Reliable Iterative Methods for Solving Convective Straight and Radial Fins with Temperature-Dependent Thermal Conductivity Problems

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Highlights
• Three iterative methods are used to solve the straight and radial fins with thermal conductivity.
• The numerical solutions were further achieved by using fourth Runge-Kutta and Euler methods.
• The convergence was discussed and proved. The maximum error remainder values are also evaluated.

Abstract

In our article, three iterative methods are performed to solve the nonlinear differential equations that represent the straight and radial fins affected by thermal conductivity. The iterative methods are the Daftardar-Jafari method namely (DJM), Temimi-Ansari method namely (TAM) and Banach contraction method namely (BCM) to get the approximate solutions. For comparison purposes, the numerical solutions were further achieved by using the fourth Runge-Kutta (RK4) method, Euler method and previous analytical methods that available in the literature. Moreover, the convergence of the proposed methods was discussed and proved. In addition, the maximum error remainder values are also evaluated which indicates that the proposed methods are efficient and reliable. Our computational works have been done by using the computer algebra system MATHEMATICA®10 to evaluate the terms in the iterative processes.

1. INTRODUCTION

Many phenomena in physics and engineering can be modeled by the non-linear differential equations. For example, the fin surfaces are used for developing the transfer of the heat between the base of the surface and the surrounding medium [1]. The surface area for the heat transference can be increased by thermal fins made by high conductivity materials at the base surface. Besides, the thermal conductivity for the fin materials should be high to dominate the temperature of the surface base difference to the tip of the fin. The thermal conductivity can be transfer the heat from the origin to the surface of the fin [2]. The thermal conductivity and the heat transfer coefficients are one of the key features of this is the effect on thermal conductivity and efficiency by fins [3].

There are different techniques for promoting heat transfer such as increasing the area of heat transfer coefficient or heat transfer surface. When thermal conductivity based on temperature is of great importance, this large temperature gradient will lead to an extra transfer of energy. The dependent temperature on thermal conductivity is represented by the nonlinear differential equations [4,5]. Nonlinear fin problems were of great importance to scientists and engineers because of their industrial importance and benefits in various fields of science and technology [6]. Several problems of the dimensional radioactive fins were studied and solved by the decomposition method by [7]. Recently, the equations of such problems can be transformed to a system of nonlinear differential equations and resolved as using the optimal homotopy analysis method (OHAM) [8]. Also, it can be treated and solved by the application of the Sumudu or the Fourier sine transformations [9]. Several authors have applied this kind of transformed homotopy methods in order to solve these kinds of problems, see [10-15].

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Many applications of the fluids in sciences or engineering are made such as: the developments of the fluid models where the robustness and accuracy have been checked and evaluated with six major two-phase flow benchmark problems [16]. These problems involve two linear advection problems, the problem of oscillation for the liquid column, the problem of a ransom water faucet, the reversed water faucet problem, and the problem of the two-phase shock tube where all of them are fundamental in the nuclear engineering T-H field. Several investigations have been introduced into a numerical accuracy for the estimation of the shallow water flow based on the ensemble Kalman filter by the investigation of the Streamline-Upwind Petrov-Galerkin (SUPG), finite element method (FEM) [17]. An experimental data and results for a direct numerical simulation for the flow have been developed in a constant-cross-section tube in a passage of a shock wave through a three-layer gas system are discussed in [18]. A numerical investigation is presented on the electroosmotic flow (EOF) and solute mixing in a microchannel when considering both of the geometric modulation and surface potential heterogeneity for a wall of the channel [19].

Bartas and Sellers tested the thermal rejection system made from parallel pipes to web panels side by side [20]. Straight and radial fins have been analyzed by Coskun and Atay with thermal conductivity based on temperature using the variational iteration method (VIM) [21]. Cuce and Cuce examined the temperature and effectiveness of convection fins with the thermal conductivity based on temperature by the implementation of the homotopy perturbation scheme [22]. In addition, Chiu and Chen verified the thermal convection fins of the radiation by Adomian's decomposition method (ADM) [23,24]. Tabet et al. have applied the Padé approximations with the ADM to solve the convective straight fins in solar air collector [25]. Patra and Ray analyzed the fins of the radiological load using the homotopy perturbation Sumudu transform approach [26]. Wilkins presented the optimal ratio at the starting point of zero for the trigonometric fins radiating into space [27]. Radiator applications were studied by Cockfield as a structural part in the application of the spacecraft [28].

In this paper, three iterative methods will be used to solve the straight and radial fins with a thermal conductivity for obtain new approximate solutions. The first one is presented in 2006 by Daftardar-Gejji and Jafari namely (DJM) [29], the second iterative method is suggested in 2011 by the Temimi-Ansari method namely (TAM) [30], the third iterative method is the Banach contraction method namely (BCM), it has been presented by Varsha and Sachin Bhalekar in 2009 [31]. Moreover, for comparison purposes the governing nonlinear problems are also solved using analytic methods available in literature such as VIM and ADM, also two numerical methods are implemented.

This paper is arranged as the following: in section 2, the mathematical formulations of radial and straight thermal fins are given. Section 3 introduces the basic concepts for the three used iterative methods. Section 4 provides a solution to the problem of straight fins using the suggested methods. Section 5 provides a solution to the problem of radial fins using the suggested methods. In section 6, the convergence of the used methods is presented. The numerical simulations and error analyses of the approximate solutions are shown in section 7. The conclusion will be given in section 8.

2. THE MATHEMATICAL FORMING OF THE GOVERNING EQUATIONS

In this part, the formulation of the straight and radiation fin problems will be presented.

2.1. Straight Fin Problem

Let consider a straight fin relies on $k\left(T\right)$ the temperature with heat transfer coefficient or the thermal conductivity, S is some random constant area (cross-sectional) L is the fin length and p represents the perimeter, T_a and T_b are the ambient and base temperatures, in a respective way, and h represents the heat transfer coefficient, see Figure 1 [32]. The 1D equation of energy balance is presented in this form [33, 34].

$$S \frac{d}{dx}\left(k\left(T\right)\frac{dT}{dx}\right) - ph\left(T - T_a\right) = 0, \tag{1}$$

where T is the distribution of the heat on the fin. However, we assume that the material of the fin is a linearly depend on a temperature, i.e.

$$k\left(T\right) = k_a\left(1 + \gamma\left(T - T_a\right)\right), \tag{2}$$

where k_a represents the heat conductivity at T_a and γ is a parameter without dimensions which describes the variance in the temperature conductivity.

$$u = \frac{T - T_a}{T_b - T_a}, \quad x = \frac{y}{b}, \quad \beta = \gamma\left(T_b - T_a\right), \quad \mu^2 = \frac{hpL^2}{k_a S}u. \tag{3}$$

The following nonlinear equation describes the straight energy balance of fins extending to the surface under the influence, where temperature-dependent thermal conductivity is acquired [35].

$$u''\left(x\right) + \beta u\left(x\right)u''\left(x\right) + \beta\left(u'\left(x\right)\right)^2 - \mu^2 u\left(x\right) = 0, \tag{4}$$

with the following boundary conditions

$$u'\left(0\right) = 0, \quad u\left(1\right) = 1, \text{ and } u\left(0\right) = a. \tag{5}$$

where u is an unknown function which represents the temperature without dimensions, μ thermal fin parameter, β parameter that describes the thermal conductivity, a will be evaluated later [36]. It is difficult to solve Equation (4) analytically and therefore the exact solution cannot be obtained. Hence, the approximate and numerical methods will be used.
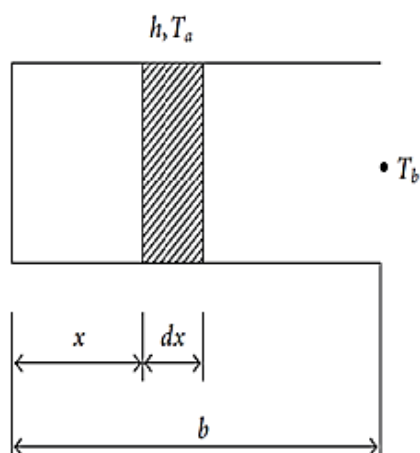

Figure 1. *Geometry of the straight fin [32]*

2.1. Radial Fin Problem

Let us recognize the heat pipe/fin space radiator as clarified in Figure 2 [9]. As the temperature is low, the fin surfaces radiate into outer space [37]. The fin has the heat conductivity $k(T)$ that is depending on temperature linearly, w is the fin diffuse with a gray balloon emission, T_b is the core temperature, and temperature of the fixed fin tube surface [38]. The equation of the energy balance for the fin

$$2S\frac{d}{dx}\left(k(T)\frac{dT}{dx}\right)-2w\,\sigma T^4=0,\tag{6}$$

where the σ represents the constant of Stefan-Boltzmann. The heat conductivity of the fin materials is considered to be a linearly depend on a temperature [39, 40].

$$k(T)=k_b\left(1+\gamma(T-T_b)\right),\tag{7}$$

where k_a reperesents the thermal conductivity for the fin when the core temperature, λ is the slope of the curve between the heat conductivity and the temperature. In order to calculate a solution for Equation (6); let use these variables without dimensions.

$$u=\frac{T}{T_b},\quad x=\frac{y}{b},\quad \beta=\gamma T_b,\quad \mu=\frac{w\,\sigma b^2 T_b^3}{k_b S},\tag{8}$$

Formulation of fin problem [41,42] reduce to this equation

$$u''(x)+\beta u(x)u''(x)+\beta\left(u'(x)\right)^2-\mu\left(u(x)\right)^4=0,\tag{9}$$

with the following boundary conditions:

$$u'(0)=0,\quad u(1)=1,\text{ and }u(0)=a.\tag{10}$$

a will be evaluated later. Equations (9) and (10) cannot be solved analytically and hence the approximate or numerical solutions will be obtained.
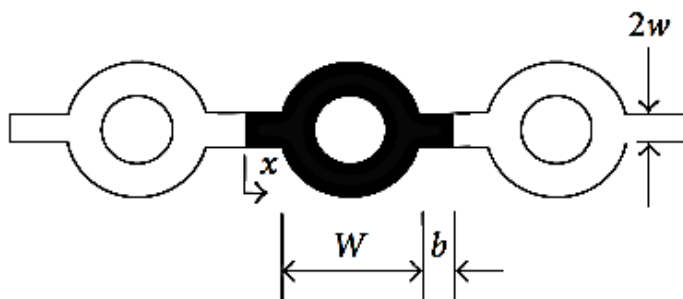

Figure 2. *Schematic display of radial fin problem [9]*

3. THE MATHEMATICAL FORMULATION OF THE GOVERNING EQUATIONS

The fundamental concepts for the three proposed techniques will be introduced in the following subsections.

3.1. The Basic Proceedings of the DJM

Let us suppose the next form for a functional equation [43-46]

$$u(x) = L(u) + N(u) + f \ , \tag{11}$$

where f is a specified function, L and N represent the linear and nonlinear operators, and u is our goal which is an unknown function and it can be decomposed as:

$$u(x) = \sum_{i=0}^{\infty} u_i \ . \tag{12}$$

Therefore, define [47]

$$G_0 = N(u_0) \ , \tag{13}$$

$$G_k = N\left(\sum_{i=0}^{k} u_j\right) - N\left(\sum_{i=0}^{k-1} u_i\right), \ k \geq 1. \tag{14}$$

So, $N(u)$ will be analyzed as follows:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = \underbrace{N(u_0)}_{G_0} + \underbrace{N(u_0 + u_1) - N(u_0)}_{G_1} + \underbrace{N(u_0 + u_1 + u_2) - N(u_0 + u_1)}_{G_2}$$
$$+ \underbrace{N(u_0 + u_1 + u_2 + u_3) - N(u_0 + u_1 + u_2)}_{G_3} + \dots \tag{15}$$

Also, this relation will be expressed by:

$$u_0 = f \ ,$$
$$u_1 = L(u_0) + G_0, \tag{16}$$
$$u_2 = L(u_0) + G_1,$$
$$\vdots$$
$$u_{k+1} = G_k + L(u_k). \tag{17}$$

As L is the linear operator where $\sum_{i=0}^{k} L(u_i) = L\left(\sum_{i=0}^{k} u_i\right)$, hence

$$\sum_{i=0}^{k+1} u_i = \left(\sum_{i=0}^{k} L(u_i)\right) + N\left(\sum_{i=0}^{k} u_i\right) = L\left(\sum_{i=0}^{k} u_i\right) + N\left(\sum_{i=0}^{k} u_i\right), \tag{18}$$

and

$$\sum_{i=0}^{\infty} u_i = f + L\left(\sum_{i=0}^{\infty} u_i\right) + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{19}$$

The k th-term approximate solution in Equations (18) and (19) is expressed by $u = u_0 + u_1 + \dots + u_{k-1}$. More details can be found in [48].

3.2. The Basic Proceedings of the TAM

Let us start with this general form of an ODE [30, 49]

$$L\big(u(x)\big)+N\big(u(x)\big)+g(x)=0, \tag{20}$$

with the boundary conditions

$$B\left(u,\frac{du}{dx}\right)=0, \tag{21}$$

where L is the linear operator and N is the non-linear operator. The boundary operator here is B, and $g(x)$ is the given function. This method can be implemented as follows, assuming that $u_0(x)$ is the initial approximation by solving the initial problem

$$L\big(u_0(x)\big)+g(x)=0 \text{ and } B\left(u_0,\frac{du_0}{dx}\right)=0, \tag{22}$$

In order to generate the next solution, let solve the presented equation

$$L\big(u_1(x)\big)+g(x)+N\big(u_0(x)\big)=0 \text{ and } B\left(u_1,\frac{du_1}{dx}\right)=0. \tag{23}$$

Thus, we get a simple iterative stride that can be used to solve a set of problems

$$L\big(u_{k+1}(x)\big)+g(x)+N\big(u_k(x)\big)=0 \text{ and } B\left(u_{k+1},\frac{du_{k+1}}{dx}\right)=0. \tag{24}$$

Note that each $u_i(x)$ is an approximated solution to the equation (24). The whole solution for the problem can be given by [50, 51]

$$u=\lim_{n\to\infty}u_n. \tag{25}$$

3.3. The Basic Proceedings of the BCM

Let us suppose the nonlinear functional equation [52, 53]
$$u(x)=f(x)+N\big(u(x)\big), \tag{26}$$

where $u(x)$ is the unknown goal, $f(x)$ represents a known function and N is the nonlinear operator for the functional Equation (26). Now we will present some consecutive approximations as:

$$u_0=f,$$
$$u_1=u_0+N(u_0),$$
$$u_2=u_0+N(u_1), \tag{27}$$
$$u_3=u_0+N(u_2),$$
$$\vdots$$

$$u_k = u_0 + N\left(u_{k-1}\right), \ k = 1, 2, \ldots. \tag{28}$$

The BCM depends on the principle of contraction which we will address later. The obtained solution for Equation (28) can be presented as:

$$u = \lim_{k \to \infty} u_k. \tag{29}$$

4. THE MATHEMATICAL FORMATION OF THE GOVERNING EQUATIONS

This section shows us the straight fins will be solved by using the iterative techniques: DJM, TAM, and BCM.

4.1. Solving the Straight Fin Problem by the DJM

To solve the straight fin formed in Equations (4) and (5) by the DJM. The following integrated form will be obtained as described in the following steps:

$$u''(x) + \beta u(x) u''(x) + \beta\left(u'(x)\right)^2 - \mu^2 u(x) = 0, \tag{30}$$

rewrite Equation (30) as follows:

$$u''(x) = N\left(u(x)\right), \tag{31}$$

where $N\left(u(x)\right) = -\beta u(x) u''(x) - \beta\left(u'(x)\right)^2 + \mu^2 u(x)$, \tag{32}

integrating Equation (32) from 0 to x, we get:

$$u'(x) = \int_0^x \int_0^x N\left(u(t_1)\right) dt_1 dt_2. \tag{33}$$

Also, integrating Equation (33) from 0 to x, we have:

$$u(x) = a + \int_0^x \int_0^x N\left(u(t_1)\right) dt_1 dt_2, \tag{34}$$

where $u(0) = a$. For simplicity, according to the rule of reducing multiple integrals [54], the integral form given in equation (34) will be reduced to the following Volterra integral equation:

$$u(x) = a + \int_0^x (x - t) N\left(u(t)\right) dt.$$

So, according to the basic proceedings of the DJM, we have:

$$\tilde{u}_0(x) = a,$$

$$\tilde{u}_1(x) = \int_0^x (x - t) N\left(\tilde{u}_0(t)\right) dt,$$

$$\tilde{u}_2(x) = \int_0^x (x - t) N\left(\tilde{u}_1(t)\right) dt - u_1(x),$$

The first approximation will be

$$\tilde{u}_1(x) = a + \frac{1}{2}ax^2\mu^2,$$

in general, we have

$$\tilde{u}_{n+1}(x) = \int_0^x (x-t)N(\tilde{u}_n(x))dt - \int_0^x (x-t)N(\tilde{u}_{n-1}(x))dt, \quad n \in \mathbb{N} \tag{35}$$

therefore

$$\tilde{u}_2(x) = -\frac{1}{2}a^2x^2\beta\mu^2 + \frac{1}{24}ax^4\mu^4 - \frac{1}{8}a^2x^4\beta\mu^4,$$

$$\tilde{u}_3(x) = \frac{1}{2}a^3x^2\beta^2\mu^2 + \frac{1}{12}a^2x^4\beta\mu^4 + \frac{3}{8}a^3x^4\beta^2\mu^4 + \frac{1}{720}ax^6\mu^6 - \frac{1}{40}a^2x^6\beta\mu^6$$

$$+ \frac{1}{12}a^3x^6\beta^2\mu^6 - \frac{1}{16}a^4x^6\beta^3\mu^6 - \frac{1}{1152}a^2x^8\beta\mu^8 + \frac{1}{192}a^3x^8\beta^2\mu^8 - \frac{1}{128}a^4x^8\beta^3\mu^8,$$

Then $u_3(x) = \tilde{u}_0(x) + \tilde{u}_1(x) + \tilde{u}_2(x) + \tilde{u}_3(x)$

$$= a + \frac{1}{2}a^3x^2\beta^2\mu^2 - \frac{1}{12}a^2x^4\beta\mu^4 + \frac{3}{8}a^3x^4\beta^2\mu^4 - \frac{1}{8}a^4x^4\beta^3\mu^4 + \frac{1}{720}ax^6\mu^6 - \frac{1}{40}a^2x^6\beta\mu^6$$

$$+ \frac{1}{12}a^3x^6\beta^2\mu^6 - \frac{1}{16}a^4x^6\beta^3\mu^6 - \frac{1}{1152}a^2x^8\beta\mu^8 + \frac{1}{192}a^3x^8\beta^2\mu^8 - \frac{1}{128}a^4x^8\beta^3\mu^8,$$

and so on. Continuing to reach the approximation till $n = 6$, for $u_n(x)$, where the terms are not noted to give brevity.

4.2. Solving the Straight Fin Problem by the TAM

To solve the problem of the straight fin problem given in Equations (4) and (5) by the TAM, we have

$$L(u) = u''(x), \quad N(u) = -\beta u(x)u''(x) - \beta(u'(x))^2 + \mu^2 u(x), g(x) = 0, \tag{36}$$

we have the following initial problem

$$L(u_0(x)) = 0, \text{ with } u_0(0) = a, \, u_0'(0) = 0, \tag{37}$$

in general, it will be

$$L(u_{n+1}(x)) + N(u_n(x)) + g(x) = 0, \quad u_{n+1}(0) = a, \, u'_{n+1}(0) = 0. \tag{38}$$

Through the steps in subsection (3.2), we solve the following initial problem:

$$u_0''(x) = 0, \qquad\qquad u_0(0) = a, \, u_0'(0) = 0, \tag{39}$$

we get,

$$u_0(x)=a.$$

In the second step, we will solve the following problem:

$$u_1''(x)=N\left(u_0(x)\right), u_1(0)=a, u_1'(0)=0, \tag{40}$$

this produces

$$u_1(x)=a+\frac{1}{2}ax^2\mu^2.$$

The same step for finding $u_2(x)$, $u_3(x)$, will be used, which means solving the following problem:

$$u_2''(x)=N\left(u_1(x)\right), \qquad u_2(0)=a, \quad u_2'(0)=0, \tag{41}$$

$$u_3''(x)=N\left(u_2(x)\right), \qquad u_3(0)=a, \quad u_3'(0)=0, \tag{42}$$

we get,

$$u_2(x)=a+\frac{1}{2}ax^2\mu^2-\frac{1}{2}a^2x^2\beta\mu^2+\frac{1}{24}ax^4\mu^4-\frac{1}{8}a^2x^4\beta\mu^4,$$

$$u_3(x)=a+\frac{1}{2}a^3x^2\beta^2\mu^2-\frac{1}{12}a^2x^4\beta\mu^4+\frac{3}{8}a^3x^4\beta^2\mu^4-\frac{1}{8}a^4x^4\beta^3\mu^4+\frac{1}{720}ax^6\mu^6-\frac{1}{40}a^2x^6\beta\mu^6$$

$$+\frac{1}{12}a^3x^6\beta^2\mu^6-\frac{1}{16}a^4x^6\beta^3\mu^6-\frac{1}{1152}a^2x^8\beta\mu^8+\frac{1}{192}a^3x^8\beta^2\mu^8-\frac{1}{128}a^4x^8\beta^3\mu^8,$$

and so on. Therefore, continue to reach the approximations at $n=6$ for $u_n(x)$.

4.3. Solving the Straight Fin Problem by the BCM

To implement the BCM to solve the straight fin problem, let us begin with the same steps of integration processes in the section (4.1) and thus we get an integrated form given in Equation (34). So according to the BCM steps given in subsection 3.3, we have:

$$u''(x)+\beta u(x)u''(x)+\beta\left(u'(x)\right)^2-\mu(u(x))^4=0, \tag{43}$$

by rewriting the Equation (43) as:

$$u''(x)=N\left(u(x)\right), \tag{44}$$

where

$$N\left(u(x)\right)=-\beta u(x)u''(x)-\beta\left(u'(x)\right)^2+\mu(u(x))^4, \tag{45}$$

therefore,

$$u(x)=a+\int_0^x(x-t)N\left(u(t)\right)dt,$$

then, we have

$$u_0(x)=a,$$

$$u_1(x)=u_0(x)+\int_0^x(x-t)N\left(u_0(t)\right)dt .$$

In general, we have

$$u_{n+1}(x)=u_0(x)+\int_0^x(x-t)N\left(u_n(t)\right)dt, \, n\geq1. \tag{46}$$

Therefore

$$u_1(x)=a+\frac{1}{2}ax^2\mu^2,$$

$$u_2(x)=a+\frac{1}{2}ax^2\mu^2-\frac{1}{2}a^2x^2\beta\mu^2+\frac{1}{24}ax^4\mu^4-\frac{1}{8}a^2x^4\beta\mu^4,$$

$$u_3(x)=a+\frac{1}{2}ax^2\mu^2+\frac{1}{2}a^3x^2\beta^2\mu^2-\frac{1}{12}a^2x^4\beta\mu^4+\frac{3}{8}a^3x^4\beta^2\mu^4-\frac{1}{8}a^4x^4\beta^3\mu^4+\frac{1}{720}ax^6\mu^6$$

$$-\frac{1}{40}a^2x^6\beta\mu^6+\frac{1}{12}a^3x^6\beta^2\mu^6-\frac{1}{16}a^4x^6\beta^3\mu^6-\frac{1}{1152}a^2x^8\beta\mu^8+\frac{1}{192}a^3x^8\beta^2\mu^8$$

$$-\frac{1}{128}a^4x^8\beta^3\mu^8,$$

and so on, continue in order to get the approximations at $n=6$ for $u_n(x)$.

5. SOLVING THE RADIAL FIN PROBLEM

In this section, the radial fin problem will be resolved by applying the three iterative processes: DJM, TAM, and BCM.

5.1. Solving the Radial Fin Problem by the DJM

Consider the problem of radial fin problem in Equations (9) and (10), the following integrated form will be obtained as described in the following steps:

$$u''(x)+\beta u(x)u''(x)+\beta\left(u'(x)\right)^2-\mu(u(x))^4=0, \tag{47}$$

Equation (47) can be written as the following:

$$u''(x)=N(u(x)), \tag{48}$$

where

$$N\left(u(x)\right)=-\beta u(x)u''(x)-\beta\left(u'(x)\right)^2+\mu(u(x))^4, \tag{49}$$

also, integrating Equation (48) from 0 to x twice, and by employing the boundary conditions:

$$u(x) = a + \int_0^x \int_0^x N\left(u(t_1)\right) dt_1 dt_2, \tag{50}$$

where $u(0) = a$, then we have

$$\tilde{u}_{n+1}(x) = \int_0^x (x - t) N\left(\sum_{i=0}^n \tilde{u}_i(t)\right) dt - \int_0^x (x - t) N\left(\sum_{i=0}^{n-1} \tilde{u}_i(t)\right) dt, \tag{51}$$

thus, we obtain

$$\tilde{u}_0(x) = a,$$

$$\tilde{u}_1(x) = \frac{1}{2} a^4 x^2 \mu,$$

$$\tilde{u}_2(x) = -\frac{1}{2} a^5 x^2 \beta\mu + \frac{1}{6} a^7 x^4 \mu^2 - \frac{1}{8} a^8 x^4 \beta\mu^2 + \frac{1}{20} a^{10} x^6 \mu^3 + \frac{1}{112} a^{13} x^8 \mu^4 + \frac{a^{16} x^{10} \mu^5}{1440}.$$

Then $u_2(x) = \tilde{u}_0(x) + \tilde{u}_1(x) + \tilde{u}_2(x)$

$$= a + \frac{1}{2} a^4 x^2 \mu - \frac{1}{2} a^5 x^2 \beta\mu + \frac{1}{6} a^7 x^4 \mu^2 - \frac{1}{8} a^8 x^4 \beta\mu^2 + \frac{1}{20} a^{10} x^6 \mu^3 + \frac{1}{112} a^{13} x^8 \mu^4$$

$$+ \frac{1}{1440} a^{16} x^{10} \mu^5,$$

and so on, continue to obtain the order at $n = 4$ for $u_n(x)$.

5.2. Solving the Radial Fin Problem by the TAM

To solve the problem of the radial fin problem given in Equations (9) and (10) by the TAM, we begin by solving the following initial problem:

$$u_0''(x) = 0, \qquad\qquad u_0(0) = a, u_0'(0) = 0, \tag{52}$$

we get,

$$u_0(x) = a.$$

In the second step, we solve the following problem:

$$u_1''(x) = N\left(u_0(x)\right), \qquad u_1(0) = a, u_1'(0) = 0, \tag{53}$$

therefore, we get

$$u_1(x) = a + \frac{1}{2} a^4 x^2 \mu.$$

In the third step, we obtain

$$u_2(x) = a + \frac{1}{2}a^4 x^2 \mu - \frac{1}{2}a^5 x^2 \beta\mu + \frac{1}{6}a^7 x^4 \mu^2 - \frac{1}{8}a^8 x^4 \beta\mu^2 + \frac{1}{20}a^{10} x^6 \mu^3 + \frac{1}{112}a^{13} x^8 \mu^4$$
$$+ \frac{1}{1440}a^{16} x^{10} \mu^5,$$

and so on, continue in this manner to have the approximation when $n = 4$ for $u_n(x)$.

5.3. Solving the Radial Fin Problem by the BCM

To apply the BCM for the radial fins problem; let us begin with the same steps of integration processes in the subsection (5.1) and thus we get an integrated form given in Equation (50). So according to the BCM steps 3.3, we get:

$$u_0(x) = a,$$
$$u_1(x) = a + \frac{1}{2}a^4 x^2 \mu,$$
$$u_2(x) = a + \frac{1}{2}a^4 x^2 \mu - \frac{1}{2}a^5 x^2 \beta\mu + \frac{1}{6}a^7 x^4 \mu^2 - \frac{1}{8}a^8 x^4 \beta\mu^2 + \frac{1}{20}a^{10} x^6 \mu^3 + \frac{1}{112}a^{13} x^8 \mu^4$$
$$+ \frac{1}{1440}a^{16} x^{10} \mu^5,$$

and so on, continue to obtain approximations at $n = 4$ for $u_n(x)$. It is worth to mention that the obtained approximate solutions by the three iterative methods are the same.

6. THE CONVERGENCE OF THE PROPOSED TECHNIQUES

In this part, let us discuss the convergence for the proposed techniques. The convergence for the DJM can be applied directly. However, to demonstrate the convergence for the TAM and BCM, we must use the following steps:

$$\begin{aligned} w_0 &= u_0(x), \\ w_1 &= F[w_0], \\ w_2 &= F[w_0 + w_1], \\ &\vdots \\ w_{n+1} &= F[w_0 + w_1 + \ldots + w_n]. \end{aligned} \tag{54}$$

where F is the operator that can be defined as

$$F[w_k] = S_k - \sum_{i=0}^{k-1} u_i(x), \ k = 1, 2, \ldots. \tag{55}$$

The term S_k expresses the solution for one of these problems.
For the TAM:

$$L(u_k(x)) + g(x) + N\left(\sum_{i=0}^{k-1} u_i(x)\right) = 0, \ k = 1, 2, \ldots. \tag{56}$$

For the BCM:

$$u_k = u_0 + N\left[\sum_{i=0}^{k-1} u_i\left(x\right)\right], \ k = 1, 2, \ldots. \tag{57}$$

The same given conditions with the used iterative method will be used. Thus, in this manner, we will have $u\left(x\right) = \lim_{n\to\infty} u_n\left(x\right) = \sum_{n=0}^{\infty} w_n$. Therefore, by using (54) and (55), one can obtain the solution by this form

$$u\left(x\right) = \sum_{i=0}^{\infty} w\left(x\right). \tag{58}$$

According to the recursive algorithm of the DJM, TAM, and BCM, We present the basic condition for conducting convergence of our methods, in these theorems [55]

Theorem 6.1. Let F presented in $\left(55\right)$, be an operator from a Hilbert space H to H. The series solution $u_n\left(x\right) = \sum_{i=0}^{n} w_i\left(x\right)$ is convergent if $\exists 0 < \gamma < 1$ when $\left\|F\left[w_0 + w_1 + \ldots + w_{i+1}\right]\right\| \leq \gamma \left\|F\left[w_0 + w_1 + \ldots + w_i\right]\right\|$ (such that $\left\|w_{i+1}\right\| \leq \gamma \left\|w_i\right\|$) $\forall i = 0, 1, 2, \ldots$.

This theorem derived from the Banach's fixed point theorem where it is a sufficient condition for studying the convergence for our proposed iterative techniques.

Theorem 6.2. If the series solution $u\left(x\right) = \sum_{i=0}^{\infty} w_i\left(x\right)$ converges, then this series represents the exact solution for the two problems given in Equations $\left(4\right)$ and $\left(5\right)$ and Equations $\left(9\right)$ and $\left(10\right)$.

Theorem 6.3. Consider the series solution $\sum_{i=0}^{\infty} w_i\left(x\right)$ which is presented in $\left(58\right)$ is convergent to the solution $u\left(x\right)$. If the truncated series $\sum_{i=0}^{n} w_i\left(x\right)$ is used as an approximation to the solution of the current problem, then the maximum error $E_n\left(x\right)$ can be estimated by:

$$E_n\left(x\right) \leq \frac{1}{1-\gamma} \gamma^{n+1} \left\|w_0\right\|. \tag{59}$$

Theorems 6.1 and **6.2** state that the obtained solutions from the DJM given in $\left(17\right)$, the TAM which is given in $\left(24\right)$, or the solution of the BCM given in $\left(28\right)$, for the nonlinear equation $\left(6\right)$ is convergent to the exact solution under the given conditions, $\exists 0 < \gamma < 1$ such that $\left\|F\left[w_0 + w_1 + \ldots + w_{i+1}\right]\right\| \leq \gamma \left\|F\left[w_0 + w_1 + \ldots + w_i\right]\right\|$ (that is $\left\|w_{i+1}\right\| \leq \gamma \left\|w_i\right\|$) $\forall i = 0, 1, 2, \ldots$. In another meaning, for each rank i, if the parameters are defined

$$\rho_i = \begin{cases} \dfrac{\left\|w_{i+1}\right\|}{\left\|w_i\right\|}, & \left\|w_i\right\| \neq 0 \\ 0, & \left\|w_i\right\| = 0 \end{cases} \tag{60}$$

Then the series solution $\sum_{i=0}^{\infty} w_i(x)$ of Equations (4) and (9) converges to the exact solution $u(x)$,

when $0 \leq \rho_i < 1, \forall i = 0,1,2,\dots$. Also, as presented in **Theorem 6.3**, the maximum truncation error can

be estimated to be $\left\| u(x) - \sum_{i=0}^{n} w_i \right\| \leq \dfrac{1}{1-\rho} \rho^{n+1} \| w_0 \|$, where $\rho = \max\{\rho_i, i = 0,1,\dots,n\}$.

6.1. The Convergence of the DJM, TAM, and BCM for Straight Fin Problem

In order to prove the convergence for the DJM, TAM, and BCM for the problem of the straight fin, we follow the following procedure:

when applying TAM, the S_k represents the following problem

$$u_k{}''(x) = N\left(\sum_{i=0}^{k-1} u_i(t) \right), \qquad u_k(0) = a, \quad u_k{}'(0) = 0, \qquad k = 1,2,\dots \tag{61}$$

Also, when applying BCM, the S_k represents the following problem

$$u_k = u_0 + N\left[\sum_{i=0}^{k-1} u_i(x) \right], \quad k = 1,2,\dots. \tag{62}$$

When applying the DJM, the iterations terms have been achieved in subsection 4.1. The value of a is calculated by substitute the given condition $u(1) = 0$ with $\beta = 0.1$ and $\mu = 0.2$ in $u_6(x)$, we get $a = 0.9820752510863351$. By setting off the values of a, β and μ in the obtained $u_6(x)$, one can get the approximate solution.

For assessing the convergent conditions of the obtained approximate solution we evaluate the ρ_i, we get

$$\rho_0 = \frac{\| w_1 \|}{\| w_0 \|} = 0.9820752510863351 < 1$$

$$\rho_1 = \frac{\| w_2 \|}{\| w_1 \|} = 0.09389211652421377 < 1$$

$$\rho_2 = \frac{\| w_3 \|}{\| w_2 \|} = 0.09272590944592346 < 1 \tag{63}$$

$$\rho_3 = \frac{\| w_4 \|}{\| w_3 \|} = 0.09250255247944411 < 1$$

$$\rho_4 = \frac{\| w_5 \|}{\| w_4 \|} = 0.09235921107655067 < 1.$$

The ρ_i's, for $i \geq 0$ and $0 < x < 1$, are less than one for the three methods (since the approximate solutions are the same). Hence, is converging according to the convergence condition.

6.2. The Convergence of the DJM, TAM, and BCM for Radial Fin Problem

To determine the convergence analysis of the DJM, TAM, and BCM for the radial fin problem, we do the following:

when applying TAM, the S_k performs the following problem

$$u_k''(x) = N\left(\sum_{i=0}^{k-1} u_i(t)\right), \qquad u_k(0) = a, \quad u_k'(0) = 0, \qquad k = 1, 2, \ldots \tag{64}$$

Also, when applying BCM, the S_k represents the following problem

$$u_k = u_0 + N\left[\sum_{i=0}^{k-1} u_i(x)\right], \quad k = 1, 2, \ldots \tag{65}$$

When applying DJM, we achieved the terms given in subsection 5.1. The value of a is calculated by substitute the given condition $u(1) = 0$ with $\beta = 0.1$ and $\mu = 0.2$ in $u_1(x)$, we get $a = 0.9263593057311732$. By adjusting the values of a, β and μ in the obtained $u_1(x)$, one has the approximate solution. In order to check the convergence, we have to evaluate the ρ_i

$$\rho_0 = \frac{\|w_1\|}{\|w_0\|} = 0.9263593057311732 < 1$$

$$\rho_1 = \frac{\|w_2\|}{\|w_1\|} = 0.6738957216238456 < 1$$

$$\rho_2 = \frac{\|w_3\|}{\|w_2\|} = 0.15375146157751043 < 1 \tag{66}$$

$$\rho_3 = \frac{\|w_4\|}{\|w_3\|} = 0.14384893447805971 < 1$$

$$\rho_4 = \frac{\|w_5\|}{\|w_4\|} = 0.13621997659656196 < 1.$$

The ρ_i's, for $i \geq 0$ and $0 < x < 1$, are less than one for the three methods (since the approximate solutions are the same). Hence, is convergent according to the convergence condition

7. THE NUMERICAL DISCUSSION AND SIMULATIONS

7.1. Straight Fin Problem

The effect of the DJM, TAM, and BCM for the straight fin problem the problem was examined to calculate the accuracy of these approximate methods of solution. We can find the appropriate approximate solutions when determining β, μ values. We observed that the numerical results obtained using three suggested iterative methods are similar to each other. Table 1 presents different values of β, μ and a.

Table 1. *The values of* $u(0)=a$ *for the approximate solution the problem straight fins*

β	μ	$a_{Approx.}$
0.1	0.2	0.9820752510863351
0.1	0.4	0.9312301454554193
0.1	0.6	0.8552043326254753
0.1	0.8	0.7640232752694894
0.1	1	0.6673364482492052
-0.1	0.2	0.9782079795428562
-0.1	0.4	0.9176057048560305
-0.1	0.6	0.8300679823446968
-0.1	0.8	0.7293792788472717
-0.1	1	0.6270320845758087
0.1	0.2	0.9820752510863351

Moreover, to check the accuracy the root mean square function (RMS) is calculated by

$$RMS(u) = \sqrt{\frac{\sum\left(u_{Approx.} - u_{RK4}\right)^2}{\sum\left(u_{RK4}\right)^2}}, \tag{67}$$

where u_{RK4} represents the numerical solution acquired by the Runge-Kutta 4 method (RK4) and $u_{Approx.}$ is the approximate solution we obtained using three suggested iterative methods. Since the exact solution is not available the numerical solution u_{RKM} is used, Figures 3(a) and (b) show the RMS which can be simply seen by raising the values of β and μ the error will be increasing.

Furthermore, the residual error function is evaluated by [45]:

$$ER_n(x) = u_n''(x) + \beta u_n(x)u_n''(x) + \beta\left(u_n'(x)\right)^2 - \mu^2 u_n(x), \tag{68}$$

The following maximal error remainder parameter

$$MER_n = \max_{0 \le x \le 1}\left|ER_n(x)\right|. \tag{69}$$

For $\beta = 0.1$ and $\mu = 0.2$, the maximal error remainder MER_n values for the numerical solutions obtained by our analytic used methods, the ADM [25,34] and the VIM [9] can be plotted as in Figure 3 (c) and the convergent is clear when increasing the rank number of iterations. Moreover, Figure 3 (d) offers the results of the proposed methods that are in a good deal with the resulted solutions given by the ADM, VIM, Runge-Kutta and Euler methods.
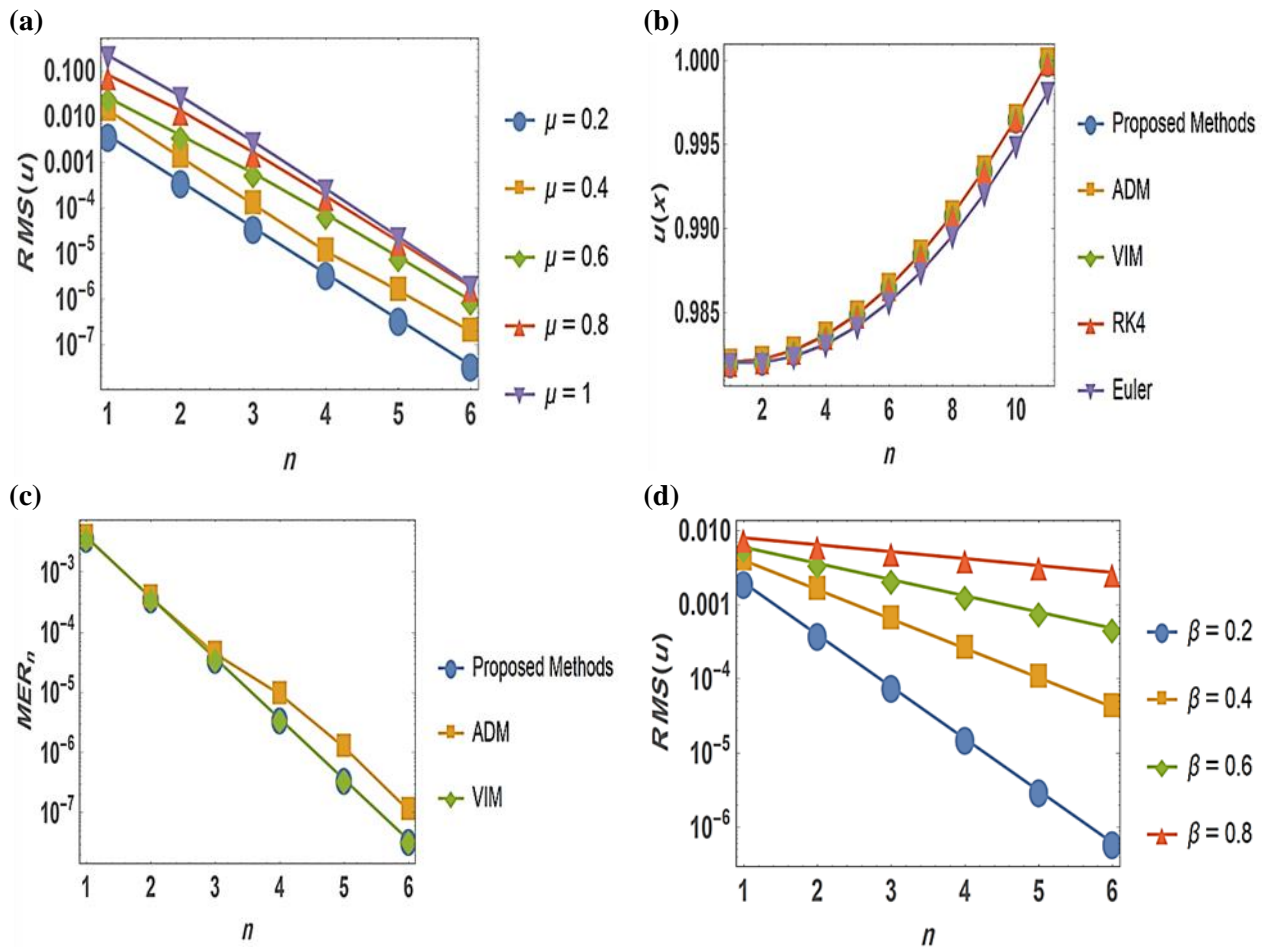
Figure 3. *a) Logarithmic plots of the RMS solution the problem straight fins function for various values of α at $\beta = 0.1$ b) Logarithmic plots of the RMS solution the problem straight fins function for various values of α at $\mu = 0.1$ c) The MER_n the problem straight fins values obtained by proposed methods, the ADM and the VIM when $\beta = 0.1$ and $\mu = 0.2$ d) The numerical solutions obtained the problem straight fins by our proposed methods, at $\beta = 0.1$ and $\mu = 0.2$. d) The numerical solutions obtained the problem straight fins by our proposed methods, at $\beta = 0.1$ and $\mu = 0.2$*

7.2. Radial Fin Problem

The effect of the proposed methods for solving the problem of the radial fin problem was studied. We can find the appropriate approximate solutions when determining β and μ values. We noted that the numerical solutions achieved by the proposed methods are the same. The values of the constant missing condition a are evaluated in Table 2 with several estimations of β and μ. Figures 4(a) and (b) show the root mean square function (RMS) it can be also seen when the values of β and μ increased the error increased as well.

Table 2. *The values of* $u(0)=a$ *for the approximate solution by using iterations*

| β	μ	$a_{Approx.}$
0.1	0.2	0.9263593057311732
0.1	0.4	0.8800394301309911
0.1	0.6	0.8461883108282183
0.1	0.8	0.8195489122835696
0.1	1	0.7976231097945159
-0.1	0.2	0.9263593057311732
-0.1	0.4	0.8800394301309911
-0.1	0.6	0.8461883108282183
-0.1	0.8	0.8195489122835696
-0.1	1	0.7976231097945159

Furthermore, the residual error function is evaluated by [45]:

$$ER_n(x)=u_n''(x)+\beta u_n(x)u_n''(x)+\beta\big(u_n'(x)\big)^2-\mu\big(u_n(x)\big)^4, \tag{70}$$

the following parameter of the maximal error remainder

$$MER_n=\max_{0\leq x\leq 1}\big|ER_n(x)\big|. \tag{71}$$

For $\beta=0.1$ and $\mu=0.2$, the MER_n values for the numerical results reached by our proposed methods, the ADM [25] and the VIM [9] can be plotted as in Figure 4(c) and it can be discerned obviously that the error decreases when increasing the number of iterations. Finally, Figure 4(d) shows a good agreement between the approximate solutions achieved by our proposed methods and ADM, VIM, RK4 and Euler method.
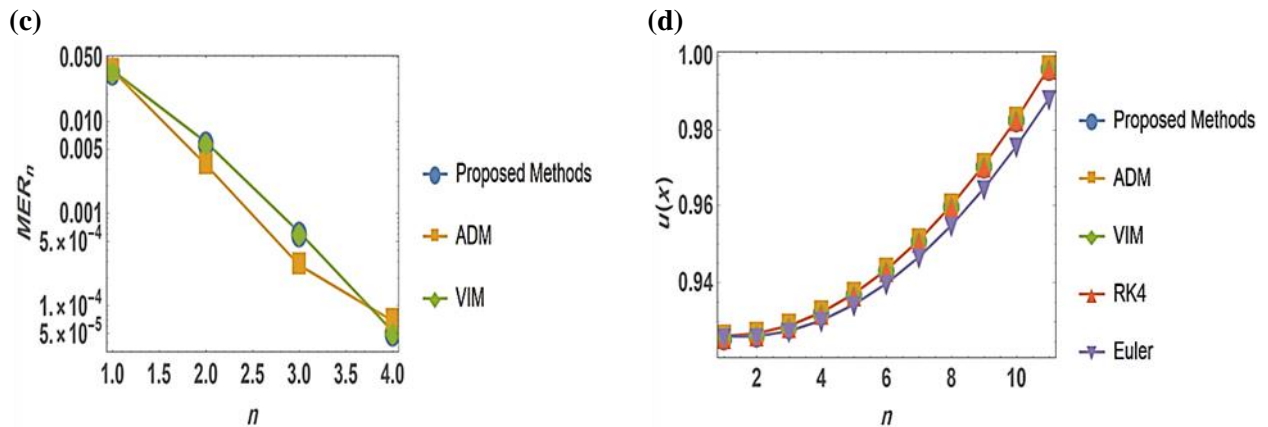
(a)

(b)

(c)

(d)
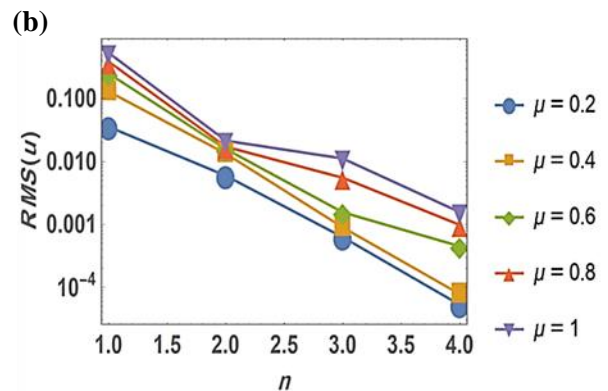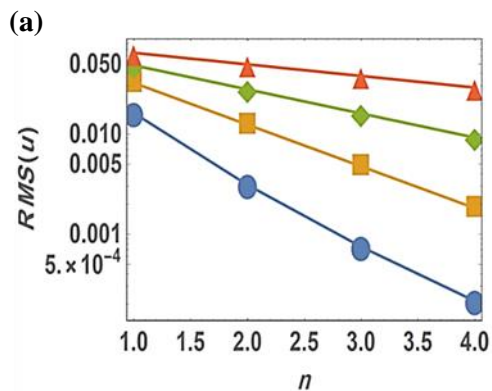
Figure 4. *a) Logarithmic plots of the RMS solution the problem Radial fins function for various values of α at $\beta = 0.1$ b) Logarithmic plots of the RMS solution the problem Radial fins function for various values of α at $\mu = 0.1$ c) The MER_n the problem radial fins values obtained by proposed methods, the ADM and the VIM when $\beta = 0.1$ and $\mu = 0.2$ d) The numerical solutions obtained the problem radial fins by our proposed methods, at $\beta = 0.1$ and $\mu = 0.2$*

It should be noted that the main benefits of the obtained results that the methods have many advantages such as being derivative-free and overcoming the inconvenience arising in calculating Adomian polynomials to handle the non-linear terms in the ADM. It does not need evaluating the multiplier of Lagrange as in the VIM in which the terms of the sequence are shifted to be complex after several iterations, thus, analytical evaluation of terms becomes very difficult or impossible in the VIM. No need to create a homotopy as in the homotopy perturbation method (HPM) and solve the identical algebraic equations.

8. CONCLUSION

In this paper, three iterative methods namely the DJM, TAM, and BCM were used to solve straight and radial fins problems with thermal conductivity associated with temperature. Approximate solutions were obtained in a converge series. Also, when the number of iterations increases, the maximum error remainder values are reduced. Moreover, the results obtained by our proposed methods are compared with other numerical results obtained by the Runge-Kutta 4 (RK4) and Euler methods and results obtained by approximate methods accessible in the literature and good agreements have noticed. Thus, the proposed methods produce accurate, reliable and effective results.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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