Erzincan Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
2019, 12(2), 863-873
ISSN: 1307-9085, e-ISSN: 2149-4584
Araştırma Makalesi

Erzincan University
Journal of Science and Technology
2019, 12(2), 863-873
DOI:10.18185/erzifbed.494358
Research Article

(s,t)-Modified Pell Sequence and Its Matrix Representation

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Geliş / Received: 10/12/2018, Kabul / Accepted: 13/05/2019

Abstract

In this paper, we investigate a generalization of modified Pell sequence, which is called (s,t)-modified Pell sequence. By considering this sequence, we define the matrix sequence whose elements are (s,t)-modified Pell numbers. Furthermore, we obtain Binet formulas, the generating functions and some sums formulas of these sequences. Finally, we give some relationships between (s,t)-Pell and (s,t)-modified Pell matrix sequences.

Keywords: Modified Pell sequence, (s,t)-modified Pell sequence, (s,t)-modified Pell matrix sequence.

(s,t)-Modified Pell Dizisi ve Matris Gösterimi

Öz

Bu çalışmada, (s,t)-modified Pell dizisi olarak adlandırılan modified Pell dizisinin bir genellemesini araştırdık. Bu diziyi dikkate alarak elemanları (s,t)-modified Pell sayıları olan matris dizisini tanımladık. Ayrıca, bu dizilerin üreteç fonksiyonlarını, Binet formüllerini ve bazı toplam formüllerini elde ettik. Son olarak, (s,t)-Pell ve (s,t)-modified Pell matris dizileri arasında bazı ilişkiler verdik.

Anahtar Kelimeler: Modified Pell dizisi, (s,t)-modified Pell dizisi, (s,t)-modified Pell matris dizisi.

1. Introduction

Recently, there are many recursive sequences that have been discussed in the literatures. The well-known examples of these sequences are Fibonacci, Lucas, Pell, Pell-Lucas and modified Pell. For $n \geq 2$, the classical Pell $\{P_n\}$, Pell-Lucas $\{Q_n\}$ and modified Pell $\{q_n\}$ sequences are defined by $P_n = 2P_{n-1} + P_{n-2}$, $Q_n = 2Q_{n-1} + Q_{n-2}$ and $q_n = 2q_{n-1} + q_{n-2}$, with initial terms $P_0 = 0, P_1 = 1, Q_0 = Q_1 = 2$ and $q_0 = q_1 = 1$, respectively. For more particular information about Fibonacci, Lucas, Pell, Pell-Lucas and modified Pell sequences can be found in references of (Benjamin et al., 2008), (Bicnell, 1975), (Horadam and Filipponi, 1995), (Koshy, 2001), (Stakhov and Rozin, 2006).

Moreover, Fibonacci, Lucas, Pell and Pell-Lucas were generalized by many authors. We refer the reader to (Civciv and Türkmen, 2008a and b), (Güleç and Taşkara, 2012). For example, Güleç and Taşkara (2012) introduced (s,t)-Pell and (s,t)-Pell-Lucas sequences as follow.

Definition 1.1. For any real numbers s,t and $n \geq 2$, let $s^2 + t > 0, s > 0$ and $t \neq 0$. Then

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the (s,t)-Pell sequence $\{P_n(s,t)\}_{n\in N}$ and the (s,t)-Pell-Lucas sequence $\{Q_n(s,t)\}_{n\in N}$ are defined respectively by

$$P_n(s,t) = 2sP_{n-1}(s,t) + tP_{n-2}(s,t),$$

$$Q_n(s,t) = 2sQ_{n-1}(s,t) + tQ_{n-2}(s,t),$$

with initial conditions $P_0(s,t) = 0$, $P_1(s,t) = 1$ and $Q_0(s,t) = 2, Q_1(s,t) = 2s$. (see [Güleç and Taşkara, 2012, Definition 1]) On the other hand, they introduced the matrix sequences which have elements of (s,t)-Pell and (s,t)-Pell-Lucas sequences.

Definition 1.2. For any real numbers s,t and $n \geq 2$, let $s^2 + t > 0, s > 0$ and $t \neq 0$. Then the (s,t)-Pell matrix sequence $\{MP_n(s,t)\}_{n\in N}$ and the (s,t)-Pell-Lucas matrix sequence $\{MQ_n(s,t)\}_{n\in N}$ are defined respectively by

$$MP_n(s,t) = 2sMP_{n-1}(s,t) + tMP_{n-2}(s,t),$$

$$MQ_n(s,t) = 2sMQ_{n-1}(s,t) + tMQ_{n-2}(s,t)$$

with initial conditions

$$MP_0(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, MP_1(s,t) = \begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix}$$

and

$$MQ_0(s,t) = \begin{pmatrix} 2s & 2 \\ 2t & -2s \end{pmatrix},$$

$$MQ_1(s,t) = \begin{pmatrix} 4s^2 + 2t & 2s \\ 2st & 2t \end{pmatrix}.$$

(see [Güleç and Taşkara, 2012, Definition 4])

Moreover, Pell-Circulant sequences were studied by many authors. We refer the reader to (Deveci, 2016), (Deveci and Shannon, 2017).

In this study, we define and study the (s,t)-modified Pell sequence. Then, by using this sequence, we also define (s,t)-modified Pell matrix sequence. We give generating functions, Binet formulas and sum formulas of them. In the last of the study, we investigate the relationships between (s,t)-Pell and (s,t)-modified Pell matrix sequences.

2. The (s,t)-Modified Pell Sequence

Firstly, we give the fundamental definition and properties for this sequence.

Definition 2.1. For any real numbers s,t and $n \geq 2$, let $s^2 + t > 0, s > 0$ and $t \neq 0$. Then the (s,t)-modified Pell sequence $\{q_n(s,t)\}_{n\in N}$ is defined recursively by

$$q_n(s,t) = 2sq_{n-1}(s,t) + tq_{n-2}(s,t) \qquad (1)$$

with initial values $q_0(s,t) = 1, q_1(s,t) = s$.

Theorem 2.1. *The generating function for $q_n(s,t)$ is*

$$M(x) = \frac{1 - sx}{1 - 2sx - tx^2}.$$

Proof. The generating function $M(x)$ has the form

$$M(x) = \sum_{n=0}^{\infty} q_n(s,t)x^n = q_0(s,t) + q_1(s,t)x + q_2(s,t)x^2 + \cdots + q_n(s,t)x^n + \cdots.$$

Since the characteristic equation of (1) $x^2 - 2sx - t = 0$, we get

$$(1 - 2sx - tx^2)M(x) = (1 - 2sx - tx^2)[q_0(s,t) + q_1(s,t)x + q_2(s,t)x^2 + \cdots$$

$$+q_n(s,t)x^n+\cdots]$$

$$=q_0(s,t)+[q_1(s,t)-2sq_0(s,t)]x.$$

By Definition 2.1, we have

$$(1-2sx-tx^2)M(x)=1-sx.$$

Thus, we get

$$M(x)=\frac{1-sx}{1-2sx-tx^2}$$

which is the desired result. ■

Theorem 2.2. *The nth term of the (s,t)-modified Pell sequence is given by*

$$q_n(s,t)=s\left(\frac{\alpha^n+\beta^n}{\alpha+\beta}\right)$$

where α and β are the roots of the equation $x^2-2sx-t=0$.

Proof. The solution of Eq. (1) is given by

$$q_n(s,t)=c\alpha^n+d\beta^n \qquad (2)$$

for some coefficients c and d.

Then, by using the initial values $q_0(s,t)=1$ and $q_1(s,t)=s$, we can write

$$c=d=\frac{1}{2}.$$

Therefore, by using c and d in Eq. (2), we obtain

$$q_n(s,t)=s\left(\frac{\alpha^n+\beta^n}{\alpha+\beta}\right).$$

■

Theorem 2.3. *For $2s+t\neq 1$, the sum of the first n terms of $q_n(s,t)$ is*

$$\sum_{i=1}^{n}q_i(s,t)=\frac{1}{2s+t-1}[q_{n+1}(s,t)+tq_n(s,t)-s-t].$$

Proof. From the recurrence relation of $q_i(s,t)$, we have

$$q_{i-1}(s,t)=\frac{1}{2s}q_i(s,t)-\frac{t}{2s}q_{i-2}(s,t). \qquad (3)$$

Applying Eq. (3), we deduce that

$$q_1(s,t)=\frac{1}{2s}q_2(s,t)-\frac{t}{2s}q_0(s,t)$$

$$q_2(s,t)=\frac{1}{2s}q_3(s,t)-\frac{t}{2s}q_1(s,t)$$

$$q_3(s,t)=\frac{1}{2s}q_4(s,t)-\frac{t}{2s}q_2(s,t)$$

$$\vdots$$

$$q_n(s,t)=\frac{1}{2s}q_{n+1}(s,t)-\frac{t}{2s}q_{n-1}(s,t).$$

Then, we get

$$\sum_{i=1}^{n} q_i(s,t) = \frac{1}{2s}\sum_{i=2}^{n+1} q_i(s,t) - \frac{t}{2s}\sum_{i=0}^{n-1} q_i(s,t).$$

After necessary calculations, we obtain

$$\sum_{i=1}^{n} q_i(s,t) = \frac{1}{2s+t-1}[q_{n+1}(s,t) + tq_n(s,t) - s - t].$$

■

We now give the relationship between the (s,t)-modified Pell and (s,t)-Pell sequences is given by the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$q_n(s,t) = sP_n(s,t) + tP_{n-1}(s,t).$$

Proof. By using the Binet formula of the (s,t)-modified Pell sequence, we get

$$sP_n(s,t) + tP_{n-1}(s,t) = s\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + t\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right)$$

$$= \frac{s(\alpha^n - \beta^n) - (\beta\alpha^n - \alpha\beta^n)}{\alpha - \beta}$$

$$= \frac{\alpha^n(s-\beta) + \beta^n(\alpha - s)}{\alpha - \beta}.$$

Since $\alpha + \beta = 2s$ and $\alpha - \beta = 2\sqrt{s^2+t}$, we obtain

$$sP_n(s,t) + tP_{n-1}(s,t) = s\left(\frac{\alpha^n + \beta^n}{\alpha + \beta}\right)$$

$$= q_n(s,t).$$

The proof is completed. ■

In the rest of paper, we denoted by q_n instead of $q_n(s,t)$ the (s,t)-modified Pell sequence.

3. The (s,t)-Modified Pell Matrix Sequence

Definition 3.1. For any real numbers s,t and $n \geq 2$, let $s^2 + t > 0, s > 0$ and $t \neq 0$. Then the (s,t)-modified Pell matrix sequence $\{Mq_n(s,t)\}_{n\in N}$ is defined recursively by

$$Mq_n(s,t) = 2sMq_{n-1}(s,t) + tMq_{n-2}(s,t)$$

with initial terms

$$Mq_0(s,t) = \begin{pmatrix} s & 1 \\ t & -s \end{pmatrix},$$

$$Mq_1(s,t) = \begin{pmatrix} 2s^2+t & s \\ st & t \end{pmatrix}.$$

Theorem 3.1. *For positive integer n, we have*

$$Mq_n(s,t) = \begin{pmatrix} q_{n+1} & q_n \\ tq_n & tq_{n-1} \end{pmatrix}.$$

Proof. We use the principle of mathematical induction on n. Since

$$Mq_1(s,t) = \begin{pmatrix} q_2 & q_1 \\ tq_1 & tq_0 \end{pmatrix} = \begin{pmatrix} 2s^2+t & s \\ st & t \end{pmatrix},$$

the statement is true for $n=1$.

Assume that it is true for $n=k$, that is,

$$Mq_k(s,t)=\begin{pmatrix} q_{k+1} & q_k \\ tq_k & tq_{k-1}\end{pmatrix}.$$

Then, we show that the formula holds for $k+1$. Indeed,

$Mq_{k+1}(s,t)=2sMq_k(s,t)+tMq_{k-1}(s,t)$

$$=2s\begin{pmatrix} q_{k+1} & q_k \\ tq_k & tq_{k-1}\end{pmatrix}+t\begin{pmatrix} q_k & q_{k-1} \\ tq_{k-1} & tq_{k-2}\end{pmatrix}$$

$$=\begin{pmatrix} 2sq_{k+1}+tq_k & 2sq_k+tq_{k-1} \\ 2stq_k+t^2q_{k-1} & 2stq_{k-1}+t^2q_{k-2}\end{pmatrix}$$

$$=\begin{pmatrix} q_{k+2} & q_{k+1} \\ tq_{k+1} & tq_k\end{pmatrix}.$$

Thus the formula works for $k+1$. Hence, the proof is completed. ∎

In the rest of paper, we denoted by Mq_n instead of $Mq_n(s,t)$ the (s,t)-modified Pell matrix sequence.

Theorem 3.2. *The generating function of the (s,t)-modified Pell sequence is*

$$N(x)=\frac{1}{1-2sx-tx^2}\left[\begin{pmatrix} s & 1 \\ t & -s\end{pmatrix}+x\begin{pmatrix} t & -s \\ -st & 2s^2+t\end{pmatrix}\right].$$

Proof. Let $N(x)$ be the generating function of the (s,t)-modified Pell matrix sequence. Then we write

$$N(x)=\sum_{n=0}^{\infty}Mq_n=Mq_0+Mq_1x+Mq_2x^2+\cdots+Mq_nx^n+\cdots,$$

$$2sxN(x)=2sMq_0x+2sMq_1x^2+2sMq_2x^3+\cdots+2sMq_{n-1}x^n+\cdots,$$

and

$$tx^2N(x)=tMq_0x^2+tMq_1x^3+tMq_2x^4+\cdots+tMq_{n-2}x^n+\cdots.$$

Thus, we have

$$N(x)(1-2sx-tx^2)=Mq_0+(Mq_1-2sMq_0)x.$$

Therefore, we obtain

$$N(x)=\frac{1}{1-2sx-tx^2}\left[\begin{pmatrix} s & 1 \\ t & -s\end{pmatrix}+x\begin{pmatrix} t & -s \\ -st & 2s^2+t\end{pmatrix}\right].$$

The proof is completed. ∎

The Binet formula of (s,t)-modified Pell matrix sequence can be given as the following theorem.

Theorem 3.3. *For $n\geq 0$, we have*

$$Mq_n=\left(\frac{Mq_1-\beta Mq_0}{\alpha-\beta}\right)\alpha^n-\left(\frac{Mq_1-\alpha Mq_0}{\alpha-\beta}\right)\beta^n.$$

Proof. The general term of (s,t)-modified Pell matrix sequence can be written in the following form:

$$Mq_n = c\alpha^n + d\beta^n$$

where c and d are coefficients.

Then, by using the initial terms $n = 0, 1$, we get

$$c = \frac{Mq_1 - \beta Mq_0}{\alpha - \beta} \text{ and } d = -\frac{Mq_1 - \alpha Mq_0}{\alpha - \beta}.$$

Hence, the Binet formula for Mq_n is obtained as follow;

$$Mq_n = \left(\frac{Mq_1 - \beta Mq_0}{\alpha - \beta}\right)\alpha^n - \left(\frac{Mq_1 - \alpha Mq_0}{\alpha - \beta}\right)\beta^n.$$

■

Now, we investigate some sums formulas of the (s,t)-modified Pell matrix sequence.

Theorem 3.4. *For $2s + t \neq 1$, the sum of the (s,t)-modified Pell matrix sequence is given as*

$$\sum_{i=1}^{n} Mq_i = \frac{1}{2s + t - 1}[Mq_{n+1} + tMq_n - Mq_2 + (2s-1)Mq_1].$$

Proof. By definition of (s,t)-modified Pell matrix sequence recurrence relation, we have

$$-tMq_{i-2} = -Mq_i + 2sMq_{i-1}. \quad (4)$$

From the Eq. (4), we can write

$$-tMq_1 = -Mq_3 + 2sMq_2,$$

$$-tMq_2 = -Mq_4 + 2sMq_3,$$

$$-tMq_3 = -Mq_5 + 2sMq_4,$$

$$\vdots$$

$$-tMq_n = -Mq_{n+2} + 2sMq_{n+1}.$$

Then, we get

$$-t\sum_{i=1}^{n} Mq_i = (2s-1)(Mq_3 + Mq_4 + \cdots + Mq_{n+1}) - Mq_{n+2} + 2sMq_2.$$

Therefore, after necessary calculations, we obtain

$$\sum_{i=1}^{n} Mq_i = \frac{1}{2s + t - 1}[Mq_{n+1} + tMq_n - Mq_2 + (2s-1)Mq_1].$$

■

Theorem 3.5. *Let us consider $s^2 + t > 0, s > 0$ and $t \neq 0$. Then, we have*

$$\sum_{k=0}^{n} \frac{Mq_k}{x^k} = \frac{1}{x^2 - 2sx - t}[xMq_1 + (x^2 - 2sx)Mq_0]$$

$$- \frac{1}{x^n(x^2 - 2sx - t)}[xMq_{n+1} + tMq_n].$$

Proof. From Theorem 3.3, we have

$$\sum_{k=0}^{n}\frac{Mq_k}{x^k}=\left(\frac{Mq_1-\beta Mq_0}{\alpha-\beta}\right)\sum_{k=0}^{n}\left(\frac{\alpha}{x}\right)^k-\left(\frac{Mq_1-\alpha Mq_0}{\alpha-\beta}\right)\sum_{k=0}^{n}\left(\frac{\beta}{x}\right)^k.$$

Considering the definition of a geometric sequence, we get

$$\sum_{k=0}^{n}\frac{Mq_k}{x^k}=\left(\frac{Mq_1-\beta Mq_0}{\alpha-\beta}\right)\left[\frac{x^{n+1}-\alpha^{n+1}}{x^{n+1}\left(\frac{x-\alpha}{x}\right)}\right]-\left(\frac{Mq_1-\alpha Mq_0}{\alpha-\beta}\right)\left[\frac{x^{n+1}-\beta^{n+1}}{x^{n+1}\left(\frac{x-\beta}{x}\right)}\right]$$

$$=\frac{1}{x^n(x^2-2sx-t)}\left[\left(\frac{Mq_1-\beta Mq_0}{\alpha-\beta}\right)(x^{n+1}-\alpha^{n+1})(x-\beta)\right.$$

$$\left.-\left(\frac{Mq_1-\alpha Mq_0}{\alpha-\beta}\right)(x^{n+1}-\beta^{n+1})(x-\alpha)\right]$$

$$=\frac{1}{x^n(x^2-2sx-t)}\left[\left(\frac{Mq_1-\beta Mq_0}{\alpha-\beta}\right)(x^{n+2}-x^{n+1}\beta-x\alpha^{n+1}+\alpha^{n+1}\beta)\right.$$

$$\left.-\left(\frac{Mq_1-\alpha Mq_0}{\alpha-\beta}\right)(x^{n+2}-x^{n+1}\alpha-x\beta^{n+1}+\beta^{n+1}\alpha)\right].$$

Since $\alpha+\beta=2s$, $\alpha.\beta=-t$ and by using the Binet formula of (s,t)-modified Pell matrix sequence, we write

$$\sum_{k=0}^{n}\frac{Mq_k}{x^k}=\frac{1}{x^n(x^2-2sx-t)}[x^{n+2}Mq_0-x^{n+1}(2sMq_0-Mq_1)-xMq_{n+1}-tMq_n].$$

After necessary calculations, we obtain

$$\sum_{k=0}^{n}\frac{Mq_k}{x^k}=\frac{1}{x^2-2sx-t}[xMq_1+(x^2-2sx)Mq_0]$$

$$-\frac{1}{x^n(x^2-2sx-t)}[xMq_{n+1}+tMq_n]$$

which is the desired result. ■

Theorem 3.6. *For $j>m$, we have*

$$\sum_{i=0}^{n}Mq_{mi+j}=\frac{Mq_{mn+m+j}+(-t)^mMq_{j-m}-(-t)^mMq_{mn+j}-Mq_j}{\alpha^m+\beta^m-(-t)^m-1}.$$

Proof. Let us consider $E = \frac{Mq_1 - \beta Mq_0}{\alpha-\beta}$ and $F = \frac{Mq_1 - \alpha Mq_0}{\alpha-\beta}$. Then, by using the Binet formula of (s,t)-modified Pell matrix sequence we can write

$$\sum_{i=0}^{n} Mq_{mi+j} = \sum_{i=0}^{n} \frac{E\alpha^{mi+j} - F\beta^{mi+j}}{\alpha-\beta}$$

$$= \frac{1}{\alpha-\beta}\left(E\alpha^j \sum_{i=0}^{n} \alpha^{mi} - F\beta^j \sum_{i=0}^{n} \beta^{mi}\right)$$

$$= \frac{1}{\alpha-\beta}\left[E\alpha^j\left(\frac{1-\alpha^{mn+m}}{1-\alpha^m}\right) - F\beta^j\left(\frac{1-\beta^{mn+m}}{1-\beta^m}\right)\right].$$

After necessary calculations, we get

$$\sum_{i=0}^{n} Mq_{mi+j} = \frac{Mq_{mn+m+j} + (-t)^m Mq_{j-m} - (-t)^m Mq_{mn+j} - Mq_j}{\alpha^m + \beta^m - (-t)^m - 1}$$

which completes the proof. ■

Theorem 3.7. *For $n \geq 0$ and $n \geq r$,*

$$Mq_{n-r}Mq_{n+r} = {Mq_n}^2.$$

Proof. Let $C = Mq_1 - \beta Mq_0$ and $D = Mq_1 - \alpha Mq_0$. Then, by using the Binet formula of (s,t)-modified Pell matrix sequence, we can write

$$Mq_{n-r}Mq_{n+r} - {Mq_n}^2 = \left(\frac{C\alpha^{n-r} - D\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{C\alpha^{n+r} - D\beta^{n+r}}{\alpha-\beta}\right) - \left(\frac{C\alpha^{n} - D\beta^{n}}{\alpha-\beta}\right)^2.$$

After necessary calculations, we obtain

$$Mq_{n-r}Mq_{n+r} - {Mq_n}^2 = \frac{CD\alpha^{n-r}\beta^{n-r}(2\alpha^r\beta^r - \alpha^{2r} - \beta^{2r})}{(\alpha-\beta)^2}.$$

Since $CD = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, we have $Mq_{n-r}Mq_{n+r} = {Mq_n}^2$ as required. ■

Theorem 3.8. *For $n \in Z^+$, we have*

$(\boldsymbol{i})$ $Mq_n = sMP_n + tMP_{n-1}$,

$(\boldsymbol{ii})$ $2Mq_1MP_n = MP_{n+2} + tMP_n$.

Proof. $(\boldsymbol{i})$ If we consider the right-hand side of equation (i) and use Theorem 6 (a) in Güleç and Taşkara, 2012, we obtain

$$sMP_n + tMP_{n-1} = s\begin{pmatrix} P_{n+1}(s,t) & P_n(s,t) \\ tP_n(s,t) & tP_{n-1}(s,t) \end{pmatrix} + t\begin{pmatrix} P_n(s,t) & P_{n-1}(s,t) \\ tP_{n-1}(s,t) & tP_{n-2}(s,t) \end{pmatrix}.$$

From Theorem 2.4 and Theorem 3.1,

$$sMP_n + tMP_{n-1} = \begin{pmatrix} q_{n+1} & q_n \\ tq_n & tq_{n-1} \end{pmatrix} = Mq_n,$$

as required.

$(\boldsymbol{ii})$ Let us consider the left-hand side of equation (ii) and use Theorem 6 (a) in Güleç and Taşkara, 2012, we obtain

$$2Mq_1MP_n = \begin{pmatrix} 4s^2+2t & 2s \\ 2st & 2t \end{pmatrix} \begin{pmatrix} P_{n+1}(s,t) & P_n(s,t) \\ tP_n(s,t) & tP_{n-1}(s,t) \end{pmatrix}.$$

After necessary matrix operations, we get

$$2Mq_1MP_n = \begin{pmatrix} P_{n+3}(s,t) & P_{n+2}(s,t) \\ tP_{n+2}(s,t) & tP_{n-1}(s,t) \end{pmatrix} + t\begin{pmatrix} P_{n+1}(s,t) & P_n(s,t) \\ tP_n(s,t) & tP_{n-1}(s,t) \end{pmatrix}$$

$$= MP_{n+2} + tMP_n.$$

So, the proof is completed. ■

Theorem 3.9. *For $m, n \in Z^+$, we have*

$(\boldsymbol{i})$ $Mq_mMq_n = Mq_nMq_m$,

$(\boldsymbol{ii})$ $Mq_1MP_n = MP_nMq_1 = Mq_{n+1}$,

$(\boldsymbol{iii})$ $Mq_nMP_1 = MP_1Mq_n = Mq_{n+1}$,

$(\boldsymbol{iv})$ $2Mq_n = MP_{n+1} + tMP_{n-1}$.

Proof. Theorem can be proven by using Theorem 3.1, Theorem 3.8, matrix multiplication and recurrence relation of (s,t)-Pell and (s,t)-modified Pell sequences. ■

Theorem 3.10. *For $m, n \in Z^+$, we have*

$$Mq_{n+1}^m = Mq_1^mMP_{mn}.$$

Proof. By induction on m, we can prove the theorem.

For $m = 1$, the proof is clear by Theorem 3.9. We now assume that the theorem holds for $m = k$, that is,

$$Mq_{n+1}^k = Mq_1^kMP_{kn}. \quad (5)$$

Finally, we have to show that the theorem is true for $m = k+1$. We now multiply the Eq. (5) with Mq_{n+1} on both sides. Then, we get

$$Mq_{n+1}^{k+1} = Mq_1^kMP_{kn}Mq_{n+1}.$$

Also, from Theorem 3.9 (ii), we write

$$Mq_{n+1}^{k+1} = Mq_1^kMP_{kn}Mq_1MP_n$$

$$= Mq_1^kMq_1MP_{kn}MP_n$$

$$= Mq_1^{k+1}MP_{kn}MP_n.$$

Since $MP_mMP_n = MP_nMP_m = MP_{m+n}$ (see [Güleç and Taşkara, 2012, Proposition 9]) where MP_n is the nth (s,t)-Pell matrix sequence we obtain

$$Mq_{n+1}^{k+1} = Mq_1^{k+1}MP_{(k+1)n}.$$

So, we obtain the desired result. ■

Theorem 3.11. *For $m, n \in Z^+$, we have*

$(\boldsymbol{i})$ $Mq_mMq_n = (s^2+t)MP_{m+n}$,

$(\boldsymbol{ii})$ $Mq_mMq_{n+1} = Mq_{m+1}Mq_n$.

Proof. $(\boldsymbol{i})$ By using Theorem 3.8 (i) we write

$$Mq_mMq_n = (sMP_m + tMP_{m-1})(sMP_n + tMP_{n-1})$$

$$= s^2MP_mMP_n + stMP_mMP_{n-1} + stMP_{m-1}MP_n + t^2MP_{m-1}MP_{n-1}.$$

Since $MP_mMP_n = MP_nMP_m = MP_{m+n}$ (see [Güleç and Taşkara, 2012, Proposition 9]) where MP_n is the nth (s,t)-Pell matrix sequence we obtain

$$Mq_mMq_n = s^2MP_{m+n} + 2stMP_{m+n-1} + t^2MP_{m+n-2}$$

$$= s^2MP_{m+n} + t(2sMP_{m+n-1} + tMP_{m+n-2}).$$

Then, from the recurrence relation (s,t)-Pell sequence, we get

$$Mq_mMq_n = s^2MP_{m+n} + tMP_{m+n}$$

$$= (s^2+t)MP_{m+n}.$$

$(\boldsymbol{ii})$ From the first identity, it is seen that

$$Mq_mMq_{n+1} = (s^2+t)MP_{m+n+1}$$

$$= (s^2+t)MP_{m+1+n}$$

$$= Mq_{m+1}Mq_n.$$

■

Theorem 3.12. *For $m,n \in Z^+$, let $m+n \geq 4$. Then we have*

$$MP_nMP_m = 4sMq_{m+n-2} + t^2MP_{m+n-4}.$$

Proof. If we consider the right-hand side of equation and use Theorem 3.8 (i), we get

$$4sMq_{m+n-2} + t^2MP_{m+n-4} = 4s(sMP_{m+n-2} + tMP_{m+n-3}) + t^2MP_{m+n-4}$$

$$= 4s^2MP_{m+n-2} + 4stMP_{m+n-3} + t^2MP_{m+n-4}$$

$$= 4s^2MP_{m+n-2} + 2stMP_{m+n-3} + 2stMP_{m+n-3} + t^2MP_{m+n-4}$$

$$= 2s(2sMP_{m+n-2} + tMP_{m+n-3}) + t(2sMP_{m+n-3} + tMP_{m+n-4}).$$

Then, from the recurrence of relation (s,t)-Pell sequence, we have

$$4sMq_{m+n-2} + t^2MP_{m+n-4} = 2sMP_{m+n-1} + tMP_{m+n-2}$$

$$= MP_{m+n}.$$

Since $MP_mMP_n = MP_nMP_m = MP_{m+n}$ (see [Güleç and Taşkara, 2012, Proposition 9]) where MP_n is the nth (s,t)-Pell matrix sequence we obtain

$$4sMq_{m+n-2} + t^2MP_{m+n-4} = MP_nMP_m.$$

■

Theorem 3.13. *For $n \in Z^+$, we have*

$$MP_nMq_{n+1} = Mq_{2n+1}.$$

Proof. We use the principle of mathematical induction on n. It can be seen clearly for $n=1$.

Now, assume that the theorem holds for $n=k$, that is

$$MP_kMq_{k+1} = Mq_{2k+1}. \quad (6)$$

Finally, we have to show that the theorem is true for $n=k+1$. We now multiply the Eq. (6) with MP_1 on both sides. Then, we get

$$MP_1MP_kMq_{k+1}MP_1 = MP_1Mq_{2k+1}MP_1.$$

Since $MP_mMP_n = MP_nMP_m = MP_{m+n}$ (see [Güleç and Taşkara, 2012, Proposition 9]) where MP_n is the nth (s,t)-Pell matrix sequence and by using Theorem 3.9 (iii), we obtain

$$MP_{k+1}Mq_{k+2} = Mq_{2k+2}MP_1$$

$$MP_{k+1}Mq_{k+2} = Mq_{2k+3}.$$

■

4. Results

In this study, we introduce (s,t)-modified Pell sequence. By using this sequence, we define (s,t)-modified Pell matrix sequence. We also give some results, such as generating functions, Binet formulas and summation formulas for these sequences. Moreover, we obtain some relationships between (s,t)-Pell and (s,t)-modified Pell matrix sequences.

5. References

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