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Research Article

3-Fibonacci Polynomials in The Family of Fibonacci Numbers

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Abstract

In this study, we define 3-Fibonacci Polynomials by using terms of a new family of Fibonacci numbers was given by Mikkawy and Sogabe (2010). We give some important properties of the polynomial. Then, we compare the polynomials with known Fibonacci polynomial. We expressed these polynomials using the Fibonacci polynomials. Furthermore, we prove some theorems related to the polynomials.

Keywords: Fibonacci Numbers, Fibonacci Polynomials, Generalized Fibonacci Polynomials.

Fibonacci Sayılarının Ailesinde 3-Fibonacci Polinomları

Öz

Bu çalışmamızda, Mikkawy and Sogabe (2010)' nin vermiş olduğu Fibonacci sayılarının yeni ailesi kullanılarak $3-$Fibonacci polinomları tanımlandı. Bu polinomun sahip olduğu bazı önemli özellikler gösterildi. Daha sonra elde ettiğimiz polinomlar ile bilinen Fibonacci polinomlar karşılaştırıldı. Elde edilen yeni polinom Fibonacci polinomu türünden ifade edildi. Ayrıca, bu polinomlarla ilgili bazı teoremlerin ifade ve ispatı verildi.

Anahtar Kelimeler: Fibonacci Sayıları, Fibonacci Polinomları, Genelleştirilmiş Fibonacci Polinomları.

1. Introduction

Fibonacci numbers are of great importance in the study of many fields such as mathematics, physics, biology, statistics, etc. (Koshy, 2001; Nalli and Haukkanen, 2009; Özkan et. al., 2017). Also, Fibonacci numbers and polynomials have numerous applications in various fields. Hoggatt and Lind (1968) obtained some more identities for Fibonacci and Lucas Polynomials. Falcon and Plaza (2009) presented the derivatives of Fibonacci polynomials as convolution of the Fibonacci polynomials and proved some relations for the derivatives of Fibonacci polynomials Then, Mikkawy and Sogabe (2010) defined a new family $k-$Fibonacci numbers.

The remainder of the paper is structured as follows. In the second section, some fundamental concepts about the 3-Fibonacci numbers and Fibonacci polynomials are given. In the third section, we define the 3-Fibonacci polynomials and show that the relationship between Pascal's triangle and the coefficient of the 3-Fibonacci polynomials. In addition, we present 3-Fibonacci polynomials using the Fibonacci polynomials and give the derivatives of the 3-Fibonacci polynomials.

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2. Materials and Methods

The Fibonacci numbers F_n are defined by

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2$$

with the initial conditions $F_1 = 1$ and $F_0 = 0$. It is well known that Binet's formula, is the explicit formula for Fibonacci numbers, is defined by

$$F_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}), \qquad n = 0,1,2,\ldots$$

where $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$.

The Fibonacci polynomials $F_n(x)$, studied by Catalan, are defined by the recurrence relation.

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad n \geq 1$$

with initial condition $F_1(x) = 1, F_2(x) = x$. In particular, for $x = 1$, the Fibonacci polynomial $F_n(1)$ is the n^{th} Fibonacci number F_n, and for $x = 2$, the Fibonacci polynomial $F_n(2)$ is the n^{th} Pell number P_n.

Ivei (1972) generated the Fibonacci polynomials by a matrix

$$\mathbb{Q}_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathbb{Q}_2^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix}.$$

Hoggat and Bicknell (1973) introduced that if one writes Pascal's triangle in left-justified form, one obtains the Fibonacci numbers by adding the elements along the rising diagonals. The coefficients of the Fibonacci polynomials are the elements of left-justified form of Pascal's triangle. That is,

$$F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1}.$$

Definition 2.1. Let n and $k \neq 0$ be positive integer, then there exist unique numbers m and r such that $n = mk + r$ $(0 \leq r < k)$. The generalized k-Fibonacci numbers $F_n^{(k)}$ are defined by

$$F_n^{(k)} = \frac{1}{(\sqrt{5})^k}(\alpha^{m+2} - \beta^{m+2})^r(\alpha^{m+1} - \beta^{m+1})^{k-r}, \qquad n = mk + r$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ (Mikkawy and Sogabe, 2010).

The first few k-Fibonacci numbers $F_n^{(3)}$ is as follows:

$$\{F_n^{(3)}\} = \{1,1,1,2,4,8,12,18,27,45,75,\ldots\}.$$

The relation between generalized k −Fibonacci and Fibonacci numbers is

$$F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r, \qquad n = mk + r.$$

Also, it is apparent that, when $k = 1$ in last equation, $m = n$ and $r = 0$, $F_n^{(1)} = F_n$.

Definition 2.2. The 2-Fibonacci polynomials are defined by

$$F_{n+1}^{(2)}(x) = xF_n^{(2)}(x) + F_{n-1}^{(2)}(x)$$

where $F_0^{(2)}(x) = 1$ and $F_1^{(2)}(x) = 1$ (Özkan et al., 2018).

The first few 2-Fibonacci polynomials and their coefficients array are showed below.

Table 1. The first few Fibonacci polynomials and their coefficients array

2-Fibonacci Polynomials	Coefficient Array
$F_1^{(2)}(x) = 1$	1
$F_2^{(2)}(x) = 1 + x$	1 1
$F_3^{(2)}(x) = 1 + x + x^2$	1 1 1
$F_4^{(2)}(x) = 1 + 2x + x^2 + x^3$	1 2 1 1
$F_5^{(2)}(x) = 1 + 2x + 3x^2 + x^3 + x^4$	1 2 3 1 1
$F_6^{(2)}(x) = 1 + 3x + 3x^2 + 4x^3 + x^4 + x^5$	1 3 3 4 1 1
$F_7^{(2)}(x) = 1 + 3x + 6x^2 + 4x^3 + 5x^4 + x^5 + x^6$	1 3 6 4 5 1 1
$F_8^{(2)}(x) = 1 + 4x + 6x^2 + 10x^3 + 5x^4 + 6x^5 + x^6 + x^7$	1 4 6 10 5 6 1 1
$F_9^{(2)}(x) = 1 + 4x + 10x^2 + 10x^3 + 15x^4 + 6x^5 + 7x^6 + x^7 + x^8$	1 4 10 10 15 6 7 1 1
$F_{10}^{(2)}(x) = 1 + 5x + 10x^2 + 20x^3 + 15x^4 + 21x^5 + 7x^6 + 8x^7 + x^8 + x^9$	1 5 10 20 15 21 7 8 1 1

Theorem 2.1 The 2-Fibonacci polynomials are given by

$$F_n^{(2)}(x) = F_{n-1}(x) + F_n(x), \quad n \geq 1.$$

Proof. For $n = 1$, the claim is true, since

$$F_1^{(2)}(x) = 1 = F_0(x) + F_1(x)$$

Suppose that it is true for $n = k$, that is

$$F_k^{(2)}(x) = F_{k-1}(x) + F_k(x).$$

We should show that the claim is true for $n = k + 1$. We have

$$F_{k+1}^{(2)}(x) = xF_k^{(2)}(x) + F_{k-1}^{(2)}(x)$$
$$F_{k+1}^{(2)}(\mathrm{x}) = \mathrm{x}[F_{k-1}(x) + F_k(x)] + [F_{k-2}(x) + F_{k-1}(x)]$$
$$F_{k+1}^{(2)}(x) = [xF_{k-1}(x) + F_{k-2}(x)] + [xF_k(x) + F_{k-1}(x)]$$
$$F_{k+1}^{(2)}(x) = F_k(x) + F_{k+1}(x)$$

□

3. Main Results

Definition 3.1. The 3-Fibonacci polynomials are defined by

$$F_{n+1}^{(3)}(x) = xF_n^{(3)}(x) + F_{n-1}^{(3)}(x), \qquad n \geq 2$$

where $F_1^{(3)}(x) = 1$, $F_2^{(3)}(x) = 1$.

The first few 3-Fibonacci polynomials and their coefficients array are showed below.

Table 2. The first few 3-Fibonacci polynomials and their coefficients array

3-Fibonacci Polynomials	Coefficient Array
$F_1^{(3)}(x) = 1$	1
$F_2^{(3)}(x) = 1$	1
$F_3^{(3)}(x) = 1 + x$	1 1
$F_4^{(3)}(x) = 1 + x + x^2$	1 1 1
$F_5^{(3)}(x) = 1 + 2x + x^2 + x^3$	1 2 1 1
$F_6^{(3)}(x) = 1 + 2x + 3x^2 + x^3 + x^4$	1 2 3 1 1
$F_7^{(3)}(x) = 1 + 3x + 3x^2 + 4x^3 + x^4 + x^5$	1 3 3 4 1 1
$F_8^{(3)}(x) = 1 + 3x + 6x^2 + 4x^3 + 5x^4 + x^5 + x^6$	1 3 6 4 5 1 1
$F_9^{(3)}(x) = 1 + 4x + 6x^2 + 10x^3 + 5x^4 + 6x^5 + x^6 + x^7$	1 4 6 10 5 6 1 1
$F_{10}^{(3)}(x) = 1 + 4x + 10x^2 + 10x^3 + 15x^4 + 6x^5 + 7x^6 + x^7 + x^8$	1 4 10 10 15 6 7 1 1
$F_{11}^{(3)}(x) = 1 + 5x + 10x^2 + 20x^3 + 15x^4 + 21x^5 + 7x^6 + 8x^7 + x^8 + x^9$	1 5 10 20 15 21 7 8 1 1

Theorem 3.1. The 3-Fibonacci polynomials are given by

$$F_n^{(3)}(x) = F_{n-2}(x) + F_{n-1}(x), \;\; n > 1.$$

Proof. For $n = 2$, the claim is true, since

$$F_2^{(3)}(x) = 1 = F_0(x) + F_1(x)$$

Suppose that it is true for $n = k$, that is

$$F_k^{(3)}(x) = F_{k-2}(x) + F_{k-1}(x).$$

We should show that the claim is true for $n = k + 1$. We have

$$\begin{aligned} F_{k+1}^{(3)}(x) &= xF_k^{(3)}(x) + F_{k-1}^{(3)}(x) \\ &= x\big(F_{k-2}(x) + F_{k-1}(x)\big) + \big(F_{k-3}(x) + F_{k-2}(x)\big) \\ &= xF_{k-2}(x) + F_{k-3}(x) + xF_{k-1}(x) + F_{k-2}(x) \\ &= F_{k-1}(x) + F_k(x). \end{aligned}$$

□

We introduce Binet's formulas for 3-Fibonacci polynomials. The solutions of the quadratic equation $t^2 - xt - 1 = 0$ are

$$\alpha(x) = \frac{x + \sqrt{x^2 + 4}}{2} \;\; \beta(x) = \frac{x - \sqrt{x^2 + 4}}{2}\ .$$

Notice that $\alpha(1) = \alpha$ and $\beta(1) = \beta$; $\alpha(2) = 1 + \sqrt{2}$ and $\beta(2) = 1 - \sqrt{2}$ are the characteristic roots of the Pell recurrence relation $x^2 - 2x - 1 = 0$. We can show that

$$F_n^{(3)}(x) = \frac{\alpha^{n-2}(x)(\alpha(x)+1) - \beta^{n-2}(x)(\beta(x)+1)}{\alpha(x) - \beta(x)}.$$

The 3-Fibonacci polynomials are obtained by a matrix $\mathbb{Q}_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$. Since $F_n^{(3)}(x) = F_{n-2}(x) + F_{n-1}(x)$,

$$\mathbb{Q}_2^n = \mathbb{Q}_2^{n-1} + \mathbb{Q}_2^{n-2}$$

$$= \begin{pmatrix} F_{n+1}^{(3)}(x) & F_n^{(3)}(x) \\ F_n^{(3)}(x) & F_{n-1}^{(3)}(x) \end{pmatrix}.$$

The following interesting determinant identity is obtained,

$$det \begin{pmatrix} F_{n+1}^{(3)}(x) & F_n^{(3)}(x) \\ F_n^{(3)}(x) & F_{n-1}^{(3)}(x) \end{pmatrix} = (-1)^n x.$$

Theorem 3.2. For $n > 2$, we have

$$F_n^{(3)}(x) = 1 + x \sum_{i=1}^{n-2} F_i(x).$$

Proof. Using the recurrence relation

$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$,

$$\sum_{i=1}^{n} F_{i+1}(x) = x \sum_{i=1}^{n} F_i(x) + \sum_{i=1}^{n} F_{i-1}(x)$$

$$x \sum_{i=1}^{n} F_i(x) = \sum_{i=1}^{n} F_{i+1}(x) - \sum_{i=1}^{n} F_{i-1}(x)$$

$$= F_{n+1}(x) + F_n(x) - F_1(x) - F_0(x)$$

Since $F_0(x) = 0$ and $F_{n+2}^{(3)}(x) = F_{n+1}(x) + F_n(x)$, it follows that

$$x \sum_{i=1}^{n} F_i(x) = F_{n+2}^{(3)}(x) - 1.$$

We then get from last equation

$$F_n^{(3)}(x) = 1 + x \sum_{i=1}^{n-2} F_i(x).$$

□

Theorem 3.3. More generally, the sum of the elements along the diagonal beginning at row n is $F_n^{(3)}(x)$; that is,

$$F_n^{(3)}(x) = \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j-2}{j} x^{n-2j-2} + \binom{n-j-3}{j} x^{n-2j-3}.$$

Proof. We have

$$\frac{y}{1 - xy - y^2} = \sum_{n=0}^{\infty} F_n(x) y^n.$$

But

$$\frac{1}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} \left[\sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \binom{n-j}{j} (2t)^{n-2j} \right] z^n$$

$$U_n(t) = \sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \binom{n-j}{j} (2t)^{n-2j}$$

Let $z = iy$ and $t = x/2i$, where $i^2 = -1$, then

$$\frac{1}{1-xy-y^2} = \sum_{n=0}^{\infty} i^n U_n(x/2i) y^n.$$

Multiplying the last equation by y, we get

$$\frac{y}{1-xy-y^2} = \sum_{n=0}^{\infty} i^n U_n(x/2i) y^{n+1}.$$

It follows that

$$F_{n+1}(x) = i^n U_n(x/2i)$$

$$= i^n \sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \binom{n-j}{j} (x/i)^{n-2j}$$

$$= \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j}{j} x^{n-2j}$$

and

$$F_n(x) = \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j-1}{j} (2t)^{n-2j-1}.$$

Since $F_n^{(3)}(x) = F_{n-2}(x) + F_{n-1}(x)$, we get that

$$F_n^{(3)}(x) = \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j-3}{j} x^{n-2j-3} + \binom{n-j-2}{j} x^{n-2j-2}.$$

□

Theorem 3.4. The derivative of 3-Fibonacci polynomial is

$$(F_n^{(3)}(x))' = \sum_{i=1}^{n-2} F_i(x).\big(F_{n-i-1}(x) + F_{n-i-2}(x)\big).$$

Proof. It is known that

$$F_n^{(3)}(x) = F_{n-1}(x) + F_{n-2}(x)$$

where $F_n'(x)$ denotes the derivative of $F_n(x)$ with respect to x and $n \geq 1$.

Computing the derivative both side of

$$(F_n^{(3)}(x))' = (F_{n-1}(x) + F_{n-2}(x))'$$

$$F_{n-1}'(x) = \sum_{i=1}^{n-1} F_i(x) F_{n-i}(x),$$

$$F_{n-2}'(x) = \sum_{i=1}^{n-2} F_i(x) F_{n-i-1}(x).$$

We get

$$(F_n^{(3)}(x))' = \sum_{i=1}^{n-2} F_i(x)\big(F_{n-i-1}(x) + F_{n-i-2}(x)\big)$$

since $F_0^{(3)}(x) = F_1^{(3)}(x) = F_2^{(3)}(x) = 1.$

□

Example: Deriving polynomial $F_6^{(3)}(x) = x^4 + x^3 + 3x^2 + 2x + 1$, it is obtained

$$\left(F_6^{(3)}(x)\right)' = 4x^3 + 3x^2 + 6x + 2\,.$$

Then, we have

$$\sum_{i=1}^{4} F_i(x)[F_{5-i}(x) + F_{4-i}(x)] = F_1(x)[F_4(x) + F_3(x)] + F_2(x)[F_3(x) + F_2(x)]$$
$$+F_3(x)[F_2(x) + F_1(x)] + F_4(x)[F_1(x) + F_0(x)]$$
$$= F_1(x)F_4(x) + F_1(x)F_3(x) + F_2(x)F_3(x) + F_2(x)F_2(x)$$
$$+F_3(x)F_2(x) + F_3(x)F_1(x) + F_4(x)F_1(x) + F_4(x)F_0(x)$$
$$= 2F_1(x)F_4(x) + 2F_1(x)F_3(x) + 2F_2(x)F_3(x) + F_2^2(x)$$
$$= 2.1.(x^3 + 2x) + 2 \cdot 1 \cdot (x^2 + 1) + 2 \cdot x \cdot (x^2 + 1) + x^2$$
$$= 2x^3 + 4x + 2x^2 + 2 + 2x^3 + 2x + x^2$$
$$= 4x^3 + 3x^2 + 6x + 2$$
$$= \left(F_6^{(3)}(x)\right)'.$$

□

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4. Results

In the present paper, we define 3-Fibonacci polynomials and give some identities between the 3-Fibonacci polynomials and Fibonacci polynomials. Also, we introduce some properties of the 3-Fibonacci polynomials and the derivative of the 3-Fibonacci polynomials. This allows us to present in a comprehensible way several formulas for the sums of suchlike polynomials. The proposed identities can be used to improve new polynomial identities.

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