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A GENERALIZED STUDY ON CLOSED LIE IDEALS WITH (α,α) −DERIVATIONS

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ABSTRACT

In this paper, we study square closed Lie ideals of semi-prime rings with generalized $(\alpha,\alpha)-$ derivations and investigate commutative properties of square closed Lie ideals under different conditions. Also, we take generalized $(\alpha,\alpha)-$ derivation H with determined $(\alpha,\alpha)-$ derivation h on prime ring and prove that h is $\alpha-$ commuting on Lie ideal. Finally, we reach the corollaries about commutativity of prime rings by using the theorems we prove.

Keywords: Semi-prime ring, Lie ideal, Generalized (α,α) −derivation

1. INTRODUCTION

Let $Z(R)$ be center of ring R. Suppose that $pRs=(0)$ for any $p,s\in R$. If $p=0$ or $s=0$, then R is said to be a prime ring. Similarly, suppose that $sRs=(0)$ for any $s\in R$. If $s=0$, then R is said to be a semi-prime ring. $[p,s]$ notation is used for commutator $ps-sp$ and $p\circ s$ notation is used for anticommutator $ps+sp$ for $p,s\in R$. An additive subgroup $L\subseteq R$ is said to be a Lie ideal of R if $[L,R]\subseteq L$. L is said to be a square closed if $p^2\in L$ for all $p\in L$. Let $\emptyset\neq S\subseteq R$. A map d from R into R that provides $[d(s),s]=0$ for all $s\in S$, is said to be commuting on S. Similarly, for α automorphism of R, a map d from R into R that provides $[d(s),\alpha(s)]=0$ for all $s\in S$, is said to be $\alpha-$ commuting on S.

After a map d that provides $d(ps)=d(p)s+pd(s)$ for any $p,s\in R$ is defined as a derivation, many authors have studied commutative property for prime rings and semi-prime rings with derivation. In [1], Bresar generalized the definition of derivation as the following: D from R into R is said to be generalized derivation with determined derivation d if $D(ps)=D(p)s+pd(s)$ for any $p,s\in R$. According to [2,3], definitions of (α,β) −derivation and generalized $(\alpha,\beta)-$ derivation are given as follows: Let d be an additive map from R into R and α,β are automorphisms of R. If $d(ps)=d(p)\alpha(s)+\beta(p)d(s)$ holds for any $p,s\in R$, then d is said to be (α,β) −derivation. Let D an additive map from R into R. If $D(ps)=D(p)\alpha(s)+\beta(p)d(s)$ holds for any $p,s\in R$, then D is said to be generalized (α,β) −derivation with determined (α,β) −derivation d.

Using these definitions, it is given definitions of (α,α) −derivation and generalized $(\alpha,\alpha)-$ derivations for $\alpha=\beta$ as the following: If $d(ps)=d(p)\alpha(s)+\alpha(p)d(s)$ holds for any $p,s\in R$, then d is said to be (α,α) −derivation. If $D(ps)=D(p)\alpha(s)+\alpha(p)d(s)$ holds for any $p,s\in R$, then D is said to be generalized (α,α) −derivation with determined (α,α) −derivation d.

Of late years, several researchers have proved commutativity theorems and lemmas for prime rings and semi-prime rings with derivation, generalized derivation, (α,α) −derivation and generalized $(\alpha,\alpha)-$ derivation. Also, many researchers have generalized previous results to ideals and Lie ideals of ring. In [4], Söğütçü and Gölbaşı proved commutativity theorems for square closed Lie ideals of prime rings and semi-prime rings with generalized derivation. In this paper, we generalize the results for generalized derivation to generalized $(\alpha,\alpha)-$ derivation.

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In this study, we generalize the previous study on Lie ideals of semi-prime rings with generalized derivation to generalized $(\alpha,\alpha)-$ derivation. Let R be a semi-prime ring, $0\neq L$ be a square closed Lie ideal of R and $0\neq D,H:R\to R$ are generalized $(\alpha,\alpha)-$ derivations with determined $(\alpha,\alpha)-$ derivations $0\neq d,h:R\to R$ respectively such that $\alpha(L)\subseteq L$ and $h(L)\subseteq L$. We investigate following conditions and prove that h is $\alpha-$commuting map on L. (i) $D(p)\alpha(p)=\alpha(p)H(p)$ for all $p\in L$. (ii) $[D(p),\alpha(s)]=[\alpha(p),H(s)]$ for all $p,s\in L$. (iii) $D(p)o\alpha(s)=\alpha(p)oH(s)$for all $p,s\in L$. (iv) $[D(p),\alpha(s)]=\alpha(p)oH(s)$ for all $p\in L$.

Also, we study above conditions for square closed Lie ideal L of prime ring R and prove that $L\subseteq Z(R)$. Finally, we adapt the theorems which we prove for two derivations to only one derivation and we reach corollaries.

2. PRELIMINARIES

Following identities is provided for commutator and anticommutator for all $p_1,p_2,p_3\in R$.

- $[p_1p_2,p_3]=p_1[p_2,p_3]+[p_1,p_3]p_2$
- $[p_1,p_2p_3]=[p_1,p_2]p_3+p_2[p_1,p_3]$
- $(p_1p_2)\circ p_3=p_1(p_2\circ p_3)-[p_1,p_3]p_2=(p_1\circ p_3)p_2+p_1[p_2,p_3]$
- $p_1\circ(p_2p_3)=(p_1\circ p_2)p_3-p_2[p_1,p_3]=p_2(p_1\circ p_3)+[p_1,p_2]p_3$

Remark *Let R be a prime ring with $charR\neq 2$ and L be a square closed Lie ideal of R. Then, $2ps\in L$ for all $p,s\in L$. Since $charR\neq 2$, if $2ps=0$ for all $p,s\in L$, then $ps=0$. Hence, it is taken $ps\in L$ instead of $2ps\in L$ in relations equal to zero.*

Lemma 2.1 [5] *Let R be a prime ring with $charR\neq 2$, $a,b\in R$. If U a noncentral Lie ideal of R and $aUb=0$, then $a=0$ or $b=0$.*

Lemma 2.2 [5] *Let R be a prime ring with $charR\neq 2$ and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that d(U)=0, then $U\subseteq Z$.*

Lemma 2.3 [6] *Let R be a $2-$torsion free semiprime ring, U a noncentral Lie ideal of R and $a,b\in U$. If $aUa=0$, then $a=0$.*

Lemma 2.4 [7] *Let R be a $2-$torsion free semiprime ring and L be a nonzero Lie ideal of R. If L is a commutative Lie ideal of R, i. e., $[x,y]=0$ for all $x,y\in L$, then $L\subseteq Z(R)$.*

3. RESULTS

3.1. Generalization on Lie Ideals of Prime Rings

Throughout this section, we take R is a prime ring with $charR\neq 2$, L is a square closed Lie ideal of R, α is an automorphism of R and $0\neq D,H:R\to R$ are generalized $(\alpha,\alpha)-$ derivations determined with $(\alpha,\alpha)-$ derivations $0\neq d,h:R\to R$ respectively such that $\alpha(L)\subseteq L$ and $h(L)\subseteq L$.

We begin with two lemmas to be used in the theorems.

Lemma 3.1.1. *If* $[\alpha(p_1),\alpha(p_2)]\alpha(p_3)h(p_2)=0$ *for all $p_1,p_2,p_3\in L$, then $L\subseteq Z(R)$.*

Proof: Let $[\alpha(p_1),\alpha(p_2)]\alpha(p_3)h(p_2)=0$ for all $p_1,p_2,p_3\in L$. Using the fact that α is automorphism, we have $\alpha[p_1,p_2]\alpha(p_3)h(p_2)=0$ for all $p_1,p_2,p_3\in L$. Also, this relation is equal to following relation:

$$[p_1,p_2]p_3\alpha^{-1}\left(h(p_2)\right)=0 \text{ for all } p_1,p_2,p_3\in L.$$

Suppose that, $L\nsubseteq Z(R)$. From Lemma 2.1, we get

$$[p_1,p_2]=0 \ or\ \alpha^{-1}\left(h(p_2)\right)=0 \text{ for all } p_1,p_2\in L.$$

Since α is automorphism, this relation is equal to following relation:

$$[p_1,p_2]=0 \ or\ h(p_2)=0 \text{ for all } p_1,p_2\in L.$$

Let $C=\{p_2\in L|[p_1,p_2]=0 \text{ for all } p_1\in L\}$ and $E=\{p_2\in L|h(p_2)=0\}$. C and E are subgroups of additive group L whose $L=C\cup E$, but L can't be written as a union of its two proper subgroups. So, $L=C$ or $L=E$. If $L=C$, then $[p_1,p_2]=0$ for all $p_1,p_2\in L$. From Lemma 2.4, we arrive that $L\subseteq Z(R)$. But this result contradicts with $L\nsubseteq Z(R)$. If $L=E$, then $h(p_2)=0$ for all $p_2\in L$. That means, $h(L)=0$. From Lemma 2.2, we arrive that $L\subseteq Z(R)$. But this result contradicts with $L\nsubseteq Z(R)$. Hence, assumption is incorrect and $L\subseteq Z(R)$.

Lemma 3.1.2. *If* $[\alpha(p_1),h(p_1)]\alpha(p_2)h(p_1)=0$ *for all* $p_1,p_2\in L$*, then* h *is* $\alpha-$*commuting on* L *or* $L\subseteq Z(R)$.

Proof: Let

$$[\alpha(p_1),h(p_1)]\alpha(p_2)h(p_1)=0 \text{ for all } p_1,p_2\in L.$$

Since α is automorphism, this relation is equal to following relation:

$$\left[p_1,\alpha^{-1}\left(h(p_1)\right)\right]p_2\alpha^{-1}\left(h(p_1)\right)=0 \text{ for all } p_1,p_2\in L.$$

Suppose that, $L\nsubseteq Z(R)$. From Lemma 2.1, we get

$$\left[p_1,\alpha^{-1}\left(h(p_1)\right)\right]=0 \ or\ \alpha^{-1}\left(h(p_1)\right)=0 \text{ for all } p_1\in L.$$

Since α is automorphism, this relation is equal to following relation:

$$[\alpha(p_1),h(p_1)]=0 \ or\ h(p_1)=0 \text{ for all } p_1\in L.$$

Let $C=\{p_1\in L|[\alpha(p_1),h(p_1)]=0\}$ and $E=\{p_1\in L|h(p_1)=0\}$. C and E are subgroups of additive group L whose $L=C\cup E$, but L can't be written as a union of its two proper subgroups. Hence, $L=C$ or $L=E$. If $L=C$, then $[\alpha(p_1),h(p_1)]=0$ for all $p_1\in L$. So, h is $\alpha-$commuting map on L and proof is complete. If $L=E$, then $h(p_1)=0$ for all $p_1\in L$. That means, $h(L)=0$. From Lemma 2.2, we arrive that $L\subseteq Z(R)$. But this result contradicts with $L\nsubseteq Z(R)$. Hence, h *is* $\alpha-$commuting on L or $L\subseteq Z(R)$.

In the following theorems, we give the results about inclusion of a Lie ideal in center of a prime ring with generalized $(\alpha,\alpha)-$ derivation by using the previous lemmas.

Theorem 3.1.3 *If* $D(p_1)\alpha(p_1)=\alpha(p_1)H(p_1)$ *for all* $p_1\in L$*, then* $L\subseteq Z(R)$.

Proof. Let $D(p_1)\alpha(p_1)=\alpha(p_1)H(p_1)$ for all $p_1\in L$. Replacing p_1 by p_1+p_2, $p_2\in L$ we have

$$D(p_1)\alpha(p_2) + D(p_2)\alpha(p_1) = \alpha(p_1)H(p_2) + \alpha(p_2)H(p_1) \text{ for all } p_1, p_2 \in L. \quad (1)$$

Replacing p_1 by p_1p_2 and using above relation, we get

$$\begin{aligned} 0 &= D(p_1)\alpha(p_2)\alpha(p_2) + \alpha(p_1)d(p_2)\alpha(p_2) + D(p_2)\alpha(p_1)\alpha(p_2) - \alpha(p_1)\alpha(p_2)H(p_2) \\ &\quad -\alpha(p_2)H(p_1)\alpha(p_2) - \alpha(p_2)\alpha(p_1)h(p_2) \\ &= (D(p_1)\alpha(p_2) + D(p_2)\alpha(p_1) - \alpha(p_2)H(p_1))\alpha(p_2) - \alpha(p_1)\alpha(p_2)H(p_2) \\ &\quad +\alpha(p_1)d(p_2)\alpha(p_2) - \alpha(p_2)\alpha(p_1)h(p_2). \end{aligned}$$

Using equation (1) in above relation, we obtain

$$\begin{aligned} 0 = \alpha(p_1)H(p_2)\alpha(p_2) - \alpha(p_1)\alpha(p_2)H(p_2) + \alpha(p_1)d(p_2)\alpha(p_2) - \\ \alpha(p_2)\alpha(p_1)h(s) \text{ for all } p_1, p_2 \in L. \end{aligned} \quad (2)$$

Replacing p_1 by p_1p_3, $p_3 \in L$ in above relation, we get

$$\begin{aligned} 0 &= \alpha(p_1)\alpha(p_3)H(p_2)\alpha(p_2) - \alpha(p_1)\alpha(p_3)\alpha(p_2)H(p_2) + \alpha(p_1)\alpha(p_3)d(p_2)\alpha(p_2) \\ &\quad - \alpha(p_2)\alpha(p_1)\alpha(p_3)h(p_2) \\ &= \alpha(p_1)\left(\alpha(p_3)H(p_2)\alpha(p_2) - \alpha(p_3)\alpha(p_2)H(p_2) + \alpha(p_3)d(p_2)\alpha(p_2)\right) \\ &\quad - \alpha(p_2)\alpha(p_1)\alpha(p_3)h(p_2). \end{aligned}$$

Using equation (2) in above relation, we have

$$0 = \alpha(p_1)\alpha(p_2)\alpha(p_3)h(p_2) - \alpha(p_2)\alpha(p_1)\alpha(p_3)h(p_2) \text{ for all } p_1, p_2, p_3 \in L.$$

Using commutator properties in this relation, we obtain

$$0 = [\alpha(p_1), \alpha(p_2)]\alpha(p_3)h(p_2) \text{ for all } p_1, p_2, p_3 \in L.$$

From Lemma 3.1.1, $L \subseteq Z(R)$.

Theorem 3.1.4 *If* $[D(p_1), \alpha(p_2)] = [\alpha(p_1), H(p_2)]$ *for all* $p_1, p_2 \in L$, *then* $L \subseteq Z(R)$.

Proof: Let $[D(p_1), \alpha(p_2)] = [\alpha(p_1), H(p_2)]$ for all $p_1, p_2 \in L$. Replacing p_2 by p_2p_1, we get

$$[D(p_1), \alpha(p_2)\alpha(p_1)] = [\alpha(p_1), H(p_2)\alpha(p_1) + \alpha(p_2)h(p_1)] \text{ for all } p_1, p_2 \in L. \quad (3)$$

Editing equation (3), we obtain

$$\begin{aligned} &[D(p_1), \alpha(p_2)]\alpha(p_1) + \alpha(p_2)[D(p_1), \alpha(p_1)] \\ &= [\alpha(p_1), H(p_2)]\alpha(p_1) + [\alpha(p_1)\alpha(p_2)]h(p_1) + \alpha(p_2)[\alpha(p_1), h(p_1)]. \end{aligned}$$

Using hypothesis in this relation, we get

$$\alpha(p_2)[D(p_1),\alpha(p_1)] = [\alpha(p_1)\alpha(p_2)]h(p_1) + \alpha(p_2)[\alpha(p_1),h(p_1)] \text{for all } p_1,p_2 \in L. \qquad (4)$$

Replacing p_2 by p_2p_3, $p_3 \in L$ in above relation, we have

$$\alpha(p_2)\alpha(p_3)[D(p_1),\alpha(p_1)] = [\alpha(p_1),\alpha(p_2)]\alpha(p_3)h(p_1) + \alpha(p_2)[\alpha(p_1),\alpha(p_3)]h(p_1)$$
$$+\alpha(p_2)\alpha(p_3)[\alpha(p_1),h(p_1)].$$

Using equation (4) in above relation, we have

$$0 = [\alpha(p_1),\alpha(p_2)]\alpha(p_3)h(p_1) \text{ for all } p_1,p_2,p_3 \in L.$$

From Lemma 3.1.1, $L \subseteq Z(R)$.

Theorem 3.1.5 *If $D(p_1)o\alpha(p_2) = \alpha(p_1)oH(p_2)$ for all $p_1,p_2 \in L$, then h is α −commuting on L or* $L \subseteq Z(R)$.

Proof: Let $D(p_1)o\alpha(p_2) = \alpha(p_1)oH(p_2)$ for all $p_1,p_2 \in L$. Replacing p_2 by p_2p_1, we have

$$D(p_1)o(\alpha(p_2)\alpha(p_1)) = \alpha(p_1)o(H(p_2)\alpha(p_1) + \alpha(p_2)h(p_1)) \text{ for all } p_1,p_2 \in L. \qquad (5)$$

Using anti-commutator properties and editing equation (5), we obtain

$$(D(p_1)o\alpha(p_2))\alpha(p_1) - \alpha(p_2)[D(p_1),\alpha(p_1)]$$
$$= \left(\alpha(p_1)oH(p_2)\right)\alpha(p_1) + \left(\alpha(p_1)o\alpha(p_2)\right)h(p_1) - \alpha(p_2)[\alpha(p_1),h(p_2)].$$

Using hypothesis in this relation, we have

$$0 = \left(\alpha(p_1)o\alpha(p_2)\right)h(p_1) + \alpha(p_2)[D(p_1),\alpha(p_1)] - \alpha(p_2)[\alpha(p_1),h(p_1)] \text{for all } p_1,p_2 \in L. \qquad (6)$$

Replacing p_2 by $\alpha^{-1}(h(p_1))s$ in above relation, we get

$$0 = (\alpha(p_1)o(h(p_1)\alpha(p_2)))h(p_1) + h(p_1)\alpha(p_2)[D(p_1),\alpha(p_1)] - h(p_1)\alpha(p_2)[\alpha(p_1),h(p_1)]$$
$$= h(p_1)\left(\alpha(p_1)o\alpha(p_2)\right)h(p_1) + [\alpha(p_1),h(p_1)]\alpha(p_2)h(p_1)$$
$$+ h(p_1)\alpha(p_2)[D(p_1),\alpha(p_1)]$$
$$-h(p_1)\alpha(p_2)[\alpha(p_1),h(p_1)]$$
$$= h(p_1)\left(\left(\alpha(p_1)o\alpha(p_2)\right)h(p_1) + \alpha(p_2)[D(p_1),\alpha(p_1)] - \alpha(p_2)[\alpha(p_1),h(p_1)]\right)$$
$$-[\alpha(p_1),h(p_1)]\alpha(p_2)h(p_1).$$

Using equation (6) in above relation, we have

$$0 = [\alpha(p_1),h(p_1)]\alpha(p_2)h(p_1) \text{ for all } p_1,p_2 \in L.$$

From Lemma 3.1.2, h is α −commuting on L or $L \subseteq Z(R)$.

Theorem 3.1.6 *If $[D(p_1),\alpha(p_2)] = \alpha(p_1)oH(p_2)$ for all $p_1, p_2 \in L$, then h is $\alpha -$commuting on L or $L \subseteq Z(R)$.*

Proof: Let $[D(p_1),\alpha(p_2)] = \alpha(p_1)oH(p_2)$ for all $p_1, p_2 \in L$. Replacing p_2 by $p_2 p_1$, we obtain

$$[D(p_1),\alpha(p_2)\alpha(p_1)] = \alpha(p_1)o\left(H(p_2)\alpha(p_1)\right) + \alpha(p_1)o\left(\alpha(p_2)h(p_1)\right). \qquad (7)$$

for all $p_1, p_2 \in L$. Using commutator and anti-commutator properties and editing equation (7), we get

$$\begin{aligned}&[D(p_1),\alpha(p_2)]\alpha(p_1) + \alpha(p_2)[D(p_1),\alpha(p_1)]\\ &= \left(\alpha(p_1)oH(p_2)\right)\alpha(p_1) + \alpha(p_2)\left(\alpha(p_1)oh(p_1)\right) + [\alpha(p_1),\alpha(p_2)]h(p_1).\end{aligned}$$

Using hypothesis in this relation, we have

$$0 = \alpha(p_2)[D(p_1),\alpha(p_1)] - \alpha(p_2)\left(\alpha(p_1)oh(p_1)\right) - [\alpha(p_1),\alpha(p_2)]h(p_1) \text{for all } p_1, p_2 \in L. \qquad (8)$$

Replacing p_2 by $\alpha^{-1}(h(p_1))s$ in above relation, we get

$$\begin{aligned}0 &= h(p_1)\alpha(p_2)[D(p_1),\alpha(p_1)] - h(p_1)\alpha(p_2)\left(\alpha(p_1)oh(p_1)\right) - [\alpha(p_1),h(p_1)\alpha(p_2)]h(p_1)\\ &= h(p_1)\alpha(p_2)[D(p_1),\alpha(p_1)] - h(p_1)\alpha(p_2)\left(\alpha(p_1)oh(p_1)\right)\\ &\quad - h(p_1)[\alpha(p_1),\alpha(p_2)]h(p_1)\\ &\quad -[\alpha(p_1),h(p_1)]\alpha(p_2)h(p_1)\\ &= h(p_1)\left(\alpha(p_2)[D(p_1),\alpha(p_1)] - \alpha(p_2)\left(\alpha(p_1)oh(p_1)\right) - [\alpha(p_1),\alpha(p_2)]h(p_1)\right)\\ &\quad -[\alpha(p_1),h(p_1)]\alpha(p_2)h(p_1).\end{aligned}$$

Using equation (8) in above relation, we have

$$0 = [\alpha(p_1),h(p_1)]\alpha(p_2)h(p_1) \text{ for all } p_1, p_2 \in L.$$

From Lemma 3.1.2, h is $\alpha -$commuting on L or $L \subseteq Z(R)$.

Corollary 3.1.7 *Let R be a prime ring with $charR \neq 2$, L a square closed Lie ideal of R, α an automorphism of R and $0 \neq H: R \to R$ a generalized $(\alpha,\alpha)-$ derivation determined with $(\alpha,\alpha)-$ derivation $0 \neq h: R \to R$ such that $\alpha(L) \subseteq L$ and $h(L) \subseteq L$. Then, following properties are provided.*

(***i***) *If $[H(p_1),\alpha(p_1)] = 0$ for all $p_1 \in L$, then $L \subseteq Z(R)$.*

(***ii***) *If $[H(p_1),\alpha(p_2)] = [\alpha(p_1),H(p_2)]$ for all $p_1, p_2 \in L$, then $L \subseteq Z(R)$.*

(***iii***) *If $H(p_1)o\alpha(p_2) = \alpha(p_1)oH(p_2)$ for all $p_1, p_2 \in L$, then h is $\alpha -$commuting on L or $L \subseteq Z(R)$.*

(***iv***) *If $[H(p_1),\alpha(p_2)] = \alpha(p_1)oH(p_2)$ for all $p_1, p_2 \in L$, then h is $\alpha -$commuting on L or $L \subseteq Z(R)$.*

3.2. Generalization on Lie Ideals of Semi-Prime Rings

Throughout this section, we take R is a semi-prime ring with $charR \neq 2$, L is a noncentral square closed Lie ideal of R, α is an automorphism of R and $0 \neq D, H: R \to R$ are generalized $(\alpha, \alpha) -$ derivations determined with $(\alpha, \alpha) -$ derivations $0 \neq d, h: R \to R$ respectively such that $\alpha(L) \subseteq L$ and $h(L) \subseteq L$.

In this section, we generalize the previous study on Lie ideals of semi-prime rings with generalized derivation to generalized $(\alpha, \alpha) -$ derivation.

Theorem 3.2.1 *If (i) or (ii) is provided for all* $p_1, p_2 \in L$, *then* h *is* $\alpha -$*commuting on* L.

$(\boldsymbol{i}) D(p_1)\alpha(p_1) = \alpha(p_1)H(p_1)$

$(\boldsymbol{ii}) [D(p_1), \alpha(p_2)] = [\alpha(p_1), H(p_2)]$

Proof. (i) Let $D(p_1)\alpha(p_1) = \alpha(p_1)H(p_1)$ for all $p_1 \in L$. Using same proof methods in Theorem 3.1.3, we get

$$[\alpha(p_1), \alpha(p_2)]\alpha(p_3)h(p_2) = 0 \text{ for all } p_1, p_2, p_3 \in L.$$

In this relation, using the fact that α is automorphism, we have $\alpha[p_1, p_2]\alpha(p_3)h(p_2) = 0$ for all $p_1, p_2, p_3 \in L$. Also, this relation is equal to following relation:

$$[p_1, p_2]p_3\alpha^{-1}\left(h(p_2)\right) = 0 \text{ for all } p_1, p_2, p_3 \in L.$$

Replacing p by $\alpha^{-1}(h(p_2))$ in above relation, we get

$$\left[\alpha^{-1}\left(h(p_2)\right), p_2\right] p_3\alpha^{-1}\left(h(p_2)\right) = 0 \text{ for all } p_2, p_3 \in L. \tag{9}$$

Right multiplication of equation (9) by p_2, we have

$$\left[\alpha^{-1}\left(h(p_2)\right), p_2\right] p_3\alpha^{-1}\left(h(p_2)\right) p_2 = 0 \text{ for all } p_2, p_3 \in L. \tag{10}$$

On the other hand, replacing p_3 by $p_3 p_2$ in equation (9), we obtain

$$\left[\alpha^{-1}\left(h(p_2)\right), p_2\right] p_3 p_2\alpha^{-1}\left(h(p_2)\right) = 0 \text{ for all } p_2, p_3 \in L. \tag{11}$$

Using equation (10) and equation (11), we get

$$\left[\alpha^{-1}\left(h(p_2)\right), p_2\right] p_3 \left[\alpha^{-1}\left(h(p_2)\right), p_2\right] = 0 \text{ for all } p_2, p_3 \in L.$$

From Lemma 2.3 we have

$$\left[\alpha^{-1}\left(h(p_2)\right), p_2\right] = 0 \text{ for all } p_2 \in L.$$

Using the fact that α is automorphism, we arrive that

$$\left[h(p_2), \alpha(p_2)\right] = 0 \text{ for all } p_2 \in L.$$

So, h is $\alpha -$commuting on L.

(ii) Let $[D(p_1), \alpha(p_2)] = [\alpha(p_1), H(p_2)]$ for all $p_1, p_2 \in L$. Using same proof methods in Theorem 3.1.4, we get

$$[\alpha(p_1), \alpha(p_2)]\alpha(p_3)h(p_2) = 0 \text{ for all } p_1, p_2, p_3 \in L.$$

Applying same methods in option *(i)*, we arrive that, h is α −commuting on L.

Theorem 3.2.2 *If (i) or (ii) is provided for all $p_1, p_2 \in L$, then h is α −commuting on L.*

$(\boldsymbol{i}) D(p_1) o \alpha(p_2) = \alpha(p_1) o H(p_2)$

$(\boldsymbol{ii}) [D(p_1), \alpha(p_2)] = \alpha(p_1) o H(p_2)$

Proof. (i) Let $D(p_1) o \alpha(p_2) = \alpha(p_1) o H(p_2)$ for all $p_1, p_2 \in L$. Using same proof methods in Theorem 3.1.5, we get

$$[\alpha(p_1), h(p_1)]\alpha(p_2)h(p_1) = 0 \text{ for all } p_1, p_2 \in L. \quad (12)$$

Replacing p_2 by $p_2 p_1$ in above relation, we get

$$[\alpha(p_1), h(p_1)]\alpha(p_2)\alpha(p_1)h(p_1) = 0 \text{ for all } p_1, p_2 \in L. \quad (13)$$

On the other hand, right multiplication of equation (12) by $\alpha(p_1)$, we have

$$[\alpha(p_1), h(p_1)]\alpha(p_2)h(p_1)\alpha(p_1) = 0 \text{ for all } p_1, p_2 \in L. \quad (14)$$

Using equation (13) and equation (14), we get

$$[\alpha(p_1), h(p_1)]\alpha(p_2)[\alpha(p_1), h(p_1)] = 0 \text{ for all } p_1, p_2 \in L.$$

Using the fact that α is automorphism and Lemma 2.3, we obtain

$$[\alpha(p_1), h(p_1)] = 0 \text{ for all } p_1 \in L.$$

So, h is α −commuting map on L.

(ii) Let $[D(p_1), \alpha(p_2)] = \alpha(p_1) o H(p_2)$ for all $p_1, p_2 \in L$. Using same proof methods in Theorem 3.1.6, we get

$$[\alpha(p_1), h(p_1)]\alpha(p_2)h(p_1) = 0 \text{ for all } p_1, p_2 \in L.$$

Applying same methods in option *(i)*, we arrive that, h is α −commuting on L.

Corollary 3.2.3 *Let R be a semi-prime ring with $charR \neq 2$, L a noncentral square closed Lie ideal of R, α an automorphism of R and $0 \neq H: R \to R$ a generalized $(\alpha, \alpha)-$ derivation determined with $(\alpha, \alpha)-$ derivation $0 \neq h: R \to R$ such that $\alpha(L) \subseteq L$ and $h(L) \subseteq L$. Then, following properties are provided.*

$(\boldsymbol{i})$ *If $[H(p_1), \alpha(p_1)] = 0$ for all $p_1 \in L$, then h is α −commuting map on L.*

$(\boldsymbol{ii})$ *If $[H(p_1), \alpha(p_2)] = [\alpha(p_1), H(p_2)]$ for all $p_1, p_2 \in L$, then then h is α −commuting on L.*

$(\boldsymbol{iii})$ *If $H(p_1) o \alpha(p_2) = \alpha(p_1) o H(p_2)$ for all $p_1, p_2 \in L$, then h is α −commuting on L.*

$(\boldsymbol{iv})$ *If $[H(p_1), \alpha(p_2)] = \alpha(p_1) o H(p_2)$ for all $p_1, p_2 \in L$, then h is α −commuting on L.*

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