



On the Exponentiated Weibull Rayleigh Distribution

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Highlights

- We introduce a new four parameter distribution called exponentiated Weibull Rayleigh distribution.
- We study some statistical properties of the new model.
- We use the maximum likelihood method to estimate the parameters of the distribution.
- We make simulation results by using Mathematica 9.
- The new model is very flexible and include some special models.

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Abstract

A new four-parameter probability model, referred to the exponentiated Weibull Rayleigh (EWR) distribution, is introduced. Essential statistical properties of the distribution are considered. The maximum likelihood estimators of population parameters are given in case of complete sample. Simulation study is carried out to estimate the model parameters of EWR distribution. Additionally, parameter estimators are given in case of Type II censored samples. We come up with two applications to confirm the usefulness of the proposed distribution.

1. INTRODUCTION

In recent times, diverse statisticians explored some new generated families of distributions by incorporate one or more extra shape parameter(s) to the baseline model to yield new flexible distributions. Some of the generated families are: the beta-G [1], Kumaraswamy (Kw)-G [2], Weibull-G [3], Garhy-G [4], exponentiated Weibull-G (EW-G) [5], additive Weibull-G [6], Kw Weibull-G [7], Type II half logistic- G [8] and exponentiated extended-G [9] among others.

Modelling lifetime phenomena is an important issue in many scientific fields. For a while, several classical models like exponential, Rayleigh, Weibull, gamma and Lomax distributions seemed suitable for modelling lifetime data. But as science improved, the old models were no longer to describe the new phenomena. So, in recent years, many researchers tried to the extend classical distributions and consequently many new flexible models appeared and proved to be more acceptable. Examples of these researches are the beta gamma distribution [10], the Weibull Rayleigh (WR) distribution [11] and the Marshall-Olkin extended generalized Rayleigh distribution [12].

We intend to introduce a new four-parameter model, called the EWR distribution. The EWR distribution is

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very flexible and it extends the WR model (see [11]). First, we explain how we arrived at this model. [5] introduced the EW-G family of distributions. This family generates many new continuous distributions that are quite flexible for the purpose of modelling. The cumulative distribution function (cdf) of this family is defined by

$$F(x) = \left[1 - \exp \left(-\alpha \left[\frac{G(x)}{1-G(x)} \right]^\beta \right) \right]^a; x \geq 0; a, \alpha, \beta > 0, \quad (1)$$

where a, β are shape parameters, $\alpha > 0$ is a scale parameter and $G(x)$ is the cdf of the baseline distribution. The probability density function (pdf) of the EW-G is

$$f(x) = \frac{a\alpha\beta g(x)(G(x))^{\beta-1}}{(1-G(x))^{\beta+1}} e^{-\alpha \left[\frac{G(x)}{1-G(x)} \right]^\beta} \left[1 - \exp \left(-\alpha \left[\frac{G(x)}{1-G(x)} \right]^\beta \right) \right]^{a-1}; x \geq 0; a, \alpha, \beta > 0. \quad (2)$$

By employing the Rayleigh distribution (as the baseline distribution) in the above family, we arrive at a new flexible distribution, namely the EWR distribution. In the following sections, we obtain the pdf, cdf and hazard rate function (hrf) of the EWR distribution besides explain the shapes of the pdf and the hrf of this model briefly. We derive general statistical properties of the proposed distribution, including the expansions for the pdf and cdf, quantile function, moments, mean deviations, inequality measures, order statistics and characterizations. The maximum likelihood (ML) estimators are also discussed. Numerical study is created to estimate the model parameters. Censored Type II of the parameters is discussed. Two real data applications are presented. Finally, we point out the conclusion of our study.

2. THE EXPONENTIATED WEIBULL RAYLEIGH DISTRIBUTION

Depend on (1), a four-parameter EWR distribution is defined. The pdf and cdf of random variable (r.v.) X has the Rayleigh distribution with scale parameter λ are given by

$$g(x; \lambda) = 2\lambda x e^{-\lambda x^2}; \quad x, \lambda > 0, \quad (3)$$

and

$$G(x; \lambda) = 1 - e^{-\lambda x^2}. \quad (4)$$

Substituting (3) into (1), we get the cdf of the EWR distribution as follows

$$F(x; \Psi) = \left[1 - \exp \left(-\alpha (e^{\lambda x^2} - 1)^\beta \right) \right]^a; \quad a, \alpha, \beta, \lambda > 0, \quad x \geq 0, \quad (5)$$

where $\Psi \equiv (a, \alpha, \beta, \lambda)$ is the parameter vector. From (2), (3) and (4), the pdf of EWR distribution is as follows

$$f(x; \Psi) = 2a\alpha\beta\lambda x [e^{\lambda x^2} - 1]^{\beta-1} \exp \left(\lambda x^2 - \alpha (e^{\lambda x^2} - 1)^\beta \right) \left[1 - \exp \left(-\alpha (e^{\lambda x^2} - 1)^\beta \right) \right]^{a-1}, x \geq 0. \quad (6)$$

We write $X \sim \text{EWR}(a, \alpha, \beta, \lambda)$ if X has pdf (6). Specified sub-models of the EWR distribution are added in Table 1.

Table 1. The special sub-models of the EWR distribution

	Model	a	α	β	λ	Distribution Function	Authors
1	Exponentaited exponential Rayleigh	-	-	1	-	$F(x) = [1 - \exp(-\alpha(e^{\lambda x^2} - 1))]^a$	
2	Exponentaited Rayleigh Rayleigh Distribution	-	-	2	-	$F(x) = [1 - \exp(-\alpha(e^{\lambda x^2} - 1)^2)]^a$	
3	Weibull Rayleigh	1	-	-	-	$F(x) = 1 - \exp(-\alpha(e^{\lambda x^2} - 1)^\beta)$	[11]
4	Exponential Rayleigh	1	-	1	-	$F(x) = 1 - \exp(-\alpha(e^{\lambda x^2} - 1))$	
5	Rayleigh Rayleigh	1	-	2	-	$F(x) = 1 - \exp(-\alpha(e^{\lambda x^2} - 1)^2)$	

The survival, hrf and reversed hrf of the EWR distribution, respectively, are

$$R(x; \Psi) = 1 - [1 - \exp(-\alpha(e^{\lambda x^2} - 1)^\beta)]^a,$$

$$h(x; \Psi) = \frac{2a\alpha\beta\lambda x [e^{\lambda x^2} - 1]^{\beta-1} \exp(\lambda x^2 - \alpha(e^{\lambda x^2} - 1)^\beta) [1 - \exp(-\alpha(e^{\lambda x^2} - 1)^\beta)]^{a-1}}{1 - [1 - \exp(-\alpha(e^{\lambda x^2} - 1)^\beta)]^a},$$

and

$$\tau(x; \Psi) = \frac{2a\alpha\beta\lambda x [e^{\lambda x^2} - 1]^{\beta-1} \exp(\lambda x^2 - \alpha(e^{\lambda x^2} - 1)^\beta)}{1 - \exp(-\alpha(e^{\lambda x^2} - 1)^\beta)}.$$

The pdf and hrf plots of the EWR distribution at $\alpha = \lambda = 1$ and for certain values of a and β are clarified in Figures 1 and 2, respectively.

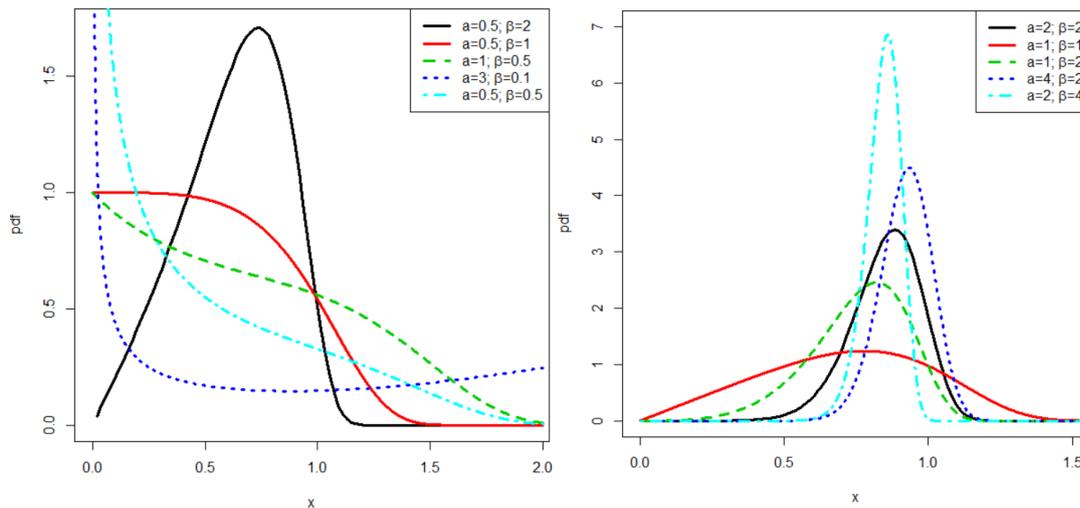


Figure 1. The pdf plots of the EWR distribution for selected parameter values at $\alpha = \lambda = 1$

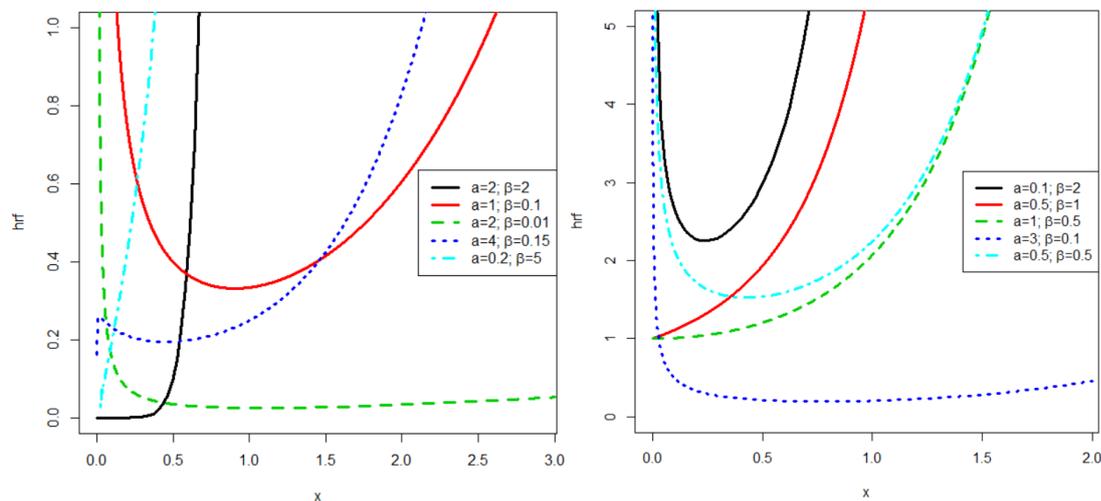


Figure 2. The hrf plots of the EWR distribution for selected parameter values at $\alpha = \lambda = 1$

From Figure 2, we observe that the hrf is increasing and bathtub shaped, showing its flexibility in modelling lifetime data sets. We note that in life phenomena, we normally encounter bathtub shaped hazard rates. We also discovered an interesting case when $a = 4$ and $\beta = 0.15$ for which the hrf is increasing-decreasing-increasing.

3. MAIN PROPERTIES

Part of the properties of the EWR distribution, in this section, is explored.

3.1. Important Expansions

Representations of the pdf and cdf of EWR distribution are provided. First, we consider the following generalized binomial expansion,

$$(1-h)^d = \sum_{i=0}^{\infty} (-1)^i \binom{d}{i} h^i, \quad |h| < 1. \tag{7}$$

The expansion (7) stops at the $(d+1)^{th}$ term if d is a positive integer value.

Now, by applying (7), we find an expansion for the cdf raised to the power m , where m is an integer, namely $[F(x; \Psi)]^m$. We have

$$[F(x; \Psi)]^m = \sum_{q=0}^{\infty} (-1)^q \binom{am}{q} e^{-\alpha q(e^{\lambda x^2} - 1)^\beta}.$$

Using power series expansion for the exponential function and the identity $e^{\lambda x^2} - 1 = (1 - e^{-\lambda x^2})e^{\lambda x^2}$, we get

$$\begin{aligned} [F(x; \Psi)]^m &= \sum_{q,w=0}^{\infty} \binom{am}{q} \frac{(-1)^{q+w} (\alpha q)^w}{w!} [e^{\lambda x^2} - 1]^{\beta w} \\ &= \sum_{q,w=0}^{\infty} \binom{am}{q} \frac{(-1)^{q+w} (\alpha q)^w}{w!} [1 - e^{-\lambda x^2}]^{\beta w} e^{\lambda \beta w x^2}. \end{aligned}$$

Again, using (7) and the identity $\binom{-r}{k} = \binom{r+k-1}{k} (-1)^k$, we have

$$e^{\lambda \beta w x^2} = \left(1 - (1 - e^{-\lambda x^2})\right)^{-\beta w} = \sum_{j=0}^{\infty} \binom{\beta w + j - 1}{j} \left(1 - e^{-\lambda x^2}\right)^j. \quad (8)$$

Thus, $[F(x; \Psi)]^m$ is written as

$$[F(x; \Psi)]^m = \sum_{k=0}^{\infty} \eta_{k,m} e^{-k\lambda x^2}, \quad (9)$$

where

$$\eta_{k,m} = \sum_{q,w,j=0}^{\infty} \frac{(-1)^{q+w+k} (\alpha q)^w}{w!} \binom{am}{q} \binom{\beta w + j - 1}{j} \binom{\beta w + j}{k}. \quad (10)$$

For $m=1$, we have the expansion for the cdf. Differentiating (9) with respect to x when $m=1$, the pdf of the EWR will be as follows

$$f(x; \Psi) = 2\lambda \sum_{k=1}^{\infty} \eta_{k,1} k x e^{-k\lambda x^2}. \quad (11)$$

From (11), we see that the density of the EWR distribution is a linear combination of Rayleigh densities. Next, we find an expansion for $[f(x; \Psi)]^\varepsilon$, ε is a positive number. Note that

$$[f(x; \Psi)]^\varepsilon = (2a\alpha\beta\lambda x)^\varepsilon [e^{\lambda x^2} - 1]^{(\beta-1)\varepsilon} \exp(\lambda\varepsilon x^2 - \alpha\varepsilon(e^{\lambda x^2} - 1)^\beta) \left[1 - \exp(-\alpha(e^{\lambda x^2} - 1)^\beta) \right]^{(a-1)\varepsilon}.$$

Using (7) and exponential expansion, we have

$$\begin{aligned} [f(x; \Psi)]^\varepsilon &= (2a\alpha\beta\lambda x)^\varepsilon [e^{\lambda x^2} - 1]^{(\beta-1)\varepsilon} e^{\lambda\varepsilon x^2} \sum_{q=0}^{\infty} (-1)^q \binom{(a-1)\varepsilon}{q} e^{-(q+\varepsilon)\alpha(e^{\lambda x^2} - 1)^\beta} \\ &= (2a\alpha\beta\lambda x)^\varepsilon e^{\lambda\varepsilon x^2} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{j+q} [\alpha(q+\varepsilon)]^j}{j!} \binom{(a-1)\varepsilon}{q} (e^{\lambda x^2} - 1)^{\beta(j+\varepsilon)-\varepsilon}. \end{aligned}$$

Again, using identity $e^{\lambda x^2} - 1 = (1 - e^{-\lambda x^2})e^{\lambda x^2}$, we have

$$[f(x; \Psi)]^\varepsilon = (2a\alpha\beta\lambda x)^\varepsilon e^{-\lambda\varepsilon x^2} \sum_{q,j=0}^{\infty} \frac{(-1)^{j+q} [\alpha(q+\varepsilon)]^j}{j!} \binom{(a-1)\varepsilon}{q} (1 - e^{-\lambda x^2})^{\beta(j+\varepsilon)-\varepsilon} e^{\lambda[\beta(j+\varepsilon)+\varepsilon]x^2}.$$

Considering (8) and using (7), we get

$$\begin{aligned} [f(x; \Psi)]^\delta &= (2a\alpha\beta\lambda x)^\varepsilon e^{-\lambda\varepsilon x^2} \sum_{q,j,w=0}^{\infty} \frac{(-1)^{j+q} [\alpha(q+\varepsilon)]^j}{j!} \binom{(a-1)\varepsilon}{q} \binom{\beta(j+\varepsilon)+\varepsilon+w-1}{w} (1 - e^{-\lambda x^2})^{\beta(j+\varepsilon)-\varepsilon+w} \\ &= (2a\alpha\beta\lambda x)^\varepsilon \sum_{q,j,w,k=0}^{\infty} \frac{(-1)^{j+q+k} [\alpha(q+\varepsilon)]^j}{j!} \binom{(a-1)\varepsilon}{q} \binom{\beta(j+\varepsilon)+\varepsilon+w-1}{w} \binom{\beta(j+\varepsilon)-\varepsilon+w}{k} e^{-(k+1)\lambda x^2}. \end{aligned}$$

From the above relations, we arrive at

$$[f(x; \Psi)]^\varepsilon = (2\lambda x)^\varepsilon \sum_{k=1}^{\infty} \eta_{k,\varepsilon}^* e^{-k\lambda x^2}, \quad (12)$$

where

$$\eta_{k,\varepsilon}^* = (a\alpha\beta)^\varepsilon \sum_{q,j,w=0}^{\infty} \frac{(-1)^{j+q+k-1} [\alpha(q+\varepsilon)]^j}{j!} \binom{(a-1)\varepsilon}{q} \binom{\beta(j+\varepsilon)+\varepsilon+w-1}{w} \binom{\beta(j+\varepsilon)-\varepsilon+w}{k-1}. \quad (13)$$

We can use (12) as an alternative expansion for the pdf of the EWR distribution after setting $\varepsilon=1$.

3.2. Quantile Function

The quantile function, say $Q(u) = F^{-1}(u)$, of X is obtained by inverting (5). So we have

$$Q(u) = \sqrt{\frac{1}{\lambda} \ln \left\{ 1 + \left(\frac{-1}{\alpha} \ln \left(1 - u^{\frac{1}{\beta}} \right) \right)^{\frac{1}{\beta}} \right\}}, \quad (14)$$

where $0 < u < 1$. Note that if U is a uniform $(0,1)$, then $Q(U)$ is an EWR r.v. Therefore, one can simulate numbers from EWR distribution by using (14). Given $u = 0.5$ in (14), the median of EWR model is obtained as follows

$$M = Q(0.5) = \sqrt{\frac{1}{\lambda} \ln \left\{ 1 + \left(\frac{-1}{\alpha} \ln \left(1 - 0.5^{\frac{1}{\beta}} \right) \right)^{\frac{1}{\beta}} \right\}}. \tag{15}$$

Furthermore, based on quantile measures, the variability analysis of the skewness and kurtosis on the shape parameters a and β is inspected. The Bowley (B) skewness (see [13]), is defined by

$$B = \frac{X_{\frac{3}{4}} - 2X_{\frac{1}{2}} + X_{\frac{1}{4}}}{X_{\frac{3}{4}} - X_{\frac{1}{4}}}.$$

The Moors (MO) kurtosis, (see [14]), is defined by

$$MO = \frac{X_{\frac{7}{8}} - X_{\frac{5}{8}} + X_{\frac{3}{8}} - X_{\frac{1}{8}}}{X_{\frac{6}{8}} - X_{\frac{2}{8}}}.$$

The B and MO measures do not rely on the moments of the distribution and are nearly insensitive to outliers. Plots of the B and MO for certain values of a as function of β and for certain choices of β as function of a are provided in Figures 3 and 4.

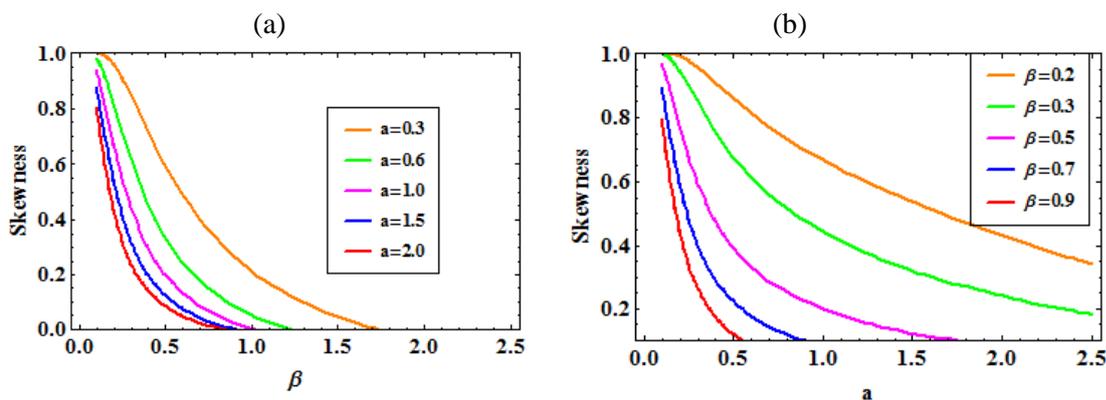


Figure 3. B skewness of the EWR distribution. (a) As function of β for certain values of a (b) As function of a for certain values of β at $\alpha = \lambda = 2$

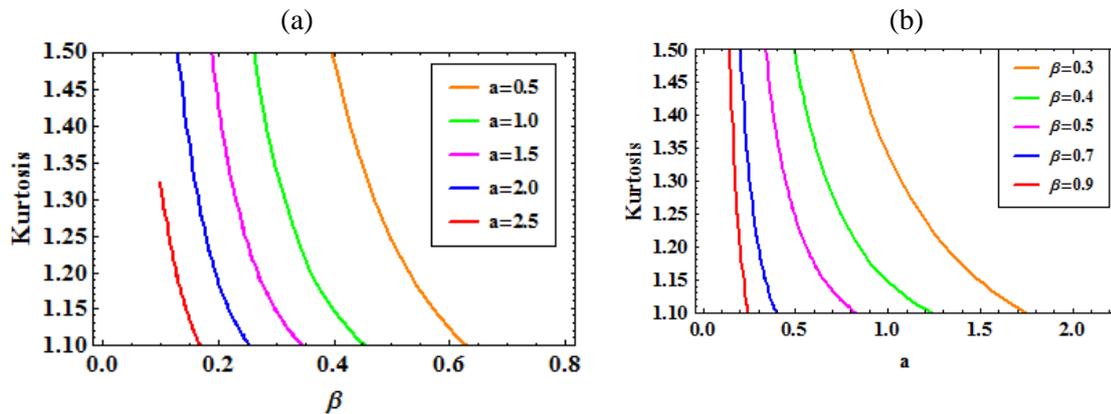


Figure 4. MO kurtosis of the EWR distribution. (a) As function of β for certain values of a (b) As function of a for certain values of β at $\alpha = \lambda = 2$

3.3. Moments

The r^{th} moment of EWR model is given by

$$\mu'_r = \int_0^\infty x^r f(x; \Psi) dx = \Gamma\left(\frac{r}{2} + 1\right) \sum_{k=1}^\infty \frac{\eta_{k,1}}{(k\lambda)^{\frac{r}{2}}}, \tag{16}$$

where $\eta_{k,m}$ is given in (10) and $\Gamma(\cdot)$ is the gamma function. In addition, the moment generating function of EWR model is given by

$$M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r) = \sum_{r=0}^\infty \sum_{k=1}^\infty \eta_{k,1} \frac{t^r \Gamma\left(\frac{r}{2} + 1\right)}{r!(k\lambda)^{\frac{r}{2}}}.$$

3.4. Mean Deviations

Two important dispersion measures are the mean deviation from the mean (μ) and the mean deviation from the median (M) denoted by δ_1 and δ_2 , are defined, respectively, by

$$\delta_1 = \int_0^\infty |x - \mu| f(x, \Psi) dx \quad \text{and} \quad \delta_2 = \int_0^\infty |x - M| f(x, \Psi) dx,$$

First, we obtain the incomplete moments, denoted by $\varphi_s(t)$, of the EWR distribution as follows, where $\varphi_s(t)$, defined by

$$\varphi_s(t) = \int_0^t x^s f(x) dx.$$

Using (11), $\varphi_s(t)$ will be as given

$$\varphi_s(t) = 2\lambda \sum_{k=1}^{\infty} \eta_{k,1} k \int_0^t x^{s+1} e^{-k\lambda x^2} dx = \sum_{k=1}^{\infty} \frac{\eta_{k,1}}{(k\lambda)^{\frac{s}{2}}} \Gamma\left(\frac{s}{2} + 1, k\lambda t^2\right), \quad (17)$$

where $\Gamma(s, t) = \int_0^t x^{s-1} e^{-x} dx$ is the lower incomplete gamma function.

Now, the mean deviations from μ and the mean deviation from M , respectively, are

$$\delta_1 = 2\mu F(\mu) - 2\varphi_1(\mu) \quad \text{and} \quad \delta_2 = \mu - 2\varphi_1(M),$$

where $\varphi_1(t)$ and M are given, in (17) and (15) respectively.

3.5. Residual Life Function

The n^{th} moment of the residual life (see [15-17]), is specified by

$$m_n(t) = E\left[(X-t)^n \mid X > t\right] = \frac{1}{R(t)} \int_t^{\infty} (x-t)^n f(x) dx.$$

Employing the pdf (11) and binomial expansion for $(x-t)^n$, then $m_n(t)$ will be

$$m_n(t) = \frac{1}{R(t; \Psi)} \sum_{k=1}^{\infty} \sum_{r=0}^n \frac{(-t)^{n-r} \eta_{k,1}}{(k\lambda)^{\frac{r}{2}}} \binom{n}{r} \gamma\left(\frac{r}{2} + 1, k\lambda t^2\right), \quad (18)$$

where $\gamma(s, t)$ is the upper incomplete gamma function. For, $n=1$, in (18), we obtain the so called mean residual life (MRL). The MRL as pointed out is the expected supplementary life length of an item that is alive at age t . Hence, if $n=1$ in (18), we get the MRL of EWR model.

On the other hand, n^{th} order moment of reversed residual life (RRL) is

$$M_n(t) = E\left[(t-X)^n \mid X \leq t\right] = \frac{1}{F(t)} \int_0^t (t-x)^n f(x) dx.$$

So, the moments of RRL for EWR model is

$$M_n(t) = \frac{1}{F(t, \Psi)} \sum_{k=1}^{\infty} \sum_{r=0}^n \frac{(-1)^r t^{n-r} \eta_{k,1}}{(k\lambda)^{\frac{r}{2}}} \binom{n}{r} \Gamma\left(\frac{r}{2} + 1, k\lambda t^2\right).$$

3.6. Inequality Measures

The Lorenz, Bonferroni and Zenga curves (see [18]) are obtained, respectively, as follows

$$L_F(x) = \frac{\varphi_1(x)}{E(X)} = \frac{2 \sum_{k=1}^{\infty} \frac{\eta_{k,1}}{(k\lambda)^{\frac{3}{2}}} \Gamma\left(\frac{3}{2}, k\lambda x^2\right)}{\sqrt{\pi} \sum_{k=1}^{\infty} \frac{\eta_{k,1}}{(k\lambda)^{\frac{3}{2}}}},$$

$$B_F(x) = \frac{L_F(x)}{F(x; \Psi)} = \frac{2 \sum_{k=1}^{\infty} \frac{\eta_{k,1}}{(k\lambda)^{\frac{3}{2}}} \Gamma\left(\frac{3}{2}, k\lambda x^2\right)}{\sqrt{\pi} \sum_{q=0}^{\infty} \sum_{k=1}^{\infty} \frac{\eta_{k,1} \eta_{q,1}}{(k\lambda)^{\frac{3}{2}}} e^{-q\lambda x^2}},$$

and

$$A_F(x) = 1 - \frac{\mu^-(x)}{\mu^+(x)},$$

where

$$\mu^-(x) = \frac{\varphi_1(x)}{F(x; \Psi)} = \frac{\sum_{k=1}^{\infty} \frac{\eta_{k,1}}{(k\lambda)^{\frac{3}{2}}} \Gamma\left(\frac{3}{2}, k\lambda x^2\right)}{\sum_{k=0}^{\infty} \eta_{k,1} e^{-k\lambda x^2}},$$

and

$$\mu^+(x) = \frac{\int_x^{\infty} t f(t; \Psi) dt}{R(x; \Psi)} = \frac{\sum_{k=1}^{\infty} \frac{\eta_{k,1}}{(k\lambda)^{\frac{3}{2}}} \gamma\left(\frac{3}{2}, k\lambda x^2\right)}{1 - \sum_{k=0}^{\infty} \eta_{k,1} e^{-k\lambda x^2}}.$$

3.7. Rényi and ε -Entropies

For a r.v. X , the Rényi entropy is given by

$$I_{\varepsilon}(X) = (1 - \varepsilon)^{-1} \log \int_0^{\infty} f(x)^{\varepsilon} dx, \quad \varepsilon > 0 \text{ and } \varepsilon \neq 1.$$

Using (12), the Rényi entropy of EWR distribution is given by

$$I_{\varepsilon}(X) = (1 - \varepsilon)^{-1} \log \left[(2\lambda x)^{\varepsilon} \sum_{k=1}^{\infty} \eta_{k,\varepsilon}^* e^{-k\lambda x^2} \right],$$

where $\eta_{k,\varepsilon}^*$ is defined in (13). Furthermore, the ε -entropy is defined by

$$H_\varepsilon(X) = (1-\varepsilon)^{-1} \log \left(1 - \int_0^\infty f(x)^\varepsilon dx \right), \quad \varepsilon > 0 \text{ and } \varepsilon \neq 1.$$

Therefore, the ε -entropy of EWR distribution is given by

$$H_\varepsilon(X) = (1-\varepsilon)^{-1} \log \left\{ 1 - (2\lambda x)^\varepsilon \sum_{k=1}^\infty \eta_{k,\varepsilon}^* e^{-k\lambda x^2} \right\}.$$

3.8. Order Statistics

Let $X_{(1)}, \dots, X_{(n)}$ be the order statistics (Os) of a random sample of size n from EWR distribution, then the pdf of the s^{th} Os, is given (see [19]) by

$$f_{X_{(s)}}(x) = \frac{f(x)}{B(s, n-s+1)} \sum_{v=0}^{n-s} (-1)^v \binom{n-s}{v} [F(x)]^{v+s-1}, \quad (19)$$

where $B(\dots)$ is the beta function. By substituting (9) and (11) into (19) and replacing m with $v+s-1$, we have

$$f_{X_{(s)}}(x) = \frac{2\lambda x}{B(s, n-s+1)} \sum_{v=0}^{n-s} \sum_{k_1=1}^\infty \sum_{k_2=0}^\infty \zeta_{v,s,k_1,k_2} e^{-\lambda(k_1+k_2)x^2}, \quad (20)$$

where $\zeta_{v,s,k_1,k_2} = k_1 (-1)^v \binom{n-s}{v} \eta_{k_2, v+s-1} \eta_{k_1, 1}$.

In addition, from (20), the r^{th} moment of $X_{(s)}$ is given by

$$E(X_{(s)}^r) = \frac{\Gamma((r/2)+1)}{\lambda^{\frac{r}{2}} B(s, n-s+1)} \sum_{v=0}^{n-s} \sum_{k_1=1}^\infty \sum_{k_2=0}^\infty \frac{\zeta_{v,s,k_1,k_2}}{(k_1+k_2)^{\frac{r}{2}+1}}.$$

4. CHARACTERIZATIONS

Here, we provide certain characterizations of EWR distribution. However, these characterizations (CHs) are based on: (i) a relation between two truncated moments; (ii) the hrf ; (iii) the reversed hrf and (iv) conditional expectation of a function of the r.v. One of the benefits of CH (i) is that the cdf does not required to be in closed form. CHs in our study are provided in the following subsections.

4.1. Characterizations via Ratio of Truncated Moments

We offer CHs of EWR distribution in terms of a simple relationship between two truncated moments. This CH result employs a theorem due to [20] (see Theorem 1- Appendix A). Note that the result holds also when the interval H is not closed. However, as previously mentioned, it can be also applied when the cdf does not have a closed form. According to [21], this CH is stable in the sense of weak convergence.

Proposition 4.1. Let $X : \Omega \rightarrow (0, \infty)$ is a continuous r.v. and let

$$q_1(x) = \left[1 - \exp(-\alpha(e^{\lambda x^2} - 1)^\beta) \right]^{1-\alpha}, \quad \text{and} \quad q_2(x) = q_1(x) \exp(-\alpha(e^{\lambda x^2} - 1)^\beta),$$

for $x > 0$. The r.v. X has pdf (6) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2} \exp(-\alpha(e^{\lambda x^2} - 1)^\beta), \quad x > 0.$$

Proof. Let X be a r.v. with pdf (6), then

$$(1 - F(x)) E[q_1(X) | X \geq x] = a \exp(-\alpha(e^{\lambda x^2} - 1)^\beta), \quad x > 0,$$

$$(1 - F(x)) E[q_2(X) | X \geq x] = \frac{1}{2} \exp(-2\alpha(e^{\lambda x^2} - 1)^\beta), \quad x > 0.$$

and

$$\eta(x)q_1(x) - q_2(x) = -\frac{1}{2} q_1(x) \exp(-\alpha(e^{\lambda x^2} - 1)^\beta) < 0 \quad \text{for } x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \alpha\beta x e^{\lambda x^2} (e^{\lambda x^2} - 1)^{\beta-1} \quad x > 0.$$

And

$$s'(x) = \alpha (e^{\lambda x^2} - 1)^{\beta-1} \quad x > 0.$$

According to Theorem 1, then X has pdf (6).

Corollary 4.1. Let $X : \Omega \rightarrow (0, \infty)$ be a continuous r.v. and let $q_1(x)$ be as in Proposition 4.1. The pdf of X is (6) if and only if there exist functions $q_2(x)$ and η defined in Theorem 1 verifying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \alpha\beta x e^{\lambda x^2} (e^{\lambda x^2} - 1)^{\beta-1} \quad x > 0.$$

The general solution of the differential equation in Corollary 4.1 is

$$\eta(x) = \exp(-\alpha(e^{\lambda x^2} - 1)^\beta) \left[-\int \alpha\beta x e^{\lambda x^2} (e^{\lambda x^2} - 1)^{\beta-1} \times \exp(-\alpha(e^{\lambda x^2} - 1)^\beta) (q_1(x))^{-1} q_2(x) + D \right],$$

where D be a constant. Note that a set of functions verifying the above differential equation is provided in Proposition 4.1 for $D = 0$. Also, it pay attention to note that there are other triplets $(q_1(x), q_2(x), \eta)$ verifying the conditions of Theorem 1.

4.2. Characterization via Hazard Function

It is known that the hrf, h_F , of a twice differentiable distribution function, F , verifies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F}{h_F} - h_F(x).$$

For various univariate continuous distributions, this is the only CH available in terms of the hrf. The following CH create a non-trivial CH of EWR distribution, when $a=1$, which is not of the above trivial form.

Proposition 4.2. Let $X : \Omega \rightarrow (0, \infty)$ be a continuous r.v. The pdf of X for $a=1$, is (6) if and only if its hrf $h_F(x)$ verifies the differential equation

$$h'_F(x) - x^{-1}h'_F(x) = 4\alpha\beta\lambda^2 x^2 e^{\lambda x^2} (e^{\lambda x^2} - 1)^{\beta-1} (2e^{\lambda x^2} - 1), \quad x > 0.$$

Proof. If X has pdf (6), then the above differential equation holds, then for $x > 0$,

$$\frac{d}{dx} \{x^{-1}h'_F(x)\} = 2\alpha\beta\lambda \frac{d}{dx} \left\{ e^{\lambda x^2} (e^{\lambda x^2} - 1)^{\beta-1} \right\},$$

from which, we obtain

$$h_F(x) = 2\alpha\beta\lambda x e^{\lambda x^2} (e^{\lambda x^2} - 1)^{\beta-1}, \quad x > 0,$$

which is the hrf of EWR distribution for $a=1$.

4.3. Characterizations on the basis of the Reversed hrf

The reversed hrf, r_F , of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in \text{support of } F.$$

Proposition 4.3. Let $X : \Omega \rightarrow (0, \infty)$ be a continuous r.v. The r.v. X has pdf (6) if and only if its reversed hrf r_F verifies the following differential equation

$$r'_F(x) - x^{-1}r'_F(x) = 2\alpha\beta\lambda x \frac{d}{dx} \left\{ \frac{(e^{\lambda x^2} - 1)^{\beta-1} \exp\left\{\lambda x^2 - \alpha(e^{\lambda x^2} - 1)^\beta\right\}}{1 - \exp\left\{-\alpha(e^{\lambda x^2} - 1)^\beta\right\}} \right\}, \quad x > 0.$$

Proof: If X has pdf (6), then clearly the above differential equation holds. That is

$$\frac{d}{dx} \{x^{-1}r_F(x)\} = 2a\alpha\beta\lambda \frac{d}{dx} \left\{ \frac{(e^{\lambda x^2} - 1)^{\beta-1} \exp\{\lambda x^2 - \alpha(e^{\lambda x^2} - 1)^\beta\}}{1 - \exp\{-\alpha(e^{\lambda x^2} - 1)^\beta\}} \right\},$$

from which, we have, for $x > 0$

$$r_F(x) = \frac{2a\alpha\beta\lambda x (e^{\lambda x^2} - 1)^{\beta-1} \exp\{\lambda x^2 - \alpha(e^{\lambda x^2} - 1)^\beta\}}{1 - \exp\{-\alpha(e^{\lambda x^2} - 1)^\beta\}}.$$

4.4. Characterization via the Conditional Expectation of Certain Functions of the Random Variable

Here, we utilize a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(x)$. Proposition 4.4 is used to characterize EWR distribution, clearly, this proposition has previously appeared in [22].

Proposition 4.4. Let $X : \Omega \rightarrow (0, \infty)$ be a continuous r.v. with cdf F . Let $\psi(x)$ be a differentiable function on (c, e) with $\lim_{x \rightarrow e^-} \psi(x) = 1$. Then, for $\delta \neq 1$,

$$E[\psi(X) | X \leq x] = \delta \psi(x), \quad x \in (c, e).$$

Implies

$$\psi(x) = (F(x))^{\frac{1}{\delta-1}}, \quad x \in (c, e).$$

Remark 4.4. For $(c, e) = (0, \infty)$, $\psi(x) = 1 - \exp\{-\alpha(e^{\lambda x^2} - 1)^\beta\}$ and $\delta = \frac{a}{a+1}$, Proposition 4.4 provides a CH of EWR distribution.

5. MAXIMUM LIKELIHOOD ESTIMATION

ML procedure is the commonly employed method of estimation. The estimators that are obtained based on this procedure enjoy desirable asymptotic properties and therefore they are often utilized to obtain confidence intervals (CI) and test of statistical hypotheses. Suppose that x_1, \dots, x_n be an observed random sample from the EWR distribution with pdf (6). Then the log-likelihood function, denoted by $\ln \ell$, for the set of parameters $\Psi = (a, \alpha, \beta, \lambda)$ is

$$\begin{aligned} \ln \ell = & n \ln 2a + n \ln \alpha + n \ln \beta + n \ln \lambda + \sum_{i=1}^n \ln(x_i) + (\beta-1) \sum_{i=1}^n \ln(e^{\lambda x_i^2} - 1) \\ & + \lambda \sum_{i=1}^n x_i^2 - \alpha \sum_{i=1}^n (e^{\lambda x_i^2} - 1)^\beta + (a-1) \sum_{i=1}^n \ln \left[1 - \exp \left[-\alpha (e^{\lambda x_i^2} - 1)^\beta \right] \right]. \end{aligned} \quad (21)$$

By differentiating (21) partially with respect to unknown parameters, then, we have

$$\frac{\partial \ln \ell}{\partial a} = \frac{n}{a} + \sum_{i=1}^n \ln \left[1 - \exp \left[-\alpha \left(e^{\lambda x_i^2} - 1 \right)^\beta \right] \right],$$

$$\frac{\partial \ln \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \left(e^{\lambda x_i^2} - 1 \right)^\beta + (a-1) \sum_{i=1}^n \frac{\left(e^{\lambda x_i^2} - 1 \right)^\beta \exp \left[-\alpha \left(e^{\lambda x_i^2} - 1 \right)^\beta \right]}{1 - \exp \left[-\alpha \left(e^{\lambda x_i^2} - 1 \right)^\beta \right]},$$

$$\frac{\partial \ln \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n \ln \left(e^{\lambda x_i^2} - 1 \right) - \alpha \sum_{i=1}^n \left(e^{\lambda x_i^2} - 1 \right)^\beta \ln \left(e^{\lambda x_i^2} - 1 \right)$$

$$+ \alpha (a-1) \sum_{i=1}^n \frac{\left(e^{\lambda x_i^2} - 1 \right)^\beta \exp \left[-\alpha \left(e^{\lambda x_i^2} - 1 \right)^\beta \right] \ln \left(e^{\lambda x_i^2} - 1 \right)}{1 - \exp \left[-\alpha \left(e^{\lambda x_i^2} - 1 \right)^\beta \right]},$$

and

$$\frac{\partial \ln \ell}{\partial \lambda} = \frac{n}{\lambda} + (\beta-1) \sum_{i=1}^n \frac{x_i^2 e^{\lambda x_i^2}}{e^{\lambda x_i^2} - 1} + \alpha \beta (a-1) \sum_{i=1}^n \frac{x_i^2 e^{\lambda x_i^2} \left(e^{\lambda x_i^2} - 1 \right)^{\beta-1} e^{-\alpha \left(e^{\lambda x_i^2} - 1 \right)^\beta}}{1 - \exp \left[-\alpha \left(e^{\lambda x_i^2} - 1 \right)^\beta \right]} + \sum_{i=1}^n x_i^2$$

$$- \alpha \beta \sum_{i=1}^n x_i^2 \left(e^{\lambda x_i^2} - 1 \right)^{\beta-1} e^{\lambda x_i^2}.$$

The ML estimates are obtained numerically as solutions of the above nonlinear equations after setting them equal zeros. The observed information matrix is required in case of interval estimation. This 4×4 symmetric matrix, denoted by $I(\Psi)$, is derived by considering the second-order derivatives of the $\ln \ell$, with respect to the parameters. Under regularity conditions, the vector of the ML estimators of the parameters, denoted by $\hat{\Psi}^T = (\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\lambda})^T$, approximately possesses a 4-variate normal distribution with the mean Ψ^T and the variance-covariance matrix $I^{-1}(\Psi)$, where $I^{-1}(\Psi)$ is the inverse matrix of $I(\Psi)$. This property of ML estimators can be used to obtain approximate CI for the model parameters. We note that the elements of $I(\Psi)$ depend on the parameters and the ML estimators may substitute for the corresponding parameters to evaluate $I(\Psi)$.

6. SIMULATION ILLUSTRATION

A numerical investigation is established to examine the behavior of ML estimates (MLE) for EWR model. For different sample sizes, measures like; biases and mean square errors (MSEs) are calculated to evaluate the performance of estimates. A numerical study is done using Mathematica (9) software. The steps of simulation procedure is listed as follows

- We generate 10000 from EWR distribution of sizes; $n = 30, 50, 75$ and 100 .
- Certain values of parameters $(a, \alpha, \beta, \lambda)$ are chosen as Set 1=(2,2,0.5,1.5), Set 2 =(1.5,2,0.5,1.5), Set 3=(2,2,0.75,1.5) and Set 4= (2,2,0.9,1.5).
- MLE of the parameters $\hat{a}, \hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ are calculated for each n and for all sets.

- Calculate the biases and MSE for each n .
- Result outcomes are listed in Tables 2 and 3.

Table 2. Parameter estimates of EWR distribution for different sample sizes

n	Parameter	Mean	Bias	MSE	Mean	Bias	MSE
		Set1=(2,2,0.5,1.5)			Set2=(1.5,2,0.5,1,5)		
30	a	2.0661	0.0661	0.1521	1.5538	0.0538	0.0881
	α	2.1413	0.1413	0.4373	2.0987	0.0987	0.2603
	β	0.5307	0.0307	0.0208	0.5328	0.0328	0.0212
	λ	1.5525	0.0525	0.0908	1.5381	0.0381	0.0627
50	a	2.0365	0.0365	0.0893	1.5290	0.0290	0.0505
	α	2.0864	0.0863	0.2112	2.0626	0.0626	0.1374
	β	0.5179	0.0179	0.0112	0.5185	0.0185	0.0111
	λ	1.5359	0.0359	0.0545	1.5267	0.0266	0.0380
75	a	2.0287	0.0287	0.0571	1.5195	0.0195	0.0322
	α	2.0495	0.0495	0.1165	2.0367	0.0367	0.0850
	β	0.5128	0.0128	0.0069	0.5115	0.0115	0.0068
	λ	1.5215	0.0215	0.0344	1.5154	0.0154	0.0250
100	a	2.0217	0.0217	0.0426	1.5152	0.0152	0.0232
	α	2.0344	0.0344	0.0825	2.0279	0.0279	0.0611
	β	0.5093	0.0093	0.0049	0.5091	0.0091	0.0048
	λ	1.5154	0.0154	0.0252	1.5121	0.0121	0.0188

Table 3. Parameter estimates of EWR distribution for different sample sizes

n	Parameter	Mean	Bias	MSE	Mean	Bias	MSE
		set 3=(2,2,0.75,1.5)			set 4=(2,2,0.9,1.5)		
30	a	2.0729	0.0729	0.1626	2.0599	0.0599	0.1557
	α	2.1398	0.1398	0.4224	2.1500	0.1500	0.3931
	β	0.8028	0.0528	0.0506	0.9570	0.0570	0.0681
	λ	1.6183	0.1183	0.2426	1.6936	0.1936	0.4927
50	a	2.0409	0.0409	0.0910	2.0381	0.0381	0.0882
	α	2.0828	0.0828	0.2004	2.0816	0.0816	0.1914
	β	0.7789	0.0289	0.0250	0.9336	0.0336	0.0357
	λ	1.5713	0.0713	0.1257	1.6026	0.1026	0.2039
75	a	2.0257	0.0257	0.0569	2.0312	0.0312	0.0595
	α	2.0511	0.0511	0.1159	2.0478	0.0478	0.1170
	β	0.7681	0.0181	0.0156	0.9243	0.0243	0.0238
	λ	1.5447	0.0447	0.0753	1.5613	0.0613	0.1175
100	a	2.0214	0.0214	0.0420	2.0187	0.0187	0.0414
	α	2.0340	0.0340	0.0835	2.0413	0.0413	0.0839
	β	0.7636	0.0136	0.0112	0.9171	0.0171	0.0162
	λ	1.5304	0.0304	0.0548	1.5508	0.0508	0.0826

The values in Tables 2 and 3 show that, in general, the MSE for the estimates of the parameters $\hat{a}, \hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ decreases as n increases.

7. TYPE-II CENSORING ESTIMATION

Consider $X_{(1)}, \dots, X_{(n)}$ be sample of size n whose life time's have the EWR distribution are set on a life test and the test is stopped at certain number of failure r before all n items have failed. The log likelihood function, denoted by $\ln \ell_1$, of TII censoring sample is,

$$\begin{aligned} \ln \ell_1 = & n \ln \frac{n!}{(n-r)!} + n \ln a + n \ln \alpha + n \ln \beta + n \ln \lambda + n \ln 2 + \sum_{i=1}^r \ln(x_i) \\ & + (\beta - 1) \sum_{i=1}^r \ln(e^{\lambda x_i^2} - 1) + \lambda \sum_{i=1}^r x_i^2 - \alpha \sum_{i=1}^r (e^{\lambda x_i^2} - 1)^\beta \\ & + (a - 1) \sum_{i=1}^r \ln \left[1 - \exp \left[-\alpha (e^{\lambda x_i^2} - 1)^\beta \right] \right] + (n - r) \ln \left(1 - \left[1 - \exp \left[-\alpha (e^{\lambda x_i^2} - 1)^\beta \right] \right]^a \right). \end{aligned}$$

Note that; for simplicity we write x_i instead of $x_{(i)}$. The ML estimators of a, α, β and λ based on TII censored sample are obtained by solving the non-linear equations which we get it from first derivatives of $\ln \ell_1$, with respect to parameters a, α, β and λ .

Table 4. Parameter estimator of EWR distribution under TII censored samples

			Mean	Bias	MSE	Mean	Bias	MSE
n	Parameter	$X_{(r)}$	set 1 (2,2,0.5,1.5)			set 2 (1.5,2,0.5,1.5)		
50	a	50%	2.230	0.230	0.673	1.631	0.131	0.306
		80%	2.079	0.079	0.156	1.553	0.053	0.095
	α	50%	2.174	0.174	0.692	1.631	0.157	0.475
		80%	2.097	0.097	0.299	1.553	0.088	0.214
	β	50%	0.572	0.072	0.068	0.561	0.061	0.060
		80%	0.526	0.026	0.014	0.525	0.025	0.016
	λ	50%	1.853	0.353	0.163	1.728	0.228	0.077
		80%	1.933	0.433	0.222	1.885	0.385	0.166
100	a	50%	2.075	0.075	0.200	1.567	0.067	0.137
		80%	2.035	0.035	0.066	1.512	0.012	0.036
	α	50%	2.109	0.109	0.252	2.073	0.073	0.155
		80%	2.047	0.047	0.118	2.044	0.044	0.078
	β	50%	0.525	0.025	0.016	0.531	0.031	0.021
		80%	0.512	0.012	0.006	0.507	0.007	0.006
	λ	50%	1.856	0.356	0.145	1.716	0.216	0.058
		80%	1.941	0.441	0.211	1.893	0.393	0.162
150	a	50%	2.058	0.058	0.111	1.547	0.047	0.073
		80%	2.029	0.029	0.048	1.517	0.017	0.024
	α	50%	2.049	0.049	0.137	2.035	0.035	0.098
		80%	2.031	0.031	0.077	2.018	0.018	0.056
	β	50%	0.518	0.018	0.009	0.520	0.020	0.011
		80%	0.510	0.010	0.004	0.507	0.007	0.004

	λ	50%	1.845	0.345	0.130	1.709	0.209	0.051
		80%	1.944	0.444	0.209	1.889	0.389	0.157

From Table 4, we see that as n increases the MSE tend to decrease.

8. APPLICATIONS

Two real data sets are utilized to show that the EWR model outperforms some of other-models. These models are the EW Weibull (EWW) that introduced in [14], WR, Rayleigh Rayleigh (RR) and exponential Rayleigh (ER) dsitributions.

We use formal goodness-of-fit criteria for the purpose of comparison. These criteria are as the minus value of $\ln \ell$, multiplied by 2, Kolmogorov-Smirnov (K-S) test and its p -value, Akaike information criterion (AIC), the corrected AIC (CAIC), the Bayesian information criterion (BIC) and the Hannan-Quinn information criterion (HQIC). The better fitness corresponds to largest p -value and the smallest values for all other measures.

Example 1.

The first data represents 30 successive values of March precipitation (in inches) for Minneapolis/St Paul (see [23]). Table 5 provides the MLEs of parameters of the EWR, EWW, WR, RR and ER distributions and their standard errors (S.Es). While, the goodness-of-fit measures are mentioned in Table 6.

Table 5. The MLEs and the S.Es (in parentheses) of the model parameters for the first data set

Model	MLEs and SEs				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$	
EWR	3.322 (0.02659)	4.918 (0.164)	0.468 (0.086)	0.042 (0.00629)	
EWW	78.61 (0.14836)	79.35 (0.561)	20.486 (0.131)	0.624 (0.024)	0.014 (0.148)
WR		3.888 (0.08635)	0.792 (0.015)	0.043 (0.02)	
RR		100.351 (0.21297)		0.014 (0.00229)	
ER		9.151 (0.09666)		0.025 (0.00306)	

We notice that the EWR distribution yields a better fit than other-models. It has largest p -value and the smallest values for the other measures. Plots of the fitted densities and cdfs of the considered distributions are represented in Figure 5. As seen from this figure, that the EWR distribution outperforms the WR, RR and ER distributions.

Table 6. The goodness-of-fit statistics for the first data set

Distribution	$-2\ln \ell$	AIC	CAIC	BIC	HQIC	K-S	p -value
EWR	114.703	122.703	124.303	122.132	126.329	0.062	0.99985
EWW	129.022	139.022	141.522	136.407	141.263	0.113	0.83514
WR	127.973	133.973	134.896	132.405	135.318	0.088	0.97545
RR	243.627	247.627	248.071	247.341	249.439	0.426	0.00004
ER	139.679	143.679	144.124	142.634	144.576	0.12	0.78339

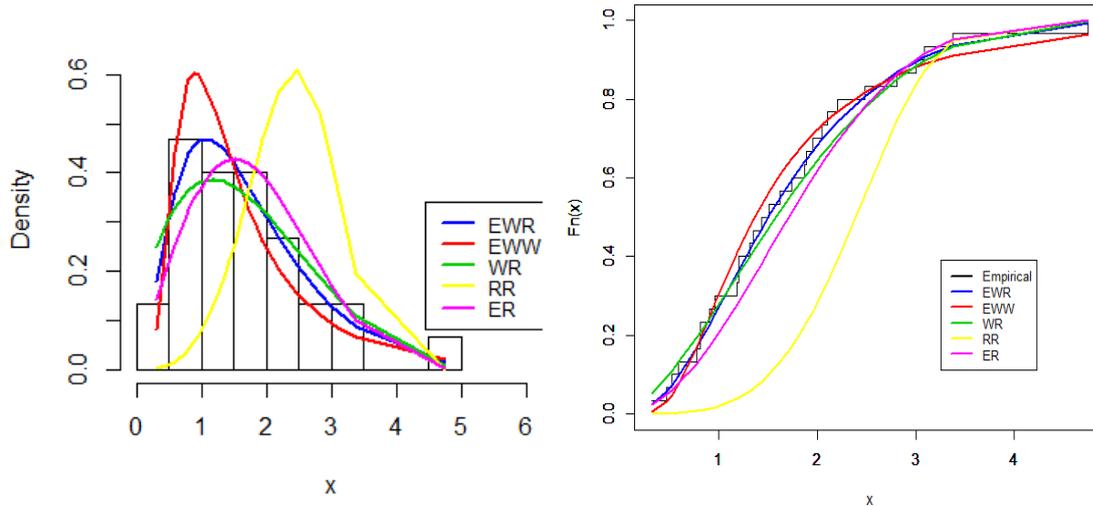


Figure 5. The fitted cdfs (left) and pdfs (right) of the EWR, WR, RR and ER distributions of first data

Example 2.

We consider the second data which represent the survival times (in days) of 72 guinea pigs infected with different amount of virulent tubercle bacilli (see [24]). Table 7 contains the MLEs of the parameters of the considered models and their corresponding S.Es. While Table 8 displays the goodness-of-fit statistics.

Table 7. The MLEs and the S.Es (in parentheses) of the parameters for the second data set

Model	MLEs				
	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$
EWR	1.415 (0.0078)	3.17 (0.105)	0.648 (0.057)	0.051 (0.00405)	
EWW	115.001 (0.0787)	125.918 (0.361)	19.125 (0.085)	0.61 (0.015)	0.013 (0.093)
WR	-	4.9 (0.05586)	0.809 (0.015)	0.031 (0.011)	
RR	-	25.417 (0.13753)	-	0.022 (0.00147)	
ER		12.812 (0.06241)		0.016 (0.00197)	

Table 8. The goodness-of-fit statistics for the second data set

Distribution	$-2\ln \ell$	AIC	CAIC	BIC	HQIC	K-S	p-value
EWR	257.833	265.833	266.43	265.263	269.459	0.097	0.50777
EWW	302.076	312.076	312.972	311.363	316.608	0.134	0.14963
WR	265.356	271.356	271.709	270.928	274.075	0.117	0.27833
RR	552.88	556.88	557.054	556.595	558.693	0.506	0.0000
ER	321.478	325.478	325.652	325.193	327.291	0.146	0.09292

We observe the EWR distribution gives the best fit among the other selected models. The fitted pdfs and cdfs plots of mentioned distributions for the second data are represented in Figure 6 for the purpose visual comparison. According to these plots, we observe that the EWR distribution is preferable than other distributions.

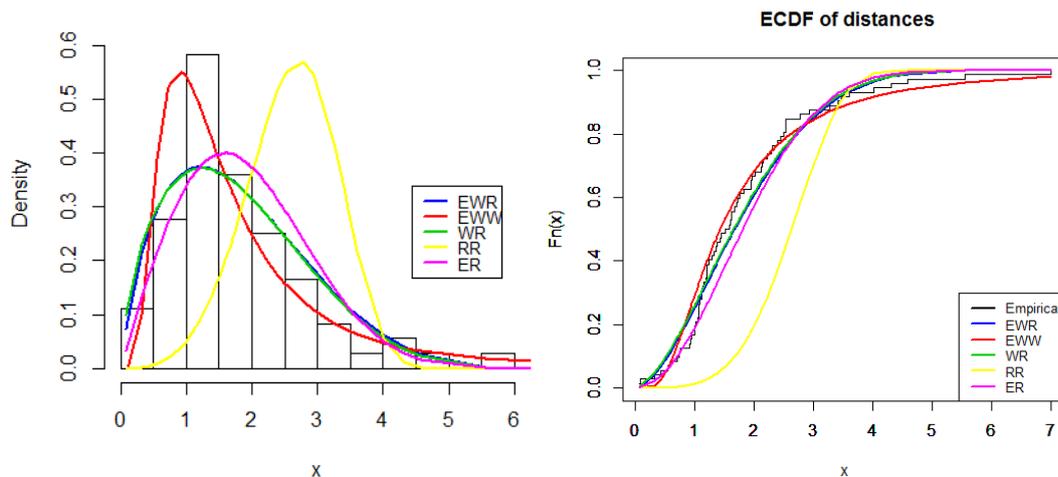


Figure 6. The fitted cdfs (left) and pdfs (right) of the EWR, WR, RR and ER distributions for the second data set

9. CONCLUSION

A four-parameter exponentiated Weibull Rayleigh distribution is introduced. Some certain properties of the proposed distribution are discussed. This model includes some new special distributions. The practical importance of the EWR model is clarified through two applications, where the EWR distribution yields the best fit among the else regarded models. We conclude that the EWR distribution can be regarded as a quite flexible gadget for modelling a big group of life data. We trust that the new distribution may serve as rather model to else life distributions and has application in lots of scientific fields.

APPENDIX A

Theorem 1. Suppose that (Ω, F, P) is a given probability space and let $H = [a, b]$ is an interval for some $d < b$ ($a = -\infty, b = \infty$ might as well be allowed): Let $X : \Omega \rightarrow H$ be a continuous r.v. with cdf F . Also, suppose q_1 and q_2 are two real functions defined on H , where

$$E[q_2(X)|X \geq x] = E[q_1(X)|X \geq x]\eta(x), \quad x \in H,$$

is defined with some real function η . Further, let $q_1, q_2 \in C^1(H), \eta \in C^2(H)$, and F is twice continuously differentiable and strictly monotone function on the set H . Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H . Then F is uniquely determined by the functions q_1, q_2 and η , specifically,

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)} \right| e^{-s(u)} du,$$

where the function is a solution of differential equation $s'(x) = \frac{\eta'q_1}{\eta q_1 - q_2}$. and C is the normalization constant, such that $\int_H dF = 1$.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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