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Generalized Regression-Cum-Exponential Estimators Using Two Auxiliary Variables for Population Variance in Simple Random Sampling

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Highlights

- Generalized regression-cum-exponential type estimators are proposed to estimate population variance.
- Bias and MSE have been derived for the proposed estimators.
- Simulation and empirical study proves better functioning of the proposed estimators than the existing one.

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Abstract

In this paper, we proposed two generalized regression-cum-exponential type estimators for the estimation of finite population variance using the information of mean and variance of the auxiliary variables in simple random sampling *(SRS)*. The expressions of approximate bias and mean square error *(MSE)* of the proposed estimators are derived. Many special cases of the proposed estimators are obtained by using different combinations of real numbers and some conventional parameters of the auxiliary variables. Algebraic comparisons of the proposed estimators have been made with some available estimators. From the numerical study, we analyzed that the proposed estimators perform well than the existing estimators available in the literature.

1. INTRODUCTION

Auxiliary information is commonly used to improve the efficiency of population parameters of interest. History filled with a lot of authors that have used auxiliary information in order to get the prices estimates. Classical ratio, product, and regression and their mixture-type of estimators are good examples and cornerstone in this background. The estimation of population variance considered to be the best measure in the variations of study variable. Various authors have been done work on estimating the population variance. [1] introduced classical ratio and regression estimators in finite population variance. The effort of [2,3] may be well-thought-out as an early effort in the estimation of population variance.[4] developed the exponential ratio and exponential product type estimators in population variance.[5] developed generalized exponential estimator. [6] developed a ratio-type estimator for finite population variance. [7] utilized a single auxiliary variable to propose an estimator in population variance and provide more efficient results as compare to the ratio estimator. The studies related to the estimation of population variance have made by different authors such as [8-22].

After a brief introduction and literature review, some basics notations and existing estimators are mentioned in Section 2. The expressions of approximate bias and *MSE* of the proposed estimators are derived in Section 3. Section 4 is based on efficiency comparisons. A numerical study is illustrated in Section 5 on three real populations. Conclusion and remarks are given in Section 6.

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2. NOTATION AND VARIOUS EXISTING ESTIMATORS

Let $U=(U_1,U_2,...,U_N)$ having the units for a finite population U for the population size N and a sample of size n is selected from the population by using the method of *SRS* without replacement. Here S_y^2, S_x^2 and S_z^2 be the population variances, while s_y^2, s_x^2 and s_z^2 be the sample variance for our study and auxiliary variables respectively. Let, e_0, e_1 and e_2 be the sampling errors such that:

$$s_y^2=S_y^2\left(1+e_0\right),\ s_x^2=S_x^2\left(1+e_1\right),s_z^2=S_z^2\left(1+e_2\right),$$
$$E\left(e_0\right)=E\left(e_1\right)=E\left(e_2\right)=E\left(e_{11}\right)=E\left(e_{22}\right)=0,$$
$$E\left(e_0^2\right)=\theta\left(\lambda_{400}-1\right),E\left(e_1^2\right)=\theta\left(\lambda_{040}-1\right),E\left(e_2^2\right)=\theta\left(\lambda_{004}-1\right),$$
$$E\left(e_0e_1\right)=\theta\left(\lambda_{220}-1\right),E\left(e_0e_2\right)=\theta\left(\lambda_{202}-1\right),E\left(e_2e_1\right)=\theta\left(\lambda_{022}-1\right),$$
$$\rho_{yx}=\frac{S_{yx}}{S_yS_x},\rho_{yz}=\frac{S_{yz}}{S_yS_z},C_y=\frac{S_y}{\bar{Y}},C_x=\frac{S_x}{\bar{X}},C_z=\frac{S_z}{\bar{Z}},w_1=\frac{\alpha_1}{\delta_1},w_2=\frac{\alpha_2}{\delta_2}.$$

Let $\bar{Y},\bar{X}$ and $\bar{Z}$ be the population means whereas $\bar{y}$, $\bar{x}$ and $\bar{z}$ be the sample means for the study and auxiliary variables respectively. Let e_{11}, e_{22} be the sampling error such that

$$\bar{x}=\bar{X}\left(1+e_{11}\right),\ \bar{z}=\bar{Z}\left(1+e_{22}\right),\ s_x^2=S_x^2\left(1+e_{11}\right),s_z^2=S_z^2\left(1+e_{22}\right),$$
$$E\left(e_{11}^2\right)=\theta C_x^2\ ,E\left(e_{22}^2\right)=\theta C_z^2,E\left(e_0e_{11}\right)=\theta\lambda_{210}C_x,E\left(e_0e_{22}\right)=\theta\lambda_{201}C_z,$$
$$E\left(e_2e_{11}\right)=\theta\lambda_{012}C_x,E\left(e_1e_{22}\right)=\theta\lambda_{021}C_z,E\left(e_{11}e_{22}\right)=\theta\rho_{xz}C_xC_z,\theta=\frac{1}{n},$$
$$\lambda_{abc}=\frac{\mu_{abc}}{\mu_{200}^{\frac{a}{2}}\mu_{020}^{\frac{b}{2}}\mu_{002}^{\frac{c}{2}}},\mu_{abc}=\frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)^a\left(X_i-\bar{X}\right)^b\left(Z_i-\bar{Z}\right)^c,$$
$${}_yf_{004}=\frac{\left(\lambda_{004}-1\right)}{\left(\lambda_{400}-1\right)},\ {}_yf_{220}=\frac{\left(\lambda_{220}-1\right)}{\left(\lambda_{400}-1\right)},\ {}_xf_{220}=\frac{\left(\lambda_{220}-1\right)}{\left(\lambda_{040}-1\right)},$$
$${}_xf_{022}=\frac{\left(\lambda_{022}-1\right)}{\left(\lambda_{040}-1\right)},\ {}_zf_{202}=\frac{\left(\lambda_{202}-1\right)}{\left(\lambda_{004}-1\right)},\ {}_zf_{022}=\frac{\left(\lambda_{022}-1\right)}{\left(\lambda_{004}-1\right)},$$
$$M_1=\frac{1}{A^2}\left[{}_yf_{004}R_1+{}_yf_{040}R_2+2\,{}_yf_{002}R_3\right]-\frac{2}{A}R_4\ ,\ P=\frac{\lambda_{012}^2R_2}{B^2\left(\lambda_{004}-1\right)}-\frac{2R_3}{B}-\frac{1}{4}C_x^2+\lambda_{210}C_x,$$
$$M_2=\frac{\lambda_{012}^2R_2}{B^2\left(\lambda_{004}-1\right)}+\frac{1}{B^2}\left(R_3-2\lambda_{012}^2\lambda_{210}^2\right)-\frac{2R_3}{B}\ ,A=1-{}_xf_{220}\,{}_zf_{022},B=\left(\frac{1-\lambda_{012}^2}{\left(\lambda_{004}-1\right)}\right),$$

where a, b and c be the non-negative integers. The quantities μ_{200}, μ_{020} and μ_{002} be the second order moments and λ_{abc} be the moment ratio.

The sample means and variances of the study and auxiliary variable may obtained as:

$$\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i\ ,\ \bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i\ \text{ and }\ \bar{z}=\frac{1}{n}\sum_{i=1}^{n}z_i,$$

$$s_y^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2,\ s_x^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2\ \text{ and }\ s_z^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(z_i-\bar{z}\right)^2.$$

Similarly, the population means and variances of the study and auxiliary variables may be obtained as:

$\bar{Y}=\frac{1}{N}\sum_{i=1}^{N}Y_i$, $\bar{X}=\frac{1}{N}\sum_{i=1}^{N}X_i$ and $\bar{Z}=\frac{1}{N}\sum_{i=1}^{N}Z_i$,

$S_y^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)^2$, $S_x^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(X_i-\bar{X}\right)^2$ and $S_z^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(Z_i-\bar{Z}\right)^2$.

In literature, the unbiased variance estimator without having auxiliary information for the finite population is

$$t_0=s_y^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2. \tag{1}$$

The variance of t_0 is given as

$$Var\left(t_0\right)=\theta S_y^4\left(\lambda_{400}-1\right). \tag{2}$$

[1] proposed the classical ratio and regression estimators for the estimation of finite population variance as

$$t_1=s_y^2\left(\frac{S_x^2}{s_x^2}\right), \tag{3}$$

$$t_2=s_y^2+b_1\left(S_x^2-s_x^2\right), \tag{4}$$

where $b_1=S_y^2\left(\lambda_{220}-1\right)/S_x^2\left(\lambda_{040}-1\right)$, which is regression coefficient.

The expressions of *MSE* of t_1 and t_2 up to the first order approximation are respectively given as

$$MSE\left(t_1\right)\approx\theta S_y^4\left[\lambda_{400}+\lambda_{040}-2\lambda_{220}\right], \tag{5}$$

$$MSE(t_2)\approx\theta S_y^4\left(\lambda_{400}-1\right)\left[1-{}_x f_{220}\,{}_y f_{220}\right]. \tag{6}$$

The traditional regression estimator for population variance using the mean of the auxiliary variable is

$$t_3=s_y^2+b_2\left(\bar{X}-\bar{x}\right), \tag{7}$$

where, $b_2=S_y^2\left(\lambda_{210}\right)/\bar{X}C_x$.which is regression coefficient.

The expression of mean square error of t_3 up to the first order approximation is given as

$$MSE(t_3)\approx\theta S_y^4\left[\left(\lambda_{400}-1\right)-\lambda_{210}^2\right]. \tag{8}$$

[4] introduced exponential ratio and exponential product-type estimators using the information of single auxiliary variable for the finite population variance are

$$t_4=s_y^2\exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right), \tag{9}$$

$$t_5=s_y^2\exp\left(\frac{s_z^2-S_z^2}{S_z^2+s_z^2}\right). \tag{10}$$

The expressions of *MSE* for t_4 and t_5 up to the first order approximation are respectively given as

$$MSE\left(t_4\right)\approx \theta S_y^4\left[\lambda_{400}+\frac{\lambda_{040}}{4}-\lambda_{220}+\frac{1}{4}\right], \tag{11}$$

$$MSE\left(t_5\right)\approx \theta S_y^4\left[\lambda_{400}+\frac{\lambda_{004}}{4}+\lambda_{202}-\frac{9}{4}\right]. \tag{12}$$

[5] introduced a generalized exponential-type estimator using the auxiliary information of finite population variance as

$$t_6=s_y^2\exp\left[\frac{S_x^2-s_x^2}{S_x^2+(\alpha-1)s_x^2}\right]. \tag{13}$$

The expression of minimum *MSE* of t_6 up to the first order approximation is

$$MSE_{min}\left(t_6\right)=\theta S_y^4\left(\lambda_{400}-1\right)\left[1-{}_{y}f_{220}\,{}_{x}f_{220}\right], \tag{14}$$

where $\alpha_{opt}=\left(\lambda_{040}-1\right)/\left(\lambda_{220}-1\right)$.

[8] introduced exponential ratio and exponential product-type estimators using the mean of the auxiliary variable as

$$t_7=s_y^2\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), \tag{15}$$

$$t_8=s_y^2\exp\left(\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}}\right). \tag{16}$$

The expressions of *MSE* of t_7 and t_8 up to the first order term are given as

$$MSE\left(t_7\right)\approx \theta S_y^2\left[\left(\lambda_{400}-1\right)-\lambda_{210}C_x+\frac{1}{4}C_x^2\right]. \tag{17}$$

$$MSE\left(t_8\right)\approx \theta S_y^2\left[\left(\lambda_{400}-1\right)+\lambda_{210}C_x+\frac{1}{4}C_x^2\right]. \tag{18}$$

[9] suggested new variance estimator as

$$t_9=\frac{s_y^2}{2}\left[\frac{S_x^2}{s_x^2}+\exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)\right]. \tag{19}$$

The expression of *MSE* of t_9 up and around to the first order term is given as

$$MSE\left(t_9\right)\approx \theta S_y^4\left[(\lambda_{400}-1)+\frac{3\left(\lambda_{040}-1\right)}{16}\left(3-8\,{}_{x}f_{220}\right)\right]. \tag{20}$$

[10] suggested a new exponential-type estimator. The estimator along with the expression of its variance is

$$t_{10} = s_y^2 \exp\left[\frac{\sqrt{\bar{X}} - \sqrt{\bar{x}}}{\sqrt{\bar{X}} + \sqrt{\bar{x}}}\right], \tag{21}$$

$$MSE\left(t_{10}\right) = \theta S_y^4\left[\lambda_{400} + \frac{1}{2}C_x\left(\frac{1}{8}C_x - \lambda_{210}\right) - 1\right]. \tag{22}$$

[11] proposed exponential-type estimator using a single auxiliary variable as

$$t_{11} = \lambda s_y^2 \exp\left[\frac{\mathrm{c}(\bar{\mathrm{X}} - \bar{x})}{\bar{X} + (\mathrm{d}-1)\bar{\mathrm{x}}}\right]\exp\left[\frac{\mathrm{e}(S_x^2 - s_x^2)}{S_x^2 + (\mathrm{f}-1)s_x^2}\right], \tag{23}$$

the expression of minimum mean square error of t_{11} up to the first order approximation is given as

$$MSE_{min}(t_{11}) = \theta S_y^4\left[\left(\lambda_{400} - 1\right) - \lambda_{210}^2 - \left\{\frac{\left(\left(\lambda_{220} - 1\right) - \lambda_{210}\left(\lambda_{030}\right)\right)^2}{\left(\lambda_{040} - 1\right) - \lambda_{030}^2}\right\}\right]. \tag{24}$$

3. PROPOSED GENERALIZED ESTIMATORS

In this section, two generalized regression-cum-exponential type estimators are proposed by using the variance and mean respectively as the auxiliary variables in simple random sampling. The estimators are defined as

$$t_{n1} = \left[s_y^2 + k_1\left(S_z^2 - s_z^2\right)\right]\exp\left[\alpha_1 \frac{S_x^2 - s_x^2}{S_x^2 + \left(\delta_1 - 1\right)s_x^2}\right] \tag{25}$$

and

$$t_{n2} = \left[s_y^2 + k_2\left(S_z^2 - s_z^2\right)\right]\exp\left[\alpha_2\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \left(\delta_2 - 1\right)\bar{x}}\right)\right], \tag{26}$$

where, $k_1, k_2, \delta_1, \delta_2 \left(\delta_1, \delta_2 > 0\right)$ be the constants and need to be optimized for the expressions of minimum mean square errors for proposed estimators $\left(t_{n1}, t_{n2}\right)$. The generalized constants α_1 and α_2 can assume values -1, 0 and 1 to generate many special cases of the suggested estimators.

3.1. Bias and *MSE* for the Proposed Estimator-I (t_{n1})

In order to derive the expression of bias of the proposed estimator t_{n1}, we may write (25) in terms of *e's* as

$$t_{n1} = \left[S_y^2\left(1 + e_0\right) + k_1\left\{S_z^2 - S_z^2\left(1 + e_2\right)\right\}\right]\exp\left[\alpha\left(\frac{S_x^2 - S_x^2\left(1 + e_1\right)}{S_x^2 + (\delta_1 - 1)S_x^2\left(1 + e_1\right)}\right)\right]. \tag{27}$$

Simplifying and applying Taylor series and ignoring the terms beyond the second order terms

$$t_{n1} \approx \left[S_y^2 + S_y^2 e_0 - k_1 S_z^2 e_2\right]\left[1-\frac{\alpha_1}{\delta_1}e_1+\frac{\alpha_1}{\delta_1}e_1^2-\frac{\alpha_1}{\delta_1^2}e_1^2+\frac{\alpha_1^2}{2\delta_1^2}e_1^2\right]. \tag{28}$$

On simplification of (28), we have

$$t_{n1}-S_y^2 \approx \left[-w_1S_y^2e_1+w_1S_y^2e_1^2\left(1-\frac{1}{\delta_1}+\frac{w_1}{2}\right)+S_y^2e_0-w_1S_y^2e_1e_0-k_1S_z^2e_2+w_1k_1S_z^2e_1e_2\right], \tag{29}$$

where

$$\xi_1=\left(1-\frac{1}{\delta_1}+\frac{w_1}{2}\right).$$

Simplifying and taking expectations of (29), we may get the expression of bias of the proposed estimator as

$$Bias\left(t_{n1}\right) \approx \theta w_1 S_y^2\left(\lambda_{040}-1\right)\left[\xi_1 - {}_x f_{220} + k_1\frac{S_z^2}{S_y^2}\,{}_x f_{022}\right]. \tag{30}$$

In order to get the expression of *MSE* of the proposed estimator t_{n1}, we expand (27) using Tylor series up to the first order of approximation as

$$t_{n1}-S_y^2 \approx S_y^2e_0-k_1S_z^2e_2-w_1S_y^2e_1. \tag{31}$$

By taking Square of (31) and ignoring the higher order terms

$$\left(t_{n1}-S_y^2\right)^2 \approx \left[S_y^4e_0^2+k_1^2S_z^4e_2^2+w_1^2S_y^4e_1^2-2k_1S_y^2S_z^2e_0e_2+2k_1w_1S_y^2S_z^2e_1e_2-2w_1S_y^4e_0e_1\right], \tag{32}$$

by taking expectations on both sides of (32), we have the expression of *MSE* as

$$MSE\left(t_{n1}\right) \approx \theta\left(\lambda_{400}-1\right)\left[S_y^4\left\{1+w_1^2\,{}_x f_{040}-2w_1\,{}_x f_{220}\right\}+k_1^2S_z^4\,{}_x f_{004}+2S_y^2S_z^2\left\{w_1\,{}_x f_{022}-{}_x f_{202}\right\}\right]. \tag{33}$$

In order to minimize the expression of (33) with respect to "w_1" and "k_1" yield its optimum values as

$$w_{1(opt)}=\frac{1}{A}\left[{}_x f_{220}-{}_x f_{022}\,{}_z f_{202}\right] \quad \text{and} \quad k_{1(opt)}=\frac{S_y^2}{S_z^2}\frac{1}{A}\left[A\,{}_z f_{202}-{}_z f_{022}\,{}_x f_{220}+{}_z f_{022}\,{}_x f_{022}\,{}_z f_{202}\right].$$

The expression of the minimized mean square error of t_{n1} after substituting the values of $w_{1\,(opt)}$ and $k_{1\,(opt)}$ obtained as,

$$MSE_{\min}\left(t_{n1}\right) \approx \theta S_y^4\left(\lambda_{400}-1\right)\frac{1}{A^2}\left[A^2+\left(R_1\,{}_y f_{004}+R_2\,{}_y f_{040}+2R_3\,{}_y f_{022}\right)-2AR_4\right], \tag{34}$$

where,

$$R_1=\left[{}_z f_{202}^2+{}_z f_{022}^2\,{}_x f_{220}^2-2\,{}_z f_{202}\,{}_z f_{022}\,{}_x f_{220}\right], R_2=\left[{}_x f_{220}^2+{}_x f_{022}^2\,{}_z f_{202}^2-2\,{}_z f_{202}\,{}_x f_{022}\,{}_x f_{220}\right],$$

$$R_3=\left[{}_x f_{220}\,{}_z f_{202}-{}_x f_{220}^2\,{}_z f_{022}-{}_z f_{202}^2\,{}_x f_{022}+{}_x f_{022}\,{}_x f_{220}\,{}_z f_{202}\,{}_z f_{022}\right].$$

3.2. Bias and *MSE* for Proposed Estimator-II (t_{n2})

In order to derive the expression of bias for the proposed estimators t_{n2}, we may express (26) in terms of *e*'s as,

$$t_{n2} = \left[S_y^2(1+e_0) + k_2\left\{S_z^2 - S_z^2(1+e_2)\right\}\right]\exp\left[\alpha_2\left(\frac{\bar{X} - \bar{X}(1+e_{11})}{\bar{X} + (\delta_2 - 1)\bar{X}(1+e_{11})}\right)\right]. \tag{35}$$

Simplifying and applying Taylor series on (35) up to the second order of approximation, we have

$$\left(t_{n2} - S_y^2\right) \approx \left[S_y^2 e_0 - w_2 S_y^2 e_{11} + w_2 S_y^2 e_{11}^2\left(1 - \frac{1}{\delta_2} + \frac{w_2}{2}\right) - w_2 S_y^2 e_0 e_{11} - k_2 S_z^2 e_2 + k_2 w_2 S_z^2 e_2 e_{11}\right]. \tag{36}$$

Applying expectation on (36), we have the final expression of the approximate bias of the proposed estimators t_{n2}, as

$$Bias(t_{n2}) \approx \theta S_y^2 C_x^2 w_2\left[\xi_2 - \lambda_{210} + k_2\frac{S_z^2}{S_y^2}\lambda_{012}\right] \text{ where } \xi_2 = \left(1 - \frac{1}{\delta_2} + \frac{w_2}{2}\right). \tag{37}$$

In order to get the expression of *MSE* for the proposed estimator t_{n2}, we expand (35) up to first order of approximation as

$$t_{n2} - S_y^2 = S_y^2 e_0 - k_2 S_z^2 e_2 - S_y^2 w_2 e_{11}. \tag{38}$$

By taking square and after simplifying the (38), we have

$$\left(t_{n2} - S_y^2\right)^2 = \left[S_y^4 e_0^2 + k_2^2 S_z^4 e_2^2 + S_y^4 w_2^2 e_{11}^2 - 2k_2 S_y^2 S_z^2 e_0 e_2 + 2k_2 w_2 S_y^2 S_z^2 e_2 e_{11} - 2w_2 S_y^4 e_0 e_{11}\right]. \tag{39}$$

By taking expectations of (39), we have

$$MSE(t_{n2}) \approx \theta\left[\begin{array}{l} S_y^4(\lambda_{400} - 1) + k_2^2 S_z^4(\lambda_{004} - 1) \\ \qquad + 2k_2 S_y^2 S_z^2\left\{w_2(\lambda_{012})C_x - (\lambda_{202} - 1)\right\} - 2w_2 S_y^4(\lambda_{210})C_x + S_y^4 w_2^2 C_x^2 \end{array}\right]. \tag{40}$$

In order to obtain the expression of minimum *MSE*, differentiating (40) with respect to "w_2" and "k_2" yield its optimum values

$$w_{2(opt)} = \frac{1}{BC_x}\left[(\lambda_{210}) - (\lambda_{012})_z f_{202}\right] \quad \text{and} \quad k_{2(opt)} = \frac{S_y^2}{S_z^2}\left[\left({}_z f_{202}\right) - Z\left\{(\lambda_{210}) - (\lambda_{012})\left({}_z f_{202}\right)\right\}\right].$$

The final expression of the minimize *MSE* of t_{n2} after substitute the values of $w_{2(opt)}$ and $k_{2(opt)}$ is,

$$MSE_{\min}(t_{n2}) \approx \theta S_y^4\left[(\lambda_{400} - 1)R_1 + \frac{1}{B^2}\left\{Z(\lambda_{012})R_2 + \left(R_3 - 2(\lambda_{012})^2(\lambda_{210})^2\right) - 2BR_3\right\}\right], \tag{41}$$

where

$$R_4=\left[1+\left({}_y f_{004}\right)\left({}_z f_{202}\right)^2-2\left({}_y f_{202}\right)\left({}_z f_{202}\right)\right],\ R_5=\left[\left(\lambda_{210}\right)^2-\left(\lambda_{012}\right)^2\left({}_z f_{202}\right)^2+2\left({}_z f_{202}\right)\left(\lambda_{210}\right)\left(\lambda_{012}\right)\right],$$

$$R_6=\left[\left(\lambda_{210}\right)^2+\left(\lambda_{012}\right)^2\left({}_z f_{202}\right)^2-2\left({}_z f_{202}\right)\left(\lambda_{210}\right)\left(\lambda_{012}\right)\right],\quad Z=\frac{\left(\lambda_{012}\right)}{B\left(\lambda_{004}-1\right)}.$$

The details of populations are presented in Appendix Table-B3 in order to judge the performance of the proposed estimators over the competing estimators at optimum conditions. Some special cases of the proposed estimators are summarized in appendix-A

4. EFFICIENCY COMPARISON OF PROPOSED ESTIMATORS WITH SOME EXISTING ESTIMATORS

The efficiency comparisons of the proposed estimators with some relevant competing estimators are given as.

i. The proposed (t_{n1}) will be more precise estimator than the [1] estimator given in (3) if

$$MSE_{\min}\left(t_{n1}\right)<MSE(t_1),$$

$${}_x f_{220}\,{}_y f_{220}+M_1<0.$$

ii. The proposed (t_{n1}) will be more precise estimator than the [4] estimator given in (9) if

$$MSE_{\min}\left(t_{n1}\right)<MSE(t_4),$$

$$M_1(\lambda_{400}-1)-\frac{\lambda_{040}}{4}+\lambda_{220}-\frac{5}{4}<0.$$

iii. The proposed (t_{n1}) will be more precise estimator than the [9] estimator given in (19)

$$MSE_{\min}\left(t_{n1}\right)<MSE(t_9),$$

$$M_1(\lambda_{400}-1)+\frac{3(\lambda_{040}-1)}{16}(3-8\,{}_x f_{220})<0.$$

iv. The proposed (t_{n2}) will be more precise estimator than the traditional regression estimator given in (13) if

$$MSE_{\min}\left(t_{n2}\right)<MSE(t_3),$$

$$(\lambda_{400}-1)\left(R_1-1\right)+\lambda_{210}^2<0.$$

v. The proposed (t_{n2}) will be more precise estimator than the [8] estimator given in (15) if

$$MSE_{\min}\left(t_{n2}\right) < MSE(t_7),$$

$$(\lambda_{400}-1)\left(R_1-1\right)+\lambda_{210}C_x-0.25C_x^2+M_2<0.$$

vi. The proposed (t_{n2}) will be more precise estimator than the [11] the estimator is given in (23) if

$$MSE_{\min}\left(t_{n2}\right) < MSE(t_{11}),$$

$$(\lambda_{400}-1)\left(R_1-1\right)+\lambda_{210}^2+M_2+P<0.$$

5. NUMERICAL STUDY

In this section, we set up two types of numerical studies in order to evaluate the performance of the estimators consider in this paper using three real populations. The judgment for the evaluation of suggested estimators are illustrated with traditional unbiased variance estimator. The amount of *ARB*, *MSE* and percent relative efficiency (*PRE's*) may be obtained by using the following mathematical formulas as

$$ARB\left(t_i\right)=\frac{\left|\frac{1}{R}\sum_{i=1}^{R}\left(t_i-S_y^2\right)\right|}{S_y^2}, \tag{42}$$

$$MSE\left(t_i\right)=\frac{1}{R}\sum_{i=1}^{R}\left(t_i-S_y^2\right)^2, \tag{43}$$

and

$$PRE=\frac{Var\left(t_0\right)}{MSE\left(t_*\right)}\times 100, \tag{44}$$

where '*R*' (*R*= 20,000) is the total number of iterations and t_i be the relevant estimators for i^{th} sample. The performance of the proposed estimators depends on *PRE*, such that the value greater than one hundred indicates that the proposed estimators are more efficient than the usual variance estimator.

5.1. Theoratical Study

In this section, we evaluated the performance of our proposed estimators on three real populations using an empirical study and the results of *PRE's* are summarized in Table 1. The sources of populations along with the descriptions of study and auxiliary variables are given in Appendix-B.

Table 1. *Percentage Relative Efficiencies of all the Estimators*

Estimators	Populations		
	1	2	3
$t_0=s_y^2$	100	100	100
t_1	41.5495	157.2156	89.4363
t_2	108.2613	180.2517	102.5925
t_3	100.0395	138.2660	109.2543
t_4	74.6328	140.2744	91.1857
t_6	108.2613	180.2517	102.5925
t_7	99.6919	80.9687	93.8899

t_9	61.1047	179.5361	96.6753
t_{10}	100.0395	109.9673	102.6304
t_{11}	120.5324	183.3319	109.2887
t_{n1} (proposed)	151.3692	323.9974	139.9441
t_{n2} (proposed)	159.4911	348.4204	149.7324

The results of the empirical study are presented in Table1 indicate that our proposed estimators t_{n1}, t_{n2} performed better and found to be the most efficient estimators as compared to [1] t_1, t_2 and t_3 estimators, [4] estimator t_4 ,[8] estimator t_5 , [9] estimators t_9 , [10] estimator t_{10} and [11] estimator t_{11} .

5.2. Simulation Study

In this simulation we consider six sample sizes *n*=8, 11,14,17,20 and 23 to evaluate the performance of proposed estimators and the results are summarized in Table 2 and Table 3 respectively.The following steps have been used to compute the *ARB's* and *PRE's* by using the *R*-Language software.

Step1 From population 3, Twenty thousand samples of size n we selected using SRSWOR.

Step2 Using the sample of step 1, Twenty thousand values of all the estimators are obtained from each sample size to achieve the efficiency in the estimation.

Step3 The *ARB*, and *PRE* of all the considered estimators are computed using the formula given in eq.(42) and eq.(44) .

Table 2. *Absolute relative bias of all the estimators for different sample sizes*

Estimators	Sample size					
	8	11	14	17	20	23
$t_0 = s_y^2$	0.6456	0.3736	0.5064	0.4935	0.3325	0.2009
t_1	0.3917	0.1893	0.2417	0.4295	0.1559	0.0258
t_2	0.5222	0.3064	0.4032	0.4802	0.2706	0.1504
t_3	0.6164	0.3903	0.4706	0.4804	0.3186	0.1842
t_4	0.5387	0.2879	0.3902	0.4611	0.2498	0.1101
t_6	0.5779	0.3214	0.4340	0.4744	0.2823	0.1480
t_7	0.6205	0.3953	0.4625	0.4524	0.3112	0.1592
t_9	0.4652	0.2386	0.3159	0.4453	0.2028	0.0679
t_{10}	0.6333	0.3846	0.4849	0.4728	0.3219	0.1799
t_{11}	0.6268	0.3749	0.4799	0.4762	0.3182	0.1795
t_{n1} (Proposed)	0.6839	0.5362	0.1652	0.5181	0.0319	0.0087
t_{n2} (Proposed)	0.6526	0.5382	0.0618	0.5017	0.0303	0.0499

Table 3: *Percentage relative efficiencies for all the estimators of different sample sizes*

Estimators	Sample size					
	8	11	14	17	20	23
$t_0 = s_y^2$	100	100	100	100	100	100
t_1	68.8873	76.4613	76.94706	76.5429	77.6677	77.3710
t_2	102.9675	103.2390	103.2839	102.9013	102.7457	102.4545
t_3	102.1620	102.2128	102.2547	102.2111	102.2037	102.1727
t_4	101.3425	100.2903	98.9985	97.5976	97.2824	96.5506
t_6	102.8320	102.1749	101.3834	100.5272	100.2806	99.8629
t_7	105.4710	104.5317	104.0685	103.5067	103.2695	103.0139
t_9	87.11435	90.0978	89.3678	88.2704	88.5735	87.9722
t_{10}	103.0179	102.5238	102.2670	101.9794	101.8474	101.7150
t_{11}	102.9033	102.4082	102.1174	101.8063	101.6656	101.5269
t_{n1} (Proposed)	132.7014	135.3150	139.5052	141.5388	141.2783	141.3261
t_{n2} (Proposed)	135.7227	137.8522	141.5803	143.2343	142.7709	142.3033

6. CONCLUSION

The results illustrated in Section-5 which are summarized in Table 2,Table 3 and Table 4 respectively shows that the proposed estimators are more efficient than the other estimators considered in this paper. The results obtained from empirical study as shown in Table 1 found to be more superior then the other existing estimators. From the results of simulation study, it is visibly acquired that the values of *ARB* for all the estimators tend to zero and the *PRE* increases by increasing the sample sizes as shown in Table 2 and Table 3 respectively. Hence the proposed estimators are most efficient and useful for the estimation of finite population variance. Some special cases of proposed estimators are given in Appendix-A. Source of populations, description of variables and results of parameters are given in Appendix-B.

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CONFLICTS OF INTEREST

No conflict of interst was declared by the authors.

REFERENCES

[1] Isaki, C.T., "Variance estimation using auxiliary information", Journal of the American Statistical Association, 78(381): 117-123, (1983).

[2] Prasad, B., Singh, H.P., "Some improved ratio-type estimators of finite population variance in sample surveys", Communications in Statistics-Theory and Methods, 19(3): 1127-1139, (1990).

[3] Prasad, B., Singh, H.P., "Unbiased estimators of finite population variance using auxiliary information in sample surveys", Communications in Statistics-Theory and Methods, 21(5): 1367-1376, (1992).

[4] Singh, R., Chauhan, P., Sawan, N., Smarandache, F., "Improved exponential estimator for population variance using two auxiliary variables", Italian Journal of Pure and Applied Mathematics, 28: 101-108, (2011).

[5] Yadav, S.K., Kadilar, C., "Improved Exponential Type Ratio Estimator of Population Variance", Revista Colombiana de Estadística,36(1): 145-152, (2013).

[6] Upadhyaya, L.N., Singh, H.P., "An estimator for population variance that utilizes the kurtosis of an auxiliary variable in sample surveys", Vikram Mathematical Journal, 19(1):14-17, (1999).

[7] Jhajj, H.S., Sharma, M.K., Grover, L.K., "An efficient class of chain estimators of population variance under sub-sampling scheme", Journal of the Japan Statistical Society, 35(2): 273-286, (2005).

[8] Asghar, A., Sanaullah, A., Hanif, M.,"Generalized exponential type estimator for population variance in survey sampling", Revista Colombiana de Estadística, 37(1): 213-224,(2014).

[9] Singh, H.P., Singh, A.K., Solanki, R.S., "Estimation of finite population variance using auxiliary information in sample surveys", Statistica, 74(1): 95-112, (2014).

[10] Ahmed, A., Singh, R. V. K., "Improved exponential type estimators for estimating population variance in survey sampling", International Journal of Advance Research, IJOAR .org, 33(1): 1-16, (2015).

[11] Gupta, S., Shabbir, J., "Variance estimation in simple random sampling using auxiliary information", Hacettepe Journal of Mathematics and Statistics, 37(1): 57-67, (2008).

[12] Cekim, H.O., Kadilar, C., "Generalized family of estimators via two auxiliary variables for population variance", Journal of Reliability and Statistical Studies, 11(1): 67-81, (2018).

[13] Grover, L.K., "A correction note on improvement in variance estimation using auxiliary information", Communications in Statistics—Theory and Methods, 39(5): 753-764, (2010).

[14] Jhajj, H.S., Lata, K., “A family of difference-cum-exponential type estimators for estimating the population variance using auxiliary information in sample surveys”, Journal of Mathematics and Statistics Sciences, 3(4): 10-14, (2015).

[15] Kadilar, C., Cingi, H., “Improvement in variance estimation using auxiliary information”, Hacettepe Journal of Mathematics and Statistics, 35(1): 111-115, (2006).

[16] Khan, M., Shabbir, J., “A ratio type estimator for the estimation of population variance using quartiles of an auxiliary variable”, Journal of Statistics Applications & Probability, 2(3): 319-325, (2013).

[17] Maqbool, S., Javaid, S., “Variance estimation using linear combination of tri-mean and quartile average”, American Journal of Biological and Environmental Statistics, 3(1): 5-9, (2017).

[18] Subramani, J., Kumarapandiyan, G., “Variance estimation using median of the auxiliary variable”, International Journal of Probability and Statistics, 1(3): 36-40, (2012).

[19] Subramani, J., Kumarapandiyan, G., “Estimation of variance using known coefficient of variation and median of an auxiliary variable”, Journal of Modern Applied Statistical Methods, 12(1):58- 64, (2013).

[20] Singh, H.P., Solanki, R.S., “A new procedure for variance estimation in simple random sampling using auxiliary information”, Statistical Papers, 54(2): 479-497, (2013).

[21] Yadav, S.K., Misra, S., Mishra, S.S., “Efficient estimator for population variance using auxiliary variable”, American Journal of Operational Research, 6(1): 9-15, (2016).

[22] Shabbir, J., Gupta, S., “A note on generalized exponential type estimator for population variance in survey sampling”, Revista Colombiana de Estadística, 38(2): 385-397, (2015).

[23] Singh, H.P., Espejo, M.R., “On linear regression and ratio–product estimation of a finite population mean”, Journal of the Royal Statistical Society: Series D (The Statistician), 52(1): 59-67, (2003).

[24] Cochran, W.G., “Sampling Techniques”, New York: John Wiley and Sons,(1977).

[25] Gujarati, D.N., “Basic econometrics”, Tata McGraw-Hill Education, (2009).

[26] Sukhatme, P.V., Sukhatme, B.V., “Sampling Theory of Surveys with Applications”, Asia Publishing House, New Delhi, (1970).

APPENDIX-A

The proposed estimators t_{n1} and t_{n2} may produce many special cases by using different values $\alpha_1, \alpha_2, k_1, k_2$ δ_1 and δ_2 . Some of them are listed below.

Table A1. *Some Special Cases of Proposed Estimators*

t_{n1}	t_{n2}	α_1	α_2	k_1	k_2	δ_1	δ_2
$t_0 = s_y^2$	$t_0 = s_y^2$	0	0	0	0	1	1
$t_{n1}^1 = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2}\right]$	$t_{n2}^1 = s_y^2 \exp\left[\frac{\bar{X} - \bar{x}}{\bar{X}}\right]$	1	1	0	0	1	1
$t_{n1}^2 = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]$	$t_{n2}^2 = s_y^2 \exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right]$	1	1	0	0	2	2
$t_{n3}^3 = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + (\delta_{1opt} - 1) s_x^2}\right]$	$t_{n2}^3 = s_y^2 \exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + (\delta_{2(opt)} - 1)\bar{x}}\right]$	1	1	0	0	$\delta_{1(opt)}$	δ_{2opt}
$t_{n1}^4 = s_y^2 \exp\left[\frac{s_x^2 - S_x^2}{S_x^2}\right]$	$t_{n2}^4 = s_y^2 \exp\left[\frac{\bar{x} - \bar{X}}{\bar{X}}\right]$	-1	-1	0	0	1	1
$t_{n1}^5 = s_y^2 \exp\left[\frac{s_x^2 - S_x^2}{S_x^2 + s_x^2}\right]$	$t_{n2}^5 = s_y^2 \exp\left[\frac{\bar{x} - \bar{X}}{\bar{X} + \bar{x}}\right]$	-1	-1	0	0	2	2
$t_{n1}^6 = s_y^2 + b_{yz}(S_z^2 - s_z^2)$	$t_{n2}^6 = s_y^2 + b_{yz}(S_z^2 - s_z^2)$	0	0	b_{yz}	b_{yz}	1	1
$t_{n1}^7 = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{S_x^2 - s_x^2}{S_x^2}\right]$	$t_{n2}^7 = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{X} - \bar{x}}{\bar{X}}\right]$	1	1	b_{yz}	b_{yz}	1	1
$t_{n1}^8 = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{s_x^2 - S_x^2}{S_x^2}\right]$	$t_{n2}^8 = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{x} - \bar{X}}{\bar{X}}\right]$	-1	-1	b_{yz}	b_{yz}	1	1
$t_{n1}^9 = \left[s_y^2 + \left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{S_x^2 - s_x^2}{S_x^2}\right]$	$t_{n2}^9 = \left[s_y^2 + \left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{X} - \bar{x}}{\bar{X}}\right]$	1	1	1	1	1	1
$t_{n1}^{10} = \left[s_y^2 + \left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{s_x^2 - S_x^2}{S_x^2}\right]$	$t_{n2}^{10} = \left[s_y^2 + \left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{x} - \bar{X}}{\bar{X}}\right]$	-1	-1	1	1	1	1
$t_{n1}^{11} = \left[s_y^2 + \left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]$	$t_{n2}^{11} = \left[s_y^2 + \left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right]$	1	1	1	1	2	2
$t_{n1}^{12} = \left[s_y^2 + \left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{s_x^2 - S_x^2}{S_x^2 + s_x^2}\right]$	$t_{n2}^{12} = \left[s_y^2 + \left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{x} - \bar{X}}{\bar{X} + \bar{x}}\right]$	-1	-1	1	1	2	2
$t_{n1}^{13} = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]$	$t_{n2}^{13} = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right]$	1	1	b_{yz}	b_{yz}	2	2
$t_{n1}^{14} = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{s_x^2 - S_x^2}{S_x^2 + s_x^2}\right]$	$t_{n2}^{14} = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{x} - \bar{X}}{\bar{X} + \bar{x}}\right]$	-1	-1	b_{yz}	b_{yz}	2	2
$t_{n1}^{15} = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + (\delta_{1opt} - 1)s_x^2}\right]$	$t_{n2}^{15} = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + (\delta_{2opt} - 1)\bar{x}}\right]$	1	1	b_{yz}	b_{yz}	$\delta_{1(opt)}$	δ_{2opt}
$t_{n1}^{16} = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{s_x^2 - S_x^2}{S_x^2 + (\delta_{1opt} - 1)s_x^2}\right]$	$t_{n2}^{16} = \left[s_y^2 + b_{yz}\left(S_z^2 - s_z^2\right)\right]\exp\left[\frac{\bar{x} - \bar{X}}{\bar{X} + (\delta_{2opt} - 1)\bar{x}}\right]$	-1	1	b_{yz}	b_{yz}	$\delta_{1(opt)}$	δ_{2opt}
$t_{n1}^{17} = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + (a-1)s_x^2}\right]$	$t_{n2}^{17} = s_y^2 \exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + (a-1)\bar{x}}\right]$	1	1	0	0	a	a

APPENDIX-B

Table-B1

Population	Sources of Population
1	Gujarati(2009, Pg 406)
2	Sukhatme and Sukhatme (1970, pg 185)
3	Cochran(1977,pg 34)

Table- B2

Population	Y	X	Z
1	Average Miles per gallon	Top, speed miles per Hour	Engine horsepower
2	The area under wheat in acres during 1937	The area under wheat in acres during 1936	The area under wheat in acres during 1931
3	Food cost	Size	Income

Table B3 *Results for Population Parameters*

parameters	Populations		
	1	2	3
N	81	34	33
n	21	15	9
$\bar{Y}$	33.7111	201.4118	27.4909
$\bar{X}$	112.4568	218.4118	3.7272
$\bar{Z}$	30.9876	765.3529	72.5455
C_y	0.2984	0.7555	0.3685
C_x	0.1256	0.7678	0.4095
C_z	0.2635	0.6169	0.1458
ρ_{yx}	-0.6883	0.9299	0.4328
ρ_{yz}	-0.9035	0.8992	0.2522
ρ_{xz}	0.6788	0.8308	-0.0660
λ_{400}	3.5699	5.1118	5.5509
λ_{040}	6.7365	4.7293	2.3154
λ_{004}	2.4758	3.9524	2.0780
λ_{220}	2.0606	3.61287	1.3889
λ_{202}	2.1238	3.8722	2.2186
λ_{022}	1.9995	2.9763	1.4466
λ_{210}	0.0319	1.0668	0.6209
λ_{201}	-0.0647	1.0686	0.5422
λ_{012}	0.4349	0.7541	0.2337
λ_{021}	0.8782	0.7404	0.2643
λ_{030}	1.8547	1.3184	0.5683