi(grad \ln \mu) \right) \Big\}.
\end{aligned} \tag{3.1}$$

Then, assume that the homothetic immersion $\varphi : (M^m, g) \to (N^{m+1}, h)$ is $\lambda\mu^2$-biminimal. So we have

$$[\tau_{2,\lambda}(\varphi, \overline{g})]^{\perp} = 0,$$

which means that $\varphi : (M^m, \overline{g}) \to (N^{m+1}, h)$ is λ-biminimal. This proves the theorem. □

Substituting $m = 2$ into Theorem 3.2, we have the following corollary:

Corollary 3.3. *The homothetic immersion $\varphi : (M^2, g) \to (N^3, h)$ is $\lambda\mu^2$-biminimal such that $\lambda \neq 0$ if and only if $\varphi : (M^2, \mu^2 g) \to (N^3, h)$ is λ-biminimal.*

Theorem 3.4. *The conformal immersion $\varphi : (M^m, g) \to (N^{m+1}, h)$ is λ-biminimal such that $\lambda = 0$ if and only if*

$$\begin{aligned}
[\tau_{2,\lambda}(\varphi, \overline{g})]^{\perp} = \mu^{-4} \{ & (m-2) trace B(\cdot, \nabla_{\cdot} d\varphi(grad \ln \mu)) \\
& + (m-2) trace \nabla^{\perp}_{\cdot} B(\cdot, d\varphi(grad \ln \mu)) \\
& - 2m \left(\Delta \ln \mu + (m-4) \left| grad \ln \mu \right|^2 \right) \left(\mu^2 \mathbf{H} \right)
\end{aligned}$$

$$+2m(m-6)\mu^2\left|grad\ln\mu\right|^2\mathbf{H}+m(m-6)\mu^2\left(\nabla^{\varphi}_{grad\ln\mu}\mathbf{H}\right)^{\perp}$$
$$-(m-2)(m-6)\left(\nabla^{\varphi}_{grad\ln\mu}d\varphi(grad\ln\mu)\right)^{\perp}$$
$$+(m-2)(m-6)B\left(grad\ln\mu,d\varphi(grad\ln\mu)\right)\}.$$

Proof. Assume that the conformal immersion $\varphi:(M^m,g)\to(N^{m+1},h)$ is λ-biminimal such that $\lambda=0$. From the equation (3.1), we have

$$\begin{aligned}\mu^4[\tau_{2,\lambda}(\varphi,\overline{g})]^{\perp}&=(m-2)traceB(\cdot,\nabla.d\varphi(grad\ln\mu))\\&+(m-2)trace\nabla^{\perp}_{\cdot}B(\cdot,d\varphi(grad\ln\mu))\\&-2\left(\Delta\ln\mu+(m-4)\left|grad\ln\mu\right|^2\right)[\tau(\varphi,g)]^{\perp}\\&+(m-6)\left(\nabla^{\varphi}_{grad\ln\mu}[\tau(\varphi,g)]\right)^{\perp}\\&+(m-2)(m-6)B\left(grad\ln\mu,d\varphi(grad\ln\mu)\right).\end{aligned}\tag{3.2}$$

From [10], it is known that the tension field is given by

$$[\tau(\varphi,\overline{g})]=F^2\left\{[\tau(\varphi,g)]-(m-2)d\varphi(grad\ln F)\right\}.$$

Then, using the equations $F=\mu^{-1}$ and $lnF=-\ln\mu$, the tension field and the normal part of the tension field of the conformal immersion $\varphi:(M^m,g)\to(N^{m+1},h)$ are

$$[\tau(\varphi,g)]=\mu^2\left(m\mathbf{H}\right)-(m-2)d\varphi(grad\ln\mu)\tag{3.3}$$

and

$$[\tau(\varphi,g)]^{\perp}=\mu^2\left(m\mathbf{H}\right),\tag{3.4}$$

respectively. Putting the equations (3.3) and (3.4) into the equation (3.2), we obtain

$$\begin{aligned}\mu^4[\tau_{2,\lambda}(\varphi,\overline{g})]^{\perp}&=(m-2)traceB(\cdot,\nabla.d\varphi(grad\ln\mu))\\&+(m-2)trace\nabla^{\perp}_{\cdot}B(\cdot,d\varphi(grad\ln\mu))\\&-2m\left(\Delta\ln\mu+(m-4)\left|grad\ln\mu\right|^2\right)\left(\mu^2\mathbf{H}\right)\\&+(m-6)\left(\nabla^{\varphi}_{grad\ln\mu}[\mu^2\left(m\mathbf{H}\right)-(m-2)d\varphi(grad\ln\mu)]\right)^{\perp}\\&+(m-2)(m-6)B\left(grad\ln\mu,d\varphi(grad\ln\mu)\right).\end{aligned}\tag{3.5}$$

Then, we calculate

$$\begin{aligned}&\left[\nabla^{\varphi}_{grad\ln\mu}[\mu^2\left(m\mathbf{H}\right)-(m-2)d\varphi(grad\ln\mu)]\right]^{\perp}\\&=\left[m\nabla^{\varphi}_{grad\ln\mu}\left(\mu^2\mathbf{H}\right)-(m-2)\nabla^{\varphi}_{grad\ln\mu}d\varphi(grad\ln\mu)\right]^{\perp}\\&=2m\mu^2\left|grad\ln\mu\right|^2\mathbf{H}+m\mu^2\left(\nabla^{\varphi}_{grad\ln\mu}\mathbf{H}\right)^{\perp}\\&-(m-2)\left(\nabla^{\varphi}_{grad\ln\mu}d\varphi(grad\ln\mu)\right)^{\perp}.\end{aligned}\tag{3.6}$$

Substituting equation (3.6) into (3.5), we have

$$\begin{aligned}\mu^4[\tau_{2,\lambda}(\varphi,\overline{g})]^{\perp}&=(m-2)traceB(\cdot,\nabla.d\varphi(grad\ln\mu))\\&+(m-2)trace\nabla^{\perp}_{\cdot}B(\cdot,d\varphi(grad\ln\mu))\\&-2m\left(\Delta\ln\mu+(m-4)\left|grad\ln\mu\right|^2\right)\left(\mu^2\mathbf{H}\right)\\&+2m(m-6)\mu^2\left|grad\ln\mu\right|^2\mathbf{H}+m(m-6)\mu^2\left(\nabla^{\varphi}_{grad\ln\mu}\mathbf{H}\right)^{\perp}\\&-(m-2)(m-6)\left(\nabla^{\varphi}_{grad\ln\mu}d\varphi(grad\ln\mu)\right)^{\perp}\\&+(m-2)(m-6)B\left(grad\ln\mu,d\varphi(grad\ln\mu)\right).\end{aligned}$$

This proves the theorem. □

For $m=2$ in Theorem 3.4, we have the following corollary:

Corollary 3.5. *The conformal immersion $\varphi:(M^2,g)\to(N^3,h)$ is λ-biminimal such that $\lambda=0$ if and only if*

$$[\tau_{2,\lambda}(\varphi,\overline{g})]^{\perp}=-4\mu^{-2}\left\{\left(\Delta\ln\mu+2\left|grad\ln\mu\right|^2\right)\mathbf{H}+2\left(\nabla^{\varphi}_{grad\ln\mu}\mathbf{H}\right)^{\perp}\right\}.$$

Theorem 3.6. *The homothetic immersion $\varphi:(M^2,g)\to(N^3(c),h)$ is $\lambda\mu^2$-biminimal such that $\lambda\neq 0$ if and only if*

$$\Delta H-H\left|B\right|^2+\mu^2\left(2c-\lambda\mu^2\right)H=0,$$

where $\mathbf{H}=H\xi$ and $N^3(c)$ is a three-dimensional space of constant sectional curvature c.

Proof. Let $\varphi:(M^2,\overline{g})\to(N^3,h)$ be the associated isometric immersion, where $\overline{g}=\varphi^*h=\mu^2g$. From [8], $\varphi:(M^m,\overline{g})\to(N^{m+1},h)$ is $\lambda\mu^2$-biminimal if and only if

$$[\tau_{2,\lambda}(\varphi,\overline{g})]^{\perp}=m\left(\Delta_{\overline{g}}H-H\left|B\right|^2_{\overline{g}}+HRic^N(\xi,\xi)-\lambda\mu^2H\right)\xi. \tag{3.7}$$

It is known from [10] that the Laplacian and the second fundamental form under a homothetic change of metrics $\overline{g}=\mu^2g$ in a two dimensional manifold are

$$\Delta_{\overline{g}}u=\mu^{-2}\Delta u,\ \ \left|B\right|^2_{\overline{g}}=\mu^{-2}\left|B\right|^2. \tag{3.8}$$

Moreover, we have

$$Ric^N(\xi,\xi)=2c. \tag{3.9}$$

By the use of equations (3.8) and (3.9) into equation (3.7), we find

$$[\tau_{2,\lambda}(\varphi,\overline{g})]^{\perp}=2\left(\mu^{-2}\Delta H-H\mu^{-2}\left|B\right|^2+2cH-\lambda\mu^2H\right)\xi. \tag{3.10}$$

From Corollary 3.3, the homothetic immersion $\varphi:(M^2,g)\to(N^3(c),h)$ is $\lambda\mu^2$-biminimal such that $\lambda\neq 0$ if and only if

$$[\tau_{2,\lambda}(\varphi,\overline{g})]^{\perp}=0. \tag{3.11}$$

By the use of the equations (3.10) and (3.11), we get the result. □

From Theorem 3.6, we have the following corollaries:

Corollary 3.7. *The homothetic immersion $\varphi:(M^2,g)\to(N^3(c),h)$ with constant mean curvature $H\neq 0$ is $\lambda\mu^2$-biminimal such that $\lambda\neq 0$ if and only if*

$$\left|B\right|^2=\mu^2\left(2c-\lambda\mu^2\right),$$

where $c>\frac{1}{2}\lambda\mu^2$.

Corollary 3.8. *The homothetic immersion $\varphi:(M^2,g)\to(\mathbb{E}^3,h)$ with constant mean curvature $H\neq 0$ is $\lambda\mu^2$-biminimal such that $\lambda\neq 0$ if and only if*

$$\left|B\right|^2=-\lambda\mu^4,$$

where $\lambda<0$.

Proposition 3.9. *The conformal immersion $\varphi:(M^2,g)\to(N^3(c),h)$ is λ-biminimal such that $\lambda=0$ if and only if*

$$\Delta H-H\left|B\right|^2+2\left(c\mu^2+\Delta\ln\mu+2\left|grad\ln\mu\right|^2\right)H+4g(grad\ln\mu,gradH)=0,$$

where $\mathbf{H}=H\xi$ and $N^3(c)$ is a three-dimensional space of constant sectional curvature c.

Proof. Let $\varphi : (M^2, \overline{g}) \to (N^3, h)$ be the associated isometric immersion, where $\overline{g} = \varphi^* h = \mu^2 g$. From [8], $\varphi : (M^m, \overline{g}) \to (N^{m+1}, h)$ is λ-biminimal such that $\lambda = 0$ if and only if

$$[\tau_{2,\lambda}(\varphi, \overline{g})]^\perp = m\left(\Delta_{\overline{g}} H - H\,|B|^2_{\overline{g}} + H Ric^N(\xi,\xi)\right)\xi. \tag{3.12}$$

Using the equations (3.8) and (3.9) into the equation (3.12), we obtain

$$[\tau_{2,\lambda}(\varphi, \overline{g})]^\perp = 2\left(\mu^{-2}\Delta H - H\mu^{-2}\,|B|^2 + (2c)\,H\right)\xi. \tag{3.13}$$

From Corollary 3.5, the conformal immersion $\varphi : (M^2, g) \to (N^3(c), h)$ is λ-biminimal such that $\lambda = 0$ if and only if

$$[\tau_{2,\lambda}(\varphi, \overline{g})]^\perp = -4\mu^{-2}\left\{\left(\Delta \ln \mu + 2\,|grad \ln \mu|^2\right)\mathbf{H} + 2\left(\nabla^{\varphi}_{grad \ln F}\mathbf{H}\right)^\perp\right\}. \tag{3.14}$$

Then, we get

$$\left(\nabla^{\varphi}_{grad \ln F}\mathbf{H}\right)^\perp = \left(\nabla^{\varphi}_{grad \ln \mu}(H\xi)\right)^\perp = g(grad \ln \mu, grad H)\xi. \tag{3.15}$$

Putting the equation (3.15) into (3.14), we have

$$[\tau_{2,\lambda}(\varphi, \overline{g})]^\perp = -4\mu^{-2}\left[\left(\Delta \ln \mu + 2\,|grad \ln \mu|^2\right)H + 2g(grad \ln \mu, grad H)\right]\xi. \tag{3.16}$$

From the equations (3.13) and (3.16), we get the result. □

From Proposition 3.9, we have the following corollary:

Corollary 3.10. *The conformal immersion* $\varphi : (M^2, g) \to (N^3(c), h)$ *with constant mean curvature* $H \neq 0$ *is* λ*-biminimal such that* $\lambda = 0$ *if and only if*

$$|B|^2 = 2\left(c\mu^2 + \Delta \ln \mu + 2\,|grad \ln \mu|^2\right).$$

Example. The immersion

$$\varphi : \left(D \subseteq \mathbb{R}^2, \mu^2 g = r^2 du^2 + d\nu^2\right) \longrightarrow \left(\mathbb{R}^3, d\rho^2 = d\sigma^2 + \sigma^2 du^2 + d\nu^2\right)$$

with $\varphi(u,\nu) = (r,u,\nu)$ for $\mu^2 = \frac{\left(c_2 e^{\mp\frac{\nu}{r}} - c_1 c_2^{-1} e^{\mp\frac{\nu}{r}}\right)}{2}$ is the circular cylinder of radius r into Euclidean space $\mathbb{R}^3$ [10]. It is easy to see that if we take $\mu^2 = \frac{1}{r^2}$, $\lambda = -1$ and $r \neq 1$, then the cylinder is a $\left(\lambda\mu^2 = -\frac{1}{r^2}\right)$-biminimal surface. Hence, Theorem 3.6 is satisfied.

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