Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 50 (2) (2021), 492 – 510
DOI : 10.15672/hujms.614217

Research Article

Existence of random attractors for strongly damped wave equations with multiplicative noise unbounded domain

Abdelmajid Ali Dafallah*[1], Qiaozhen Ma[2], Ahmed Eshag Mohamed[2]

[1] *Department of Mathematics, AL Salam University, El Muglad, Sudan*
[2] *Department of Mathematics, Northwest Normal University, Lanzhou 730070, P.R. China*

Abstract

In this paper, we establish the existence of a random attractor for a random dynamical system generated by the non-autonomous wave equation with strong damping and multiplicative noise when the nonlinear term satisfies a critical growth condition.

Mathematics Subject Classification (2020). 35R60, 35B40, 35B41, 35B45, 37L30

Keywords. stochastic strong damped wave equation, asymptotic behavior of solutions, random attractor, random dynamical system

1. Introduction

In this work, we consider the asymptotic behavior of non-autonomous stochastic strong damped wave equations with multiplicative noise defined on

$$u_{tt} - \alpha\Delta u_t - \Delta u + g(u)u_t + f(u) = q(x,t) + cu \circ \frac{dW}{dt}, \tag{1.1}$$

with initial data

$$u(\tau,x) = u_\tau(x) \quad u_t(\tau,x) = u_{1,\tau}(x) \quad x \in \mathbb{R}^n,\ t \geq \tau,\ \tau \in \mathbb{R}, \tag{1.2}$$

where $-\Delta$ is the Laplacian operator with respect to the variable $x \in \mathbb{R}^n$, with $n \leq 3$, $u = u(t,x)$ is a real-valued function on $\mathbb{R}^n \times [\tau,\infty)$, α and c are positive constants. The given function $q(x,t) \in Ł^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, W(t) are independent two sided real-valued Wiener processes on probability space. Here we show the following conditions.

(a) The function $g \in \mathbb{C}^1(\mathbb{R})$ is not identically equal to zero and satisfies the following condition:

$$-\beta < \beta_1 \leq g(s) \leq \beta_2 < +\infty, \tag{1.3}$$

where $\beta,\ \beta_1$ and β_2 are positive constants.

*Corresponding Author.
Email addresses: majid_dafallah@yahoo.com(A.D. Abdelmajid), mazh@nwnu.edu.cn (Q. Ma), ahmedesag@gmail.com (A.E. Mohamed)
Received: 02.09.2019; Accepted: 27.08.2020

(b) We assume that the nonlinear function $f \in \mathbb{C}^2(\mathbb{R})$ satisfies the following conditions:

$$|f'(s)| \le C_1(1+|s|^p), \text{ with } \begin{cases} 0 \le p < \infty, \text{ when } n = 1,2 \\ 0 \le p < 2, \text{ when } n = 3, \end{cases} \tag{1.4}$$

and

$$\lim_{|s|\to\infty} \sup \frac{f(s)}{s} \le 0, \ \forall\, s \in \mathbb{R}, \tag{1.5}$$

$$\lim_{|s|\to\infty} \inf \frac{sf(s) - c_3F(s)}{s^2} \ge 0, \tag{1.6}$$

where C_1, C_2, C_3 are positive constants.

(c) $q(x,t)$ is external force term satisfying the following conditions:

$$\int_{-\infty}^{\tau} e^{\sigma r}\|q(x,r)\|^2 dr < \infty, \quad \forall \tau \in \mathbb{R}, \tag{1.7}$$

and

$$\lim_{k\to\infty} \int_{-\infty}^{\tau} \int_{|x|\ge k} e^{\sigma r}|q(x,r)|^2 dx dr = 0, \ \forall \tau \in \mathbb{R}. \tag{1.8}$$

The asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics and the theory has been greatly developed over the last decade or so. In the deterministic case the global attractor, a compact invariant and attracting set, occupy a central position see, for example, Temam [26]. In this paper, we study random attractors of equation (1.1) when the forcing term is time dependent. In this case, we want to introduce two parametric spaces to describe the dynamics of the equations: one is responsible for deterministic forcing and the other is responsible for stochastic perturbations. Existence and upper semi-continuity of the global attractor, pullback attractor (or kernel sections) for deterministic autonomous and non-autonomous dynamical systems were studied widely related with this problem (see, e.g., [6, 7, 16, 18, 20–22, 24, 32, 33, 35]).

In order to study the corresponding random dynamical system (RDS), some authors have introduced a different notion of an attractor from the view of stochastic partial differential equations, for example, see G.Da Prato, J. Zabczyk [11], Morimoto [23], L. Arnold [2] and H. Crauel, F. Fladoli Crauel [10], Duan, Lu, and Schmalfuß [12] and T. Caraballo, J. Langa [4], in which the authors studied the existence and the upper semi-continuity of attractors for deterministic and random dynamical systems, respectively. They obtained a general criteria for the existence and upper semi-continuous of attractors. For non-autonomous stochastic evolution equations with the time-dependent external term and multiplicative noise, Wang [29] established a useful theory about the existence and upper semi-continuity of random attractors by introducing two parametric spaces and giving some applications to non-autonomous stochastic reaction-diffusion equations and wave equations, see also [1, 3–5, 13, 19, 30, 31, 34] for more details.
Note that the stochastic equation (1.1) is defined in unbounded domains. Since Sobolev embeddings are not compact on unbounded domains, we have an extra difficulty to prove our main results.

This article is organized as follows. In Section 2 we recall some basic concepts related to RDS and a random attractor for RDS. In Section 3, we first provide some basic settings about (1.1) and (1.2) and show that it generates an RDS in proper function space and existence and uniqueness of solutions. We devote the Section 4 to uniform energy estimates on the solutions of (1.1) and (1.2) defined on $\mathbb{R}^n$ when $t \to \infty$ with the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness of the RDS associated with the equation. In Section 5, we prove the existence of a random attractor.

2. Preliminaries

In this section, we recall some basic concepts related to RDS and a random attractor for RDS [9, 10, 15, 27], which are important for getting our main results. Let $(\Omega, \mathcal{F}, P)$ be a probability space and (X, d) be a Polish space with the Borel σ-algebra $B(X)$. Then, the distance between $x \in X$ and $B \subseteq X$ is denoted by $d(x, B)$. If $B \subseteq X$ and $C \subseteq X$, the Hausdorff semi-distance from B to C is denoted by $d(B, C) = \sup_{x \in B} d(x, C)$.

Definition 2.1. $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s, \forall\ (s,t) \in \mathbb{R}$ and $\theta_0 P = P$, $\forall\ t \in \mathbb{R}$.

Definition 2.2. A mapping $\Phi(t, \tau\ , \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$, if for all $\tau \in \mathbb{R}\ , \omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied:
i) $\Phi(t, \tau, \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable mapping,
ii) $\Phi(0, \tau, \omega, x)$ is identity on X,
iii) $\Phi(t + s, \tau, \omega, x) = \Phi(t, \tau + s, \theta_s\omega, x) \circ \Phi(s, \tau, \omega, x)$,
iv) $\Phi(t, \tau, \omega, x) : X \to X$ is continuous.

Definition 2.3. Let 2^X be the collection of all subsets of X. A set valued mapping $(\tau, \omega) \mapsto \mathcal{D}(t\ \omega) : \mathbb{R} \times \Omega \mapsto 2^X$ is called measurable with respect to $\mathcal{F} \in \Omega$, if $\mathcal{D}(t\ \omega)$ is a (usually closed) nonempty subset of X and the mapping $\omega \in \Omega \mapsto d(X, B(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$. Let $B = \{B(t, \omega) \in \mathcal{D}(t, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be called a random set.

Definition 2.4. A random bounded set $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ of X is called tempered with respect to $\{\theta(t)\}_{t\in\Omega}$, if for p-a.e $\omega \in \Omega$,

$$\lim_{t\ \mapsto\infty}\ e^{-\beta t}\ d(B(\theta_{-t}\omega)) = 0\ ,\ \forall\ \beta\ > 0,$$

where

$$d(B) = \sup_{x\in B} \|x\|_X.$$

Definition 2.5. Let $\mathcal{D}$ be a collection of random subset of X and $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, then K is called an absorbing set of $\Phi \in \mathcal{D}$, if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $B \in \mathcal{D}$, there exists, $T = T(\tau, \omega, B) > 0$ such that

$$\Phi(t\ , \tau, \theta_{-t}\omega, B(\tau, \theta_{-t}\omega)) \subseteq K(\tau, \omega)\ ,\ \forall\ t\ \geq T.$$

Definition 2.6. Let $\mathcal{D}$ be a collection of random subset of X, the Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for p-a.e $\omega \in \Omega$, $\{\Phi(t_n\ , \theta_{-t_n}\omega\ , x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X when $t_n \mapsto \infty$ and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 2.7. Let $\mathcal{D}$ be a collection of random subset of X and $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,then $\mathcal{A}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for Φ, if the following conditions are satisfied:

For all $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

i) $\mathcal{A}(\tau, \omega)$ is compact, and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$,
ii) $\mathcal{A}(\tau, \omega)$ is invariant, that is

$$\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) =\ \mathcal{A}(\tau + t, \theta_t\omega), \forall\ t\ \geq \tau,$$

iii) $\mathcal{A}(\tau,\omega)$ attracts every set in $\mathcal{D}$, that is for every $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$,

$$\lim_{t\mapsto\infty} d_X(\Phi(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega)),\mathcal{A}(\tau,\omega))=0,$$

where d_X is the Hausdorff semi-distance given by

$$d_X(Y,Z)=\sup_{y\in Y}\inf_{z\in Z}\|y-z\|_X,\quad \forall\ (Y,Z)\in X.$$

Lemma 2.8. *Let $\mathcal{D}$ be a neighborhood-closed collection of (τ,ω)-parameterized families of nonempty subsets of X and Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$. Then Φ has a pullback $\mathcal{D}$-attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if Φ is pullback $\mathcal{D}$-asymptotically compact in X and Φ has a closed, $\mathcal{F}$-measurable pullback $\mathcal{D}$-absorbing set $K\in\mathcal{D}$, the unique pullback $\mathcal{D}$-attractor $\mathcal{A}=\mathcal{A}(\tau\ ,\omega)$ is given by*

$$\mathcal{A}(\tau,\omega)=\bigcap\nolimits_{r\geq 0}\overline{\bigcup\nolimits_{t\geq r}\Phi(t,\tau-t,\theta_{-t}\omega,K(\tau-t,\theta_{-t}\omega))}\ \tau\in\mathbb{R}\ ,\omega\in\Omega.$$

3. Existence and uniqueness of solutions

In this section, we study the existence and uniqueness of solutions for the system (1.1) and (1.2) on an unbounded set of $\mathbb{R}^n,(n=3)$.

Let $A=-\triangle, D(A)=H^1(\mathbb{R}^n)\cap H^2(\mathbb{R}^n)\rightarrow L^2(\mathbb{R}^n)$. It is self-adjoint, positive and linear, and the eigenvalue $\{\lambda_i\}_{i\in N}$ of A satisfies, $0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_i\leq\cdots,\ \lambda_i\rightarrow+\infty\ (i\rightarrow+\infty)$.

Define the set of Hilbert space $V_{2r}=D(A^r), r\in\mathbb{R}$, with standard inner products and norms, respectively, $((\cdot,\cdot))_{D(A^r)}=(A^r\cdot,A^r\cdot)$, $\|\cdot\|_{D(A^r)}=\|A^r\cdot\|$, $((u,v))=\int_{\mathbb{R}^n}\nabla u\nabla v dx$, $\|\nabla u\|=((u,u))^{\frac{1}{2}},\ \forall\ u,v\in H^1(\mathbb{R}^n)$.

Especially, $(\cdot,\cdot)$ and $\|\cdot\|$ denote the $L^2(\mathbb{R}^n)$ inner product and norm respectively as,

$$(u,v)=\int_{\mathbb{R}^n}uvdx,\ \|u\|=(u,u)^{\frac{1}{2}},\ \forall u,v\in L^2(\mathbb{R}^n).$$

Thus we have $D(A^r)\rightarrow D(A^s)$, for $r>s$, and the continuous embedding $D(A^r)\rightarrow L^{\frac{2n}{n-4r}}(\mathbb{R}^n),\ \forall r\in[0,\frac{n}{2}]$.

Let $E=H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, endowed with norms on E and $E_\nu=H_{2\nu+1}\times H_{2\nu}$,

$$\|\varphi\|_E^2=\|\varphi\|^2_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}.$$

For a convenient study of dynamical behavior of the problem (1.1) and (1.2), we need to convert the stochastic system into deterministic with a random parameter, then show that it generates an RDS.

Consider Ornstein-Uhlenbeck process driven by the Brownian motion, which satisfies the Itô differential equation

$$dz+\mu z dt=dW,\ \mu>0,$$

and hence the solution is given by

$$\begin{aligned}&\theta_t\omega(s)=\omega(t+s)-\omega(t),\\ &z(\theta_t\omega)=z(t,\omega)\ =\ -\varepsilon\textstyle\int_{-\infty}^0 e^{\varepsilon s}(\theta_t\omega)sds, s\in\mathbb{R},\omega\in\Omega.\end{aligned}\tag{3.1}$$

From [33], it is known that the random variable $|z(\omega)|$ is tempered and there is an invariant set $\overline{\Omega}\subseteq\Omega$ of full P measure, such that $z(\theta_t\omega)=z(t,\omega)$ is continuous in t for every

$\omega \in \bar{\Omega}$. For convenience, we shall write $\bar{\Omega}$ as Ω.

For the Ornstien-Uhlenbeck process $z(\theta_t\omega)$ in (3.1) (see [8, 14]), we have

$$\begin{cases} \lim\limits_{t\to\pm\infty} \dfrac{|z(\theta_t\omega)|}{|t|} = 0, \\ \lim\limits_{t\to\pm\infty} \dfrac{1}{t}\displaystyle\int_{-t}^{0} z(\theta_s\omega)ds = E[z(\theta_s\omega)] = 0, \\ \lim\limits_{t\to\pm\infty} \dfrac{1}{t}\displaystyle\int_{-t}^{0} z(\theta_s\omega)ds = E[z(\theta_s\omega)] = \dfrac{1}{\sqrt{\pi\delta}}, \\ \lim\limits_{t\to\pm\infty} \dfrac{1}{t}\displaystyle\int_{-t}^{0} |z(\theta_s\omega)|^2 ds = E[|z(\theta_s\omega)|^2] = \dfrac{1}{2\delta}. \end{cases} \tag{3.2}$$

By (3.2), there exists $T_1(\omega) > 0$, such that, for all $t \geq T_1(\omega)$,

$$\int_{-t}^{0} z(\theta_s\omega)ds < \frac{2}{\sqrt{\pi\delta}}t \ , \ \int_{-t}^{0} |z(\theta_s\omega)|^2 ds < \frac{1}{2\delta}t. \tag{3.3}$$

To define a cocycle associated with the problem (1.1) and (1.2), let $v = u_t + \varepsilon u - cuz(\theta_t\omega)$, then we get

$$\begin{cases} \dfrac{du}{dt} = v - \varepsilon u + cuz(\theta_t\omega), \\ \dfrac{dv}{dt} = (\varepsilon - \alpha A)v - (\varepsilon^2 - \varepsilon\alpha A + A)u - g(u)(v - \varepsilon u + cuz(\theta_t\omega)) \\ \qquad + c\left((3\varepsilon - \alpha A)u - v - cuz(\theta_t\omega)\right) z(\theta_t\omega) - f(u) + q(x,t), \\ u(x,\tau) = u_\tau(x), \ v(x,\tau) = v_\tau(x) = u_1(x) + \varepsilon u_\tau(x) - cu_\tau(x)z(\theta_t\omega). \end{cases} \tag{3.4}$$

We define

$$\psi_1 = u, \ \psi_2 = \frac{du}{dt} + \varepsilon u - cuz(\theta_t\omega), \tag{3.5}$$

where ε is a given positive constant, then the system (3.4) can be rewritten as the following equivalent system with random coefficients, but without multiplicative noise on E

$$\begin{cases} \dfrac{d\psi_1}{dt} = \psi_2 - \varepsilon\psi_1 + c\psi_1 z(\theta_t\omega), \\ \dfrac{d\psi_2}{dt} = (\varepsilon - \alpha A)\psi_2 - (\varepsilon^2 - \varepsilon\alpha A + A)\psi_1 - g(\psi_1)(\psi_2 - \varepsilon\psi_1 + c\psi_1 z(\theta_t\omega)) \\ \qquad + c((3\varepsilon - \alpha A)\psi_1 - \psi_2 - c\psi_1 z(\theta_t\omega))z(\theta_t\omega) - f(\psi_1) + q(x,t), \\ \psi_1(x,\tau) = u_\tau(x) \ , \ \psi_2(x,\tau) = u_1(x) + \varepsilon u_\tau(x) - cu_\tau(x)z(\theta_t\omega), \end{cases} \tag{3.6}$$

then the random differential equation (3.6) can be written as

$$\begin{cases} \psi' + L\psi = Q(\psi_1, t, \omega) \\ \psi_\tau = (\psi_1(x,\tau), \psi_2(x,\tau)) = (u_\tau(x), u_{1\tau}(x) + \varepsilon u_\tau(x) - cu_\tau z(\theta_\tau\omega))^\top, \end{cases} \tag{3.7}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad L = \begin{pmatrix} \varepsilon I & -I \\ \varepsilon^2 I - \varepsilon\alpha A + A & -\varepsilon I + \alpha A \end{pmatrix},$$

and

$$Q(\psi, \omega, t) = \begin{pmatrix} c\psi_1 z(\theta_t\omega) \\ -g(\psi_1)\psi_{1t} + c((3\varepsilon - \alpha A)\psi_1 - \psi_2 - c\psi_1 z(\theta_t\omega))z(\theta_t\omega) - f(\psi_1) + q(x,t) \end{pmatrix}.$$

Consider the equation (3.7), we know that $-L$ is the infinitesimal generator of semigroup e^{-Lt} on E for $t > 0$, by the assumptions (1.3)-(1.8). It is easy to check that $Q(\psi, t, \omega)$: $E \to E$ is locally Lipschitz continuous with respect to ψ, then by the classical semigroup theory concerning the (local) existence and uniqueness solution of evolution differential equation [25], we have the following theorem.

Theorem 3.1. *Assume that the conditions (1.3)-(1.8) hold, for each* $\tau \in \mathbb{R}, \omega \in \Omega$ *and for any* $\psi_\tau \in E$*, there exists* $T > 0$*, such that (3.7) has a unique mild function* $\psi(t, \tau, \omega, \psi_\tau) \in C([\tau, \tau + T); E)$*, such that* $\psi(t)$ *satisfies the integral equation*

$$\psi(t, \tau, \omega, \psi_\tau) = e^{-L(t-\tau)}\psi_\tau(\omega) + \int_\tau^t e^{-L(t-r)}Q(\psi, r, \omega)dr. \tag{3.8}$$

In this case, $\psi(t, \tau, \omega, \psi_\tau)$ *is called mild solution of (3.7),* $\psi(t, \tau, \omega, \psi_\tau)$ *is jointly continuous into* t *and measurable in* ω*, moreover* $(u, u_t) \in C([\tau, +\infty, E])$*,* $\forall\ T > 0$*.*

From Theorem 3.1, the solution $\psi(t, \tau, \omega, \psi_\tau)$ of (3.7) can define a continuous RDS over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

$$\begin{aligned} &\Phi_\varepsilon(t, \omega) : \mathbb{R} \times \Omega \times E \mapsto E, t \geq \tau, \\ &\psi(\tau, \omega) = (u_\tau(\omega), v_\tau(\omega))^\top \mapsto (u(t, \omega), v(t, \omega))^\top = \psi(t, \omega), \end{aligned} \tag{3.9}$$

it is easy to see that

$$\Phi(t, \omega) : \psi(\tau, \omega) + (0, cuz(\theta_\tau\omega))^\top \mapsto \psi(t, \omega) + (0, cuz(\theta_t\omega))^\top. \tag{3.10}$$

We will also use the transformation

$$\varphi_1 = u\ ,\ \varphi_2 = u_t + \varepsilon u. \tag{3.11}$$

Thus, like as (3.6), it yields

$$\begin{cases} \varphi' + H\varphi = F_1(\varphi, t, \omega), \\ \varphi_\tau = (u_\tau, v_\tau)^\top = (u_\tau(x), u_{1\tau}(x) + \varepsilon u_\tau(x))^\top, \end{cases} \tag{3.12}$$

where

$$H\varphi = \begin{pmatrix} v - \varepsilon u \\ \varepsilon(\varepsilon - \alpha A)u + Au - (\varepsilon - \alpha A)v \end{pmatrix},$$

and

$$F_1(\varphi, t, \omega) = \begin{pmatrix} 0 \\ -g(u)(v - \varepsilon u) - f(u) + q(x, t) + cuz(\theta_t\omega) \end{pmatrix}.$$

We introduce the isomorphism $T_\epsilon\varphi = (\varphi_1, \varphi_2 - \varepsilon\varphi_1)^\top$, $\varphi = (\varphi_1, \varphi_2)^\top \in E$ which has inverse isomorphism $T_{-\epsilon}\psi = (\varphi_1, \varphi_2 + \varepsilon\varphi_1)^\top$, it follows that (θ, φ) with mapping

$$\Psi = T_\epsilon\Phi(t, \omega)T_{-\epsilon} = \Psi(t, \omega) \tag{3.13}$$

is an RDS corresponding, such that the two RDS are equivalent.

4. Uniform estimates of solutions

In this section, we will show the existence of a random absorbing set for the RDS $\{\varphi(t, \tau, \omega, \varphi_\tau), t \geq 0\}$ in the space E and uniform estimates for the solutions of (3.4) defined on $\mathbb{R}^n$. For this purpose, we introduce a new weight inner product and norm in the Hilbert space E that is, $(\varphi, \tilde{\varphi})_E = \mu(A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}u_2) + (v_1, v_2)$ and $\|\varphi\|_E = (\varphi, \varphi)_E^{\frac{1}{2}}$ for any $\varphi = (u_1, v_1)^\top$, $\tilde{\varphi} = (u_2, v_2)^\top \in E$, where μ is chosen as

$$\mu = \frac{4 + (\alpha\lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1}{4 + 2(\alpha\lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1} \in (\frac{1}{2}, 1). \tag{4.1}$$

It is clear that, the norm $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^1\times L^2}$ of E. Let $\varphi = (u, v)^\top = (u, u_t + \varepsilon u - cuz(\theta_t\omega))^\top$, where ε is chosen as

$$\varepsilon = \frac{\alpha\lambda_1 + \beta_1}{4 + 2(\alpha\lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1}, \tag{4.2}$$

and then the system (3.6) can be rewritten as

$$\begin{cases} \varphi' + H\varphi = F_1(t, \varphi, \omega), \\ \varphi_\tau = (u_\tau, v_\tau)^\top = (u_\tau(x), u_{1\tau}(x) + \varepsilon u_\tau(x) - cu_\tau z(\theta_\tau \omega))^\top, \end{cases} \tag{4.3}$$

where

$$H\varphi = \begin{pmatrix} v - \varepsilon u \\ \varepsilon(\varepsilon - \alpha A)u + Au - (\varepsilon - \alpha A)v \end{pmatrix},$$

and

$$F_1(t, \varphi, \omega) = \begin{pmatrix} cuz(\theta_t\omega) \\ -g(u)(v - \varepsilon u - cuz(\theta_t\omega)) + c\left((3\varepsilon - \alpha A)u - v - cuz(\theta_t\omega)\right) z(\theta_t\omega) - f(u) + q(x,t) \end{pmatrix}.$$

Lemma 4.1. *Suppose* $-\alpha\lambda_1 < \beta_1 \le \beta_2 < +\infty$ *and*

$$\beta_2 \ge \beta_1 + \min[\frac{1}{\alpha}, \frac{\alpha\lambda_1 + \beta_1}{2}], \tag{4.4}$$

for any $\varphi = (u, v)^\top \in E$*, it follows that*

$$\begin{aligned} (H\varphi, \varphi)_E &\ge \sigma\|\varphi\|_E^2 + \frac{\alpha}{2}\|\nabla v\|^2 + \frac{\beta_1}{2}\|v\|^2 \\ &\ge \sigma\|\varphi\|_E^2 + \tfrac{\alpha\lambda_1+\beta_1}{2}\|v\|^2, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \sigma &= \frac{\alpha\lambda_1 + \beta_1}{\gamma_1 + \sqrt{\gamma_1\gamma_2}} \\ \gamma_1 &= 4 + (\alpha\lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1, \\ \gamma_2 &= (\alpha\lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1. \end{aligned} \tag{4.6}$$

Proof. This can be easily obtained after simple computations. □

Lemma 4.2. *Assume that (1.3)-(1.8) hold, then there exists closed tempered random absorbing set* $r(\omega)$ *and a bounded ball* $B_0(\tau, \omega) \subset E$*, centered at* 0 *with a radius* $\varrho(\omega) > 0$*,* $B_E(0, \varrho(\omega) \in \mathcal{D}(E)$*, such that for any bounded non-random set* $B \in \mathcal{D}(E)$*, there exists* $T = T(\tau, \omega, B) \ge 0$*, forall* $\tau \in \mathbb{R}, \omega \in \Omega, B \in \mathcal{D}$*, such that the mild solution* $\varphi \in B_0$ *of system (4.3) with initial value* $\varphi_\tau = (u_\tau, v_\tau) \in B$ *satisfies*

$$\begin{gathered} \|\varphi(t, \tau, \theta_{-t}\omega, \varphi_\tau(\theta_{-t}\omega))\|_E^2 \le \varrho^2(\omega), \\ \Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)) \subseteq B_0(\tau, \omega)), \ \forall\, t \ge T. \end{gathered} \tag{4.7}$$

Proof. For any $\tau \in \mathbb{R}, \omega \in \Omega, t \ge 0$, suppose that $\varphi(s) = (u(s), v(s)) \in E,\ t \ge s \ge \tau$ be a mild solution of (4.3), taking the inner product of (4.3) with $\varphi(s) = (u, v) = (u, u_t + \varepsilon u - cuz(\theta_t\omega))^\top$. According to Lemma 4.1, we find that

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_E^2 + \sigma\|\varphi\|_E^2 + \frac{\alpha\lambda_1 + \beta_1}{2}\|v\|^2 \le (F_1(t, \varphi, \omega), \varphi). \tag{4.8}$$

We estimate the right hand side of (4.8)

$$\begin{aligned} (F_1(\varphi, t, \omega), \varphi) =& c((uz(\theta_t\omega), u)) - (g(u)(v - \varepsilon u - cuz(\theta_t\omega), v) - \alpha c(Au, v)z(\theta_t\omega) \\ &+ c\left((3\varepsilon u - v - cuz(\theta_t\omega))\, z(\theta_t\omega), v\right) - (f(u) - q(x,t), v), \end{aligned} \tag{4.9}$$

we deal with the term in (4.9) one by one as follows. For the first term of the right hand side of (4.9)

$$c\left((uz(\theta_t\omega), u)\right) \le c|z(\theta_t\omega)|\, \|\nabla u\|^2. \tag{4.10}$$

By (1.3), we have

$$\begin{aligned} -\left(g(u)(v - \varepsilon u - cuz(\theta_t\omega), v\right) &= -g(u)(v, v) + \varepsilon g(u)(u, v) - cg(u)(u, v)z(\theta_t\omega) \\ &\le -\beta_2\|v\|^2 + \varepsilon\beta_1(u, v) - \beta_2 c(u, v)z(\theta_t\omega) \\ &\le -\beta_2\|v\|^2 + \tfrac{\varepsilon\beta_1}{2\sqrt{\lambda_1}}\|\varphi\|_E^2 + \beta_2 c(u, v)z(\theta_t\omega). \end{aligned} \tag{4.11}$$

For the fourth term on the right-hand side of (4.9), the last term of (4.11), using the Cauchy-Schwartz inequality and Young's inequality, we obtain

$$\begin{aligned}
&c\left((3\varepsilon+\beta_2)u-v-cuz(\theta_t\omega),v\right)z(\theta_t\omega)\\
&\le (c(3\varepsilon+\beta_2)z(\theta_t\omega)+c^2|z(\theta_t\omega)|^2)(u,v)-c|z(\theta_t\omega)|\|v\|^2\\
&\le \tfrac{1}{2}(c(3\varepsilon+\beta_2)z(\theta_t\omega)+c^2|z(\theta_t\omega)|^2)(\|u\|^2+\|v\|^2)-c|z(\theta_t\omega)|\|v\|^2\\
&\le \tfrac{1}{2\sqrt{\lambda_1}}(c(3\varepsilon+\beta_2)|z(\theta_t\omega)|+c^2|z(\theta_t\omega)|^2)\,\|\varphi\|_E^2-c|z(\theta_t\omega)|\|v\|^2,
\end{aligned} \tag{4.12}$$

$$-\alpha c(Au,v)z(\theta_t\omega)\le \alpha c\left(\nabla uz(\theta_t\omega),\nabla v\right)\le \frac{c\alpha\sqrt{\lambda_1}}{2}|z(\theta_t\omega)|\,\|\varphi\|_E^2, \tag{4.13}$$

$$\begin{aligned}
(q(x,t),v)\ &\le \|q(x,t)\|\ \|v\|\\
&\le \tfrac{1}{2(\alpha\lambda_1+\beta_1-2\beta_2-\varepsilon)}\|q(x,t)\|^2+\tfrac{(\alpha\lambda_1+\beta_1-2\beta_2-\varepsilon)}{2}\|v\|^2.
\end{aligned} \tag{4.14}$$

Let $\tilde{F}(u)=\int_{\mathbb{R}^n}F(u)dx$, then using (1.5) and (1.6), and the Hölder inequality, we get that

$$\begin{aligned}
-\left(f(u),v\right)\ &=-\left(f(u),\tfrac{du}{dt}+\varepsilon u-cuz(\theta_t\omega)\right)\\
&\le -\tfrac{d}{dt}\tilde{F}(u)-\varepsilon(f(u),u)+\left(f(u),cuz(\theta_t\omega)\right).
\end{aligned} \tag{4.15}$$

Due to (1.6) and Poincarè inequality, there exists positive constants ϑ_1,ϑ_2, such that

$$(f(u),u)-C_3\tilde{F}(u)+\vartheta_1\|\nabla u\|^2+\vartheta_2\ge 0. \tag{4.16}$$

According to [17], follows (1.5), for each given $C_{\vartheta_3}>0$,

$$(f(u),u)+\vartheta_3\|\nabla u\|^2+C_{\vartheta_3}\ge 0. \tag{4.17}$$

Then, by (4.14)-(4.17) we obtain

$$\begin{aligned}
-\left(f(u),v\right)\le &-\tfrac{d}{dt}\tilde{F}(u)-\varepsilon C_3\tilde{F}(u)+(\varepsilon\vartheta_1-\vartheta_3c|z(\theta_t\omega)|)\|\nabla u\|^2\\
&+\varepsilon\vartheta_2-C_{\vartheta_3}c|z(\theta_t\omega)|.
\end{aligned} \tag{4.18}$$

Thus, we show that

$$\begin{aligned}
(F_1(t,\varphi,\omega),\varphi)\le &\left(\alpha_1c|z(\theta_t\omega)|+\tfrac{c^2|z(\theta_t\omega)|^2}{\sqrt{\lambda_1}}-\sigma_1\right)\|\varphi\|_E^2-\tfrac{d}{dt}\tilde{F}(u)-\varepsilon C_3\tilde{F}(u)\\
&+\varepsilon\vartheta_2-C_{\vartheta_3}c|z(\theta_t\omega)|+\tfrac{1}{2(\alpha\lambda_1+\beta_1-2\beta_2-\varepsilon)}\|q(x,t)\|^2+\tfrac{(\alpha\lambda_1+\beta_1)}{2}\|v\|^2,
\end{aligned} \tag{4.19}$$

where α_1 depends on $1,\frac{1}{2\sqrt{\lambda_1}},\frac{1}{2\sqrt{\lambda_1}}((3\varepsilon+\beta_2),\frac{\alpha\sqrt{\lambda_1}}{2},\frac{\beta_1}{4\sqrt{\lambda_1}}$ and $\sigma_1=\min[\varepsilon\vartheta_1,\frac{\varepsilon}{2}]$. Hence we conclude that

$$\begin{aligned}
\tfrac{d}{dt}\left(\|\varphi\|_E^2+2\tilde{F}(u)\right)\le &-2\left(\sigma-\alpha_1c|z(\theta_t\omega)|-\tfrac{c^2|z(\theta_t\omega)|^2}{\sqrt{\lambda_1}}+\sigma_1\right)\|\varphi\|_E^2\\
&-2C_3\varepsilon\tilde{F}(u)+C_{\vartheta_3}c|z(\theta_t\omega)|+\tfrac{1}{2(\alpha\lambda_1+\beta_1-2\beta_2-\varepsilon)}\|q(x,t)\|^2+\varepsilon\vartheta_2.
\end{aligned} \tag{4.20}$$

Put

$$r_0(t,\theta_t\omega)=\kappa(1+c|z(\theta_t\omega)|+\|q(x,t)\|^2),$$

where κ is a positive constant depends only on $\varepsilon\vartheta_2,C_{\vartheta_3},\frac{1}{2(\alpha\lambda_1+\beta_1-2\beta_2-\varepsilon)}$, such that

$$\begin{aligned}
\tfrac{d}{dt}\left(\|\varphi\|_E^2+2\tilde{F}(u)\right)\le &-2\left(\sigma-\alpha_1c|z(\theta_t\omega)|-\tfrac{c^2|z(\theta_t\omega)|^2}{\sqrt{\lambda_1}}+\sigma_1\right)\|\varphi\|_E^2\\
&-2C_3\varepsilon\tilde{F}(u)+r_0(t,\theta_t\omega).
\end{aligned} \tag{4.21}$$

By using Gronwall's inequality in (4.21) on $[\tau,t]$, and then replacing ω to $\theta_{-t}\omega$ yield

$$\begin{aligned}
\|\varphi\|_E^2\le &\left(\|\varphi\|_E^2+2\tilde{F}(u)\right)\\
\le &\left(\|\varphi_\tau\|_E^2+2\tilde{F}(u_\tau)\right)e^{-2C_m(\bar{\sigma}-c|z(\theta_t\omega)|-c^2|z(\theta_t\omega)|^2)(t-\tau)}\\
&+\int_{\tau-t}^0 r_0(\theta_{s-t}\omega)e^{-2C_m(\bar{\sigma}-c|z(\theta_{s-t}\omega)|-c^2|z(\theta_{s-t}\omega)|^2)(s-t,\omega)}ds,
\end{aligned} \tag{4.22}$$

where $\bar{\sigma}=\min\{\sigma,\frac{c_3\varepsilon}{2},\sigma_1\}$, and $y(t,\omega)=\|\varphi\|_E^2+2\tilde{F}(u)\ge\|\varphi\|_E^2\ge 0$.
Using (1.4) and Young inequality, there exists a constant $c_4\ge 0$, such that

$$\tilde{F}(u_\tau)=\int_{\mathbb{R}^n}F(u_\tau)dx\le c_4\left(1+\|u_\tau\|^2+\|u_\tau\|_{p+2}^{p+2}\right). \tag{4.23}$$

Due to $\varphi_\tau = (u_\tau, v_\tau)^\top \in B(\tau, \theta_{-t}\omega)$ and $B \in \mathcal{D}$, which is tempered with respect to the norm of E, since $\varphi_\tau(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, there exists $T = T(\tau, \omega, B) > 0$, such that, for all $t \geq T$.

$$\begin{cases} e^{-\sigma t}\left(\|\varphi_\tau\|_E^2 + \tilde{F}(u_\tau)\right) \leq 1, \\ \lim\limits_{r\to-\infty} e^{-\sigma r}\left(\|\varphi_\tau\|_E^2 + \tilde{F}(u_\tau)\right) = 0. \end{cases} \tag{4.24}$$

Thus, by Lemma 3.1 and for any set $\{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, $\varphi_\tau = (u_\tau(x), u_{1,\tau}(x) + \varepsilon u_\tau(x) - cu_\tau z(\theta_\tau\omega))^\top \in \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D(E)$. We deduce that

$$\begin{aligned} &\limsup_{t\to\infty}(\|\varphi_\tau(\theta_{-t}\omega)\|_E^2 + 2\varepsilon C_3(\|u_\tau\|^2 + \|u_\tau\|_{H^1}^{p+2}))e^{-2c_m(\bar{\sigma}-c|z(\theta_t\omega)|-c^2|z(\theta_t\omega)|^2)(t-\tau)} = 0, \\ &\lim_{t\to\infty}\textstyle\int_{\tau-t}^0 r_0(\theta_{s-t}\omega)e^{-2c_m(\bar{\sigma}-c|z(\theta_{s-t}\omega)|-c^2|z(\theta_{s-t}\omega)|^2)(s-t,\omega)}ds < \infty. \end{aligned} \tag{4.25}$$

When $q(x,t)$ only satisfying (1.7) and (1.8) which is a tempered random variable, then by (4.23)-(4.25), for any non-random bounded $B \subseteq E$ with radius $r(\omega)$, there exists a random variable $T = T(\tau, B, \omega) > 0$, such that for any $\varphi_\tau \in B_0(\omega)$, $\{\varphi \in E : \|\varphi_\tau(\theta_{-t}\omega)\|_E \leq \varrho^2(\omega)\}$ is closed measurable absorbing ball in $\mathcal{D}(E)$, $\forall\, t \geq T$.
The proof is completed. □

Next we construct the uniform estimates on the tail parts of the solutions for large space variables when time is sufficiently large in order to prove the pullback asymptotic compactness of the cocycle associated with equation (3.4) on the unbounded domain $\mathbb{R}^n$. We can choose a smooth function ρ defined on $\mathbb{R}^+$, such that $0 \leq \rho(s) \leq 1$, for all $s \in \mathbb{R}^+$ and

$$\rho(s) = \begin{cases} 0, & \text{for } 0 \leq s \leq 1, \\ 1, & \text{for } s \geq 2. \end{cases} \tag{4.26}$$

Then there exist constant μ_1 and μ_2, such that $|\rho^{'}(s)| \leq \phi_1, |\rho^{''}(s)| \leq \phi_2$, for any $s \in \mathbb{R}^+$. Given $k \geq 1$, denote by $H_k = \{x \in \mathbb{R}^n : |x| < k\}$ and $\mathbb{R}^n \backslash H_k$ the complement of H_k. To prove the asymptotic compactness of the RDS, we prove the following lemma.

Lemma 4.3. *Let conditions (1.3)-(1.8) hold.* B=$\{B(\tau,\omega) : \tau \in \mathbb{R},\ \omega \in \Omega\} \in \mathcal{D}$ *and* $\varphi_\tau(\omega) \in B(\tau,\omega)$. *Then for every* $\eta > 0$ *and P-a.e* $\omega \in \Omega$, *there exists* $\bar{T} = \bar{T}(\tau, B, \omega, \eta) > 0$ *and* $\bar{K} = \bar{K}(\tau, \omega, \eta) \geq 1$, *such that* $\varphi(t, \omega, \varphi_\tau(\omega))$ *is a solution of (3.4) satisfying, for all* $t \geq \bar{T}, k \geq \bar{K}$,

$$\|\varphi(\tau, \tau - t, \theta_{-t}\omega, \varphi_\tau(\theta_{-t}\omega))\|^2_{E(\mathbb{R}^n\backslash H_k)} \leq \eta. \tag{4.27}$$

Proof. Taking the inner product of the second equation of (3.4) with $\rho[\frac{|x|^2}{k^2}]v$ in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]|v|^2\,dx &= \int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right][(\varepsilon - g(u))v - \alpha Av - \varepsilon(\varepsilon - g(u))u + (\varepsilon\alpha - 1)Au]\,vdx \\ &+ \int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right][c\left((3\varepsilon - g(u))u - v - cuz(\theta_t\omega)\right)z(\theta_t\omega) - c\alpha Auz(\theta_t\omega) - f(u) + q(x,t))]\,vdx. \end{aligned} \tag{4.28}$$

In order to estimate the left hand side, we are substituting v in the first term of (3.4), so it follows that

$$\begin{aligned} &\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]\varepsilon(-\varepsilon + g(u))uvdx \\ &= \textstyle\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right](\varepsilon - g(u))u\left[\frac{du}{dt} + \varepsilon u - cuz(\theta_t\omega)\right]dx \\ &\leq \varepsilon(-\varepsilon + \beta_1)\textstyle\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]\left[\frac{1}{2}\frac{d}{dt}|u|^2 + (\varepsilon - c|z(\theta_t\omega)|)|u|^2\right]dx, \end{aligned} \tag{4.29}$$

$$
\begin{aligned}
&(\varepsilon\alpha-1)\int_{\mathbb{R}^n}(-\triangle u)\,\rho\left[\frac{|x|^2}{k^2}\right]vdx\\
&=(\varepsilon\alpha-1)\int_{\mathbb{R}^n}(\nabla u)\nabla\left[\rho\left[\tfrac{|x|^2}{k^2}\right]\left[\tfrac{du}{dt}+\varepsilon u-cuz(\theta_t\omega)\right]\right]dx\\
&=(\varepsilon\alpha-1)\int_{\mathbb{R}^n}\nabla u\left[\tfrac{2x}{k^2}\rho'\left[\tfrac{|x|^2}{k^2}\right]v\right]dx\\
&\quad+(\varepsilon\alpha-1)\int_{\mathbb{R}^n}(\nabla u)\left[\rho\left[\tfrac{|x|^2}{k^2}\right]\nabla\left[\tfrac{1}{2}\tfrac{du}{dt}+\varepsilon u-cuz(\theta_t\omega)\right]\right]dx\\
&\leq(\varepsilon\alpha-1)\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{k^2}\right]\left[\tfrac{1}{2}\tfrac{d}{dt}|\nabla u|^2+\varepsilon|\nabla u|^2-c|\nabla u|^2|z(\theta_t\omega)|\right]dx\\
&\quad+(\varepsilon\alpha-1)\tfrac{\sqrt{2}}{k}\phi_1\left(\|\nabla u\|^2+\|v\|^2\right),
\end{aligned}\tag{4.30}
$$

and

$$
\begin{aligned}
&-\alpha\int_{\mathbb{R}^n}(-\triangle v)\rho\left[\frac{|x|^2}{k^2}\right]vdx\\
&=-\alpha\int_{\mathbb{R}^n}(\nabla v)\nabla\left[\rho\left[\tfrac{|x|^2}{k^2}\right]v\right]dx\\
&\leq\alpha\int_{\mathbb{R}^n}\nabla v\left(\tfrac{2x}{k^2}\rho'\left[\left[\tfrac{|x|^2}{k^2}\right]v\right)dx-\alpha\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{k^2}\right]|\nabla v|^2dx\\
&\leq\alpha\int_{k<|x|<\sqrt{2}k}\tfrac{2x}{k^2}\phi_1|\nabla v||v|dx-\alpha\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{k^2}\right]|\nabla v|^2dx\\
&\leq\alpha\tfrac{\sqrt{2}}{k}\phi_1(\|\nabla v\|^2+\|v\|^2)-\alpha\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{k^2}\right]|\nabla v|^2dx.
\end{aligned}\tag{4.31}
$$

For the nonlinear term, according to (1.6) and (1.7), (4.20), and applying Young inequality, after detailed computations, we obtain

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]f(u)vdx\\
&\geq-\tfrac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{r^2}\right]F(u)dx-\varepsilon k\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{k^2}\right]F(u)dx\\
&\quad+\left(\varepsilon\vartheta_1-\vartheta_3c|z(\theta_t\omega)|\right)\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{r^2}\right]|\nabla u|^2dx\\
&\quad+\left(\varepsilon\vartheta_2-c_{\vartheta_3}c|z(\theta_t\omega)|\right)\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{r^2}\right]dx.
\end{aligned}\tag{4.32}
$$

By the Cauchy-Schwartz inequality, the Young inequality and $\|\nabla v\|^2\geq\lambda_1\|v\|^2$, we deduce that

$$
\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]q(x,t)vdx\leq\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\frac{|q(x,t)|^2}{4\alpha\lambda_1}dx+\alpha\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla v|^2dx,\tag{4.33}
$$

$$
\begin{aligned}
&cz(\theta_t\omega)\left((3\varepsilon+\beta_2)-cz(\theta_t\omega)\right)\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]uvdx\\
&\leq cz(\theta_t\omega)\left(3\varepsilon+\beta_2)+cz(\theta_t\omega)\right)\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{r^2}\right]|u||v|dx\\
&\leq\tfrac{1}{2}\left((3\varepsilon+\beta_2)c|z(\theta_t\omega)|+c^2|z(\theta_t\omega)|^2\right)\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{r^2}\right]\left[|u|^2+|v|^2\right]dx,
\end{aligned}\tag{4.34}
$$

and

$$
\begin{aligned}
\alpha cz(\theta_t\omega)\int_{\mathbb{R}^n}(-\triangle u)\rho\left[\frac{|x|^2}{r^2}\right]vdx&\leq\alpha c|z(\theta_t\omega)|\int_{\mathbb{R}^n}(\nabla u)\nabla\left[\rho\left[\frac{|x|^2}{r^2}\right]v\right]dx\\
&=\alpha c|z(\theta_t\omega)|\int_{\mathbb{R}^n}\frac{2|x|}{r^2}\rho'\left[\frac{|x|^2}{r^2}\right]|\nabla u|vdx\\
&\quad+\alpha c|z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u||\nabla v|dx\\
&\leq\alpha c|z(\theta_t\omega)|\frac{\sqrt{2}}{r}\phi_1(\|\nabla u\|^2+\|v\|^2)+\frac{\alpha c|z(\theta_t\omega)|}{2}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|v|^2dx\\
&\quad+\frac{\alpha\lambda_1c|z(\theta_t\omega)|}{2}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]|\nabla u|^2dx.
\end{aligned}\tag{4.35}
$$

Collecting all (4.29)-(4.35), from (4.28), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]\left(|v|^2+\varepsilon(\varepsilon-\beta_1)|u|^2+(1-\alpha\varepsilon)|\nabla u|^2+2\tilde{F}(u)\right)dx\\
&\le-(\varepsilon-c|z(\theta_t\omega)|)\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]\left[|v|^2+\varepsilon(\varepsilon-\beta_1)|u|^2+(1-\alpha\varepsilon-\varepsilon\vartheta_1)|\nabla u|^2-\varepsilon k\tilde{F}(u)\right]dx\\
&\quad+\frac{\sqrt{2}}{k}\phi_1\left[\alpha(\|\nabla v\|^2+\|v\|^2)+(1-\alpha\varepsilon)(\|\nabla u\|^2+\|v\|^2)+\alpha c|z(\theta_t\omega)|(\|\nabla u\|^2+\|v\|^2\right]\\
&\quad+\frac{1}{2}\left(c(3\varepsilon+\beta_2)|z(\theta_t\omega)|+c^2|z(\theta_t\omega)|^2\right)\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]\left[|u|^2+|v|^2\right]dx\\
&\quad+\frac{1}{4(\alpha\lambda_1-\beta_2)}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]|g(x,t)|^2dx+(\vartheta_2\varepsilon-c_{\vartheta_3}c|z(\theta_t\omega)|)\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]dx. \qquad (4.36)
\end{aligned}$$

Moreover, using Lemma 4.1 and $\|\nabla u\|^2\ge\lambda_1\|u\|^2$, when ε is small enough, need

$$\begin{cases}
\varrho=\min[\varepsilon,\frac{\varepsilon k}{2}],\\
1-\alpha\varepsilon-\varepsilon\vartheta_1\ge 1-\alpha\varepsilon,\\
\alpha_1=\min\frac{\sqrt{2}}{k}\phi_1(1-\alpha\varepsilon,\alpha),\\
\Upsilon(t,\omega)=\frac{1}{4(\alpha\lambda_1-\beta_2}|q(x,t)|^2+(\vartheta_2\varepsilon-C_{\vartheta_3}c|z(\theta_t\omega)|)\,.
\end{cases} \qquad (4.37)$$

By all the above inequalities, we can write that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]\left[|v|^2+\varepsilon(\varepsilon-\beta_1)|u|^2+(1-\alpha\varepsilon)|\nabla u|^2+2\tilde{F}(u)\right]dx\\
&\le-(\varrho-c|z(\theta_t\omega)|)\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{k^2}\right]\left[|v|^2+\varepsilon(\varepsilon-\beta_1)|u|^2+(1-\alpha\varepsilon)|\nabla u|^2+\tilde{F}(u)\right]dx\\
&\quad+\tfrac{1}{2}\left(c(3\varepsilon+\beta_2)|z(\theta_t\omega)|+c^2|z(\theta_t\omega)|^2\right)\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{r^2}\right]\left[|u|^2+|v|^2\right]dx\\
&\quad+\alpha_1\left[\|\nabla v\|^2+\|\nabla u\|^2+\|v\|^2\right]+\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{r^2}\right]\Upsilon(t,\omega)dx.
\end{aligned} \qquad (4.38)$$

Setting

$$\mathbb{X}=\|v\|^2+\varepsilon(-\varepsilon+\beta_1)\|u\|^2+(1-\alpha\varepsilon)\|\nabla u\|^2. \qquad (4.39)$$

So, it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}(t,\tau,\omega,\mathbb{X}_\tau(\omega))|+\tilde{F}(u)\right]dx\\
&\le-2\left[\varrho-\frac{1}{2}\left(c(3\varepsilon+\beta_2)|z(\theta_t\omega)|+c^2|z(\theta_t\omega)|^2\right)\right]\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}(t,\tau,\omega,\mathbb{X}_\tau(\omega))|+\tilde{F}(u)\right]dx\\
&\quad+\alpha_1\left[\|\nabla v\|^2+\|\nabla u\|^2+\|v\|^2\right]+\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\Upsilon(t,\omega)dx. \qquad (4.40)
\end{aligned}$$

Integrating (4.40) over $[\tau,t]$, we find that, for all $t\ge\tau$

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}(t,\tau,\omega,\mathbb{X}_\tau)|_E^2+2\tilde{F}(u(t,\tau,\omega,u_\tau)\right]dx\\
&\le e^{2\Gamma(t-\tau)}\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{r^2}\right]\left[|\mathbb{X}_\tau|_E^2+\tilde{F}(u_\tau)\right]dx\\
&\quad+\alpha_1\int_\tau^t e^{2\Gamma(r-t)}\left(\|\nabla v(r,\tau,\omega,v_\tau)\|^2+\|\nabla u(r,\tau,\omega,u_\tau)\|^2+\|v(r,\tau,\omega,v_\tau)\|^2\right)dr\\
&\quad+\int_\tau^t e^{2\Gamma(r-t)}\int_{\mathbb{R}^n}\rho\left[\tfrac{|x|^2}{r^2}\right]\Upsilon(r,\theta_r\omega)dxdr,
\end{aligned} \qquad (4.41)$$

where $\Gamma = \varrho - \frac{1}{2}\left(c(3\varepsilon+\beta_2)|z(\theta_t\omega)| + c^2|z(\theta_t\omega)|^2\right)$. By replacing ω *by* $\theta_{-t}\omega$, we have

$$\begin{array}{l}\displaystyle\int_{\mathbb{R}^n} \rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}(t,\tau,\theta_{-t}\omega,\mathbb{X}_\tau(\theta_{-t}\omega))|_E^2 + 2\tilde{F}(u(t,\tau,\theta_{-t}\omega,u_\tau))\right]dx \\ \le e^{2\Gamma(t-\tau)}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}_\tau(\theta_{-t}\omega)|_E^2 + \tilde{F}(u_\tau)\right]dx \\ \quad + \int_{\tau-t}^0 e^{2\Gamma(r-t)}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\Upsilon(r,\theta_r\omega)dxdr \\ \quad +\alpha_1\int_{\tau-t}^0 e^{2\Gamma(r-t)}\left(\|\nabla v(r,\tau,\theta_r\omega,v_\tau)\|^2\right. \\ \quad \left.+\|\nabla u(r,\tau,\theta_r\omega,u_\tau)\|^2 + \|v(r,\tau,\theta_r\omega,v_\tau)\|^2\right)dr.\end{array} \tag{4.42}$$

Let $\mathbb{X}_\tau = (u_\tau, v_\tau)^\top \in B(\tau,\theta_{-t}\omega), B \in \mathfrak{D}$ is tempered, by (4.23) and (4.24), we know that the first term on the right-hand side of (4.42) goes to zero as $t \to -\infty$. So, there exist $\bar{T}_1(\tau,B,\omega) > 0$ and $\bar{k}_1 = \bar{k}_1(\tau,\omega,B)$, for all $t \ge \bar{T}_1$,

$$\lim_{r\to\infty} e^{-\Gamma r}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{r^2}\right]\left[|\mathbb{X}_\tau(\theta_{-t}\omega)|_E^2 + \tilde{F}(u_\tau)\right]dx \le 2\zeta. \tag{4.43}$$

Together with the Lemma 4.1 and (1.8), there are $\bar{T}_2 = \bar{T}_2(\tau,B,\omega) > 0$ and $\bar{k}_1 = \bar{k}_1(\tau,\omega) \ge 1$ such that for all $t \ge \bar{T}_2$ and $k \ge \bar{k}_1$,

$$\int_{\tau-t}^0 e^{2\Gamma(r-t)}\int_{\mathbb{R}^n}\rho\left[\frac{|x|^2}{k^2}\right]\Upsilon(r,\theta_r\omega)dxdr \le \zeta. \tag{4.44}$$

From Lemma 4.2, there are $\bar{T}_3 = \bar{T}_3(\tau,B,\omega) > 0$ and $\bar{k}_2 = \bar{k}_2(\tau,\omega) \ge 1$, such that for all $t \ge \bar{T}_3$ and $k \ge \bar{k}_2$,

$$\alpha_1\int_{\tau-t}^0 e^{2\Gamma(r-t)}\left(\|\nabla v(r,\tau,\theta_r\omega,v_\tau)\|^2 + \|\nabla u(r,\tau,\theta_r\omega,u_\tau)\|^2 + \|v(r,\tau,\theta_r\omega,v_\tau)\|^2\right)dr \le \zeta. \tag{4.45}$$

Setting

$$\begin{cases}\bar{T} = \max\{\bar{T}_1,\bar{T}_2,\bar{T}_3\}, \\ \bar{k} = \{\bar{k}_1,\bar{k}_2\},\end{cases} \tag{4.46}$$

by (4.43)-(4.45), we arrive at

$$\|\mathbb{X}(t,\tau,\theta_{-t}\omega,\mathbb{X}_\tau(\theta_{-t}\omega))\|_{E(\mathbb{R}^n\setminus H_k)}^2 \le 4\zeta. \tag{4.47}$$

Then we complete the proof. □

Now we derive uniform estimates on the high-mode parts of the solution in bounded domains $H_{2k} = \{x \in \mathbb{R}^n : |x| < 2k\}$, these estimates will also be used to establish pullback asymptotic compactness. Denote $q(s) = I - p(s)$, where p(s) is the cut-off function defined by (4.24). Given positive integer r, we define two new variables $\tilde{u}$ and $\tilde{v}$ by

$$\begin{cases}\tilde{u}(t,\tau,\omega,u_\tau) = q\left[\dfrac{|x|^2}{k^2}\right]u(t,\tau,\omega,u_\tau), \\ \tilde{v}(t,\tau,\omega,v_\tau) = q\left[\dfrac{|x|^2}{k^2}\right]v(t,\tau,\omega,v_\tau).\end{cases} \tag{4.48}$$

Multiplying (3.4) by $q\left[\frac{|x|^2}{k^2}\right]$ and using (4.48), we have that

$$\begin{cases} \tilde{u}_t = \tilde{v} - \varepsilon\tilde{u} + c\tilde{u}z(\theta_t\omega), \\ \tilde{v}_t = (\varepsilon - g(u))\tilde{v} + \alpha\left[\triangle\tilde{v} - v\triangle q\left[\frac{|x|^2}{k^2}\right] - 2\nabla v\nabla q\left[\frac{|x|^2}{k^2}\right]\right] \\ \quad + \varepsilon(\varepsilon - g(u))\tilde{u} + (1-\varepsilon\alpha)\left[\triangle\tilde{u} - u\triangle q\left[\frac{|x|^2}{k^2}\right] - 2\nabla u\nabla q\left[\frac{|x|^2}{k^2}\right]\right] \\ \quad + c\left((3\varepsilon - g(u)) - cz(\theta_t\omega)\right)\tilde{u}z(\theta_t\omega) + \alpha c\left[\triangle\tilde{u} - u\triangle q\left[\frac{|x|^2}{k^2}\right] - 2\nabla u\nabla q\left[\frac{|x|^2}{k^2}\right]\right]z(\theta_t\omega) \\ \quad - cvz(\theta_t\omega) - q\left[\frac{|x|^2}{k^2}\right]f(u) + q\left[\frac{|x|^2}{k^2}\right]q(x,t). \end{cases} \tag{4.49}$$

Suppose that $\tilde{u} = \tilde{v} = 0$ for $|x| = 2k$, consider the eigenvalue problem

$$A\tilde{u} = \lambda\tilde{u} \ in \ H_{2k}, \text{ with } \tilde{u} = 0, \text{ on } \partial H_{2k}, \tag{4.50}$$

the problem has a family of eigenfunctions $\{e_i\}_{i\in\mathbb{N}}$ with the eigenvalue $\{\lambda_i\}_{i\in\mathbb{N}}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \lambda_i \to +\infty \ (i \to +\infty),$$

such that $\{e_i\}_{i\in N}$, is an orthonormal basis of $L^2(H_{2k})$ for given n, let

$$X_n = \{e_1, \cdots, e_n\} \text{ and } P_n : L^2(H_{2k}) \to X_n$$

be the projection operator.

Lemma 4.4. *Suppose that the condition (1.3)-(1.8) hold. Let B=*$\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ *and* $\varphi_\tau(\tau,\omega) \in B(\tau,\omega)$. *Then for every* $\eta > 0$ *and P-a.e* $\omega \in \Omega$, *there exists* $\bar{T} = \bar{T}(B,\eta,\omega) > 0$ *and* $\bar{\mathbb{K}} = \bar{\mathbb{K}}(\omega,\eta) > 1$ *and* $\bar{N} = \bar{N}(\eta,\omega) > 0$, *such that the solution* $\varphi(t,\omega,\varphi_\tau(\tau,\omega))$ *of (3.4) satisfies, for all* $t \geq \bar{T}, k \geq \bar{\mathbb{K}}, k \geq \bar{N}$,

$$\|(I - P_n)\psi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_\tau)\|^2_{E(H_{2k})} \leq \eta. \tag{4.51}$$

Proof. Let

$$\begin{bmatrix} \tilde{u}_{n,1} = P_n\tilde{u}, \\ \tilde{v}_{n,1} = P_n\tilde{v}, \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{u}_{n,2} = (I - P_n)\tilde{u}, \\ \tilde{v}_{n,2} = (I - P_n)\tilde{v}. \end{bmatrix} \tag{4.52}$$

Multiplying the first equation of (4.50) with $(I - P_n)$, we get

$$\tilde{v}_{n,2} = \frac{d}{dt}\tilde{u}_{n,2} + \varepsilon\tilde{u}_{n,2} - c\tilde{u}_{n,2}z(\theta_t\omega), \tag{4.53}$$

then applying $(1 - P_n)$ to the second equation of (4.50) and taking the inner product for results with $\tilde{v}_{n,2}$ in $L^2(H_{2k})$, we show that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\tilde{v}_{n,2}\|^2 &= (\varepsilon - g(u))\|\tilde{v}_{n,2}\|^2 - \alpha\|\nabla\tilde{v}_{n,2}\| - \alpha\left(v\triangle q\left[\frac{|x|^2}{k^2}\right] + 2\nabla v\nabla q\left[\frac{|x|^2}{k^2}\right], \tilde{v}_{n,2}\right) \\ &+ \varepsilon(\varepsilon - g(u))(\tilde{u}_{n,2}, \tilde{v}_{n,2}) + (1-\varepsilon\alpha)\left(\triangle\tilde{u}_{n,2} - u\triangle q\left[\frac{|x|^2}{k^2}\right] - 2\nabla u\nabla q\left[\frac{|x|^2}{k^2}\right], \tilde{v}_{n,2}\right) \\ &+ c\left((3\varepsilon - g(u)) - cz(\theta_t\omega)\right)(\tilde{u}_{n,2}, \tilde{v}_{n,2})z(\theta_t\omega) + (I - P_n)q\left[\frac{|x|^2}{k^2}\right](q(x,t) - f(u), \tilde{v}_{n,2}) \\ &+ \alpha c\left(\triangle\tilde{u}_{n,2} - u\triangle q\left[\frac{|x|^2}{k^2}\right] - 2\nabla u\nabla q\left[\frac{|x|^2}{k^2}\right], \tilde{v}_{n,2}\right)z(\theta_t\omega) - c\|\nabla\tilde{v}_{n,2}\||z(\theta_t\omega)|. \end{aligned} \tag{4.54}$$

Next, we need to estimate each term on the right-hand side of (4.54)

$$\begin{aligned}\varepsilon(\varepsilon-g(u))(\tilde{u}_{n,2},\tilde{v}_{n,2}) &= \varepsilon(\varepsilon-g(u))\left(\tilde{u}_{n,2},\frac{d}{dt}\tilde{u}_{n,2}+\varepsilon\tilde{u}_{n,2}-c\tilde{u}_{n,2}z(\theta_t\omega)\right)\\ &\le \varepsilon(\varepsilon-\beta_1)\left(\tfrac{1}{2}\tfrac{d}{dt}\|\tilde{u}_{n,2}\|^2+(\varepsilon-c|z(\theta_t\omega)|)\|\tilde{u}_{n,2}\|^2\right),\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}&(\varepsilon\alpha-1)(-\triangle\tilde{u}_{n,2},\tilde{v}_{n,2})\\ &=(\varepsilon\alpha-1)(-\triangle\tilde{u}_{n,2},\tfrac{d}{dt}\tilde{u}_{n,2}+\varepsilon\tilde{u}_{n,2}-c\tilde{u}_{n,2}z(\theta_t\omega))\\ &\le(\varepsilon\alpha-1)(\nabla\tilde{u}_{n,2},\nabla(\tfrac{d}{dt}\tilde{u}_{n,2}+\varepsilon\tilde{u}_{n,2}-c\tilde{u}_{n,2}z(\theta_t\omega)))\\ &\le(\varepsilon\alpha-1)(\nabla\tilde{u}_{n,2},\nabla(\tfrac{d}{dt}\tilde{u}_{n,2}-(-\varepsilon+c|z(\theta_t\omega)|)\tilde{u}_{n,2})\\ &\le(\varepsilon\alpha-1)\tfrac{1}{2}\left(\tfrac{d}{dt}\|\nabla\tilde{u}_{n,2}\|^2-(-\varepsilon+c|z(\theta_t\omega)|)\|\nabla\tilde{u}_{n,2}\|^2\right).\end{aligned} \tag{4.56}$$

For the nonlinear term, by (1.4)-(1.7) we have

$$\begin{aligned}&\left((I-P_n)\,q\left[\frac{|x|^2}{k^2}\right]f(u),\tilde{v}_{n,2}\right)\\ &\le\left((I-P_n)\,q\left[\tfrac{|x|^2}{k^2}\right]f(u),\tfrac{d}{dt}\tilde{u}_{n,2}+\varepsilon\tilde{u}_{n,2}-c\tilde{u}_{n,2}z(\theta_t\omega)\right)\\ &\le\tfrac{d}{dt}\left((I-P_n)q\left[\tfrac{|x|^2}{k^2}\right]f(u),\tilde{u}_{n,2}\right)-\left((I-P_n)q\left[\tfrac{|x|^2}{k^2}\right]\acute{f}_u(u)u_t,\tilde{u}_{n,2}\right)\\ &\quad+(\varepsilon c_3-c|z(\theta_t\omega)|)\left((I-P_n)q\left[\tfrac{|x|^2}{k^2}\right]f(u),\tilde{u}_{n,2}\right).\end{aligned} \tag{4.57}$$

By (1.4), using Hölder inequality and Gagliardo-Nirenberg inequality, the nonlinear term in (4.54) satisfies

$$\begin{aligned}&2\varepsilon\left((I-P_n)q(\frac{|x|^2}{k^2})\acute{f}_u(u)u_t,\tilde{u}_{n,2}\right)\\ &\le c_1 2\varepsilon(I-P_n)q(\tfrac{|x|^2}{k^2})\,(1+\|u\|^p)\,\|u_t\|\|\tilde{u}_{n,2}\|\\ &\le M_1\|u_t\|\|\tilde{u}_{n,2}\|+M_2\|u\|_6^p\|u_t\|\|\tilde{u}_{n,2}\|_{\frac{6}{3-p}}\\ &\le M_1\lambda_{n+1}^{-\frac{1}{2}}\|u_t\|\|\nabla\tilde{u}_{n,2}\|+M_3\|u\|_{H^1}^p\|u_t\|\|\tilde{u}_{n,2}\|^{\frac{2-p}{2}}\|\nabla\tilde{u}_{n,2}\|^{\frac{p}{2}}\\ &\le M_1\lambda_{n+1}^{-1}\|u_t\|\|\nabla\tilde{u}_{n,2}\|+M_3\lambda_{n+1}^{\frac{p-2}{4}}\|u\|_{H^1}^p\|u_t\|\|\tilde{u}_{n,2}\|\\ &\le M_4\|\nabla\tilde{u}_{n,2}\|+M_5\lambda_{n+1}^{-1}\|u_t\|^2+M_5\lambda_{n+1}^{\frac{p-2}{2}}\|u\|_{H^1}^{2p}\|u_t\|^2\\ &\le M_4\|\nabla\tilde{u}_{n,2}\|+M_6\lambda_{n+1}^{-1}\left(\|u\|^2+\|v\|^2+\|u\|^4+|z(\theta_t\omega)|^4\right)\\ &\quad+M_6\lambda_{n+1}^{\frac{p-2}{2}}\left(\|u\|_{H^1(\mathbb{R}^n)}^6+\|v\|^6+|z(\theta_t\omega)|^{\frac{12}{3-p}}\right).\end{aligned} \tag{4.58}$$

Similarly, we also have

$$\begin{aligned}2\varepsilon\left((I-P_n)q(\tfrac{|x|^2}{k^2})f(u),\tilde{u}_{n,2}\right)&\le c_1 2\varepsilon(I-P_n)q(\tfrac{|x|^2}{k^2})\left(\|u\|+\|u\|^{p+1}\right)\|\tilde{u}_{n,2}\|\\ &\le M_7\|u\|\|\tilde{u}_{n,2}\|+M_8\|u\|_{2(p+1)}^{p+1}\|\tilde{u}_{n,2}\|_{\frac{6}{3-p}}\\ &\le M_7\lambda_{n+1}^{-\frac{1}{2}}\|u\|\|\nabla\tilde{u}_{n,2}\|+M_9\lambda_{n+1}^{-\frac{1}{2}}\|u\|_{H^1}^p\|\nabla\tilde{u}_{n,2}\|\\ &\le M_{10}\|\nabla\tilde{u}_{n,2}\|+M_{11}\lambda_{n+1}^{-1}\left(\|u\|^2+\|u\|_{H^1}^{2p}\right).\end{aligned} \tag{4.59}$$

By using the Cauchy-Schwartz inequality and the Young inequality, we have that

$$\begin{aligned}\left((I-P_n)q\left[\frac{|x|^2}{k^2}\right]q(x,t),\tilde{v}_{n,2}\right)\le&\frac{2\lambda_1}{2\alpha\lambda_1+\delta\varepsilon+1}\left\|(I-P_n)q\left[\frac{|x|^2}{k^2}\right]q(x,t)\right\|^2\\ &+\frac{2\alpha\lambda_1+\delta\varepsilon+1}{2}\|\tilde{v}_{n,2})\|^2,\end{aligned} \tag{4.60}$$

$$\begin{aligned}&c\left((3\varepsilon+\beta_2)-cz(\theta_t\omega)\right)(\tilde{u}_{n,2},\tilde{v}_{n,2})\,z(\theta_t\omega)\\ &\qquad\le\tfrac{1}{2}((3\varepsilon+\beta_2)c|z(\theta_t\omega)|+\tfrac{1}{2}c^2|z(\theta_t\omega)|^2)(\|\tilde{u}_{n,2}\|^2+\|\tilde{v}_{n,2}\|^2).\end{aligned} \tag{4.61}$$

Next, we have

$$\begin{aligned}
&\left(uAq\left[\frac{|x|^2}{k^2}\right]-2\nabla u\nabla q\left[\frac{|x|^2}{k^2}\right],\tilde{v}_{n,2}\right)\\
&=\left(-u\left(\tfrac{4x^2}{k^4}q''\left[\tfrac{|x|^2}{k^2}\right]+\tfrac{2}{k^2}q'\left[\tfrac{|x|^2}{k^2}\right]\right)-\tfrac{4x}{k^2}\nabla uq'\left[\tfrac{|x|^2}{k^2}\right],\tilde{v}_{n,2}\right)\\
&\le\textstyle\int_{k<|x|<2\sqrt{2}k}\left(-u\left(\tfrac{4x^2}{k^4}q''\left[\tfrac{|x|^2}{k^2}\right]+\tfrac{2}{k^2}q'\left[\tfrac{|x|^2}{k^2}\right]\right)-\tfrac{4x}{k^2}\nabla uq'\left[\tfrac{|x|^2}{k^2}\right],\tilde{v}_{n,2}\right)\\
&=\left(-u\left(\tfrac{8}{k^2}\phi_2+\tfrac{2}{k^2}\phi_1\right)-\tfrac{4\sqrt{2}}{k}\nabla u\phi_1,\tilde{v}_{n,2}\right)\\
&\le\left(\tfrac{8\phi_2+2\phi_1}{k^2}\|u\|\ \|\tilde{v}_{n,2}\|+\tfrac{4\sqrt{2}}{k}\phi_1\|\nabla u\|\ \|\tilde{v}_{n,2}\|\right)\\
&\le\tfrac{1}{2}\left((\tfrac{8\phi_2+2\phi_1}{k^2})^2\|u\|^2+\tfrac{32\phi_1^2}{k^2}\|\nabla u\|^2\right)+\tfrac{1}{2}\|\tilde{v}_{n,2}\|^2,
\end{aligned}\tag{4.62}$$

and

$$\begin{aligned}
&\alpha(vAq\left[\frac{|x|^2}{k^2}\right]-2\nabla v\nabla q\left[\frac{|x|^2}{k^2}\right],\tilde{v}_{n,2})\\
&=\alpha(-v\left(\tfrac{4x^2}{k^4}q''\left[\tfrac{|x|^2}{k^2}\right]+\tfrac{2}{k^2}q'\left[\tfrac{|x|^2}{k^2}\right]\right)-\tfrac{4x}{k^2}\nabla vq'\left[\tfrac{|x|^2}{k^2}\right],\tilde{v}_{n,2})\\
&\le\alpha\textstyle\int_{k<|x|<2\sqrt{2}k}\left(-v\left(\tfrac{4x^2}{k^4}q''\left[\tfrac{|x|^2}{k^2}\right]+\tfrac{2}{k^2}q'\left[\tfrac{|x|^2}{k^2}\right]\right)-\tfrac{4x}{k^2}\nabla vq'\left[\tfrac{|x|^2}{k^2}\right],\tilde{v}_{n,2}\right)\\
&=\alpha\left(-u\left(\tfrac{8}{k^2}\phi_2+\tfrac{2}{k^2}\phi_1\right)-\tfrac{4\sqrt{2}}{k}\nabla v\phi_1,\tilde{v}_{n,2}\right)\\
&\le\alpha\left(\tfrac{8\phi_2+2\phi_1}{k^2}\|v\|\ \|\tilde{v}_{n,2}\|+\tfrac{4\sqrt{2}}{k}\phi_1\|\nabla v\|\ \|\tilde{v}_{n,2}\|\right)\\
&\le\tfrac{\alpha}{2}\left((\tfrac{8\phi_2+2\phi_1}{k^2})^2\|v\|^2+\tfrac{32\phi_1^2}{k^2}\|\nabla v\|^2\right)+\tfrac{\alpha}{2}\|\tilde{v}_{n,2}\|^2.
\end{aligned}\tag{4.63}$$

By applying (4.55)-(4.63) and (4.54), after detailed computations we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{v}_{n,2}\|^2+\varepsilon(\varepsilon-\beta_1)\|\tilde{u}_{n,2}\|^2+(1-\varepsilon\alpha)\|\nabla\tilde{u}_{n,2}\|^2+\left(2(I-P_n)q\left[\frac{|x|^2}{k^2}\right]f(u),\tilde{u}_{n,2}\right)\right)\\
&\le-\left(\varepsilon-c|z(\theta_t\omega)|\right)(\|\tilde{v}_{n,2}\|^2+\varepsilon(\varepsilon-\beta_1)\|\tilde{u}_{n,2}\|^2+(1-\varepsilon\alpha)\|\nabla\tilde{u}_{n,2}\|^2)\\
&\quad-(\varepsilon-c|z(\theta_t\omega)|)\left((I-P_n)q\left[\frac{|x|^2}{k^2}\right]f(u),\tilde{u}_{n,2}\right)+\frac{2\lambda_1}{2\alpha\lambda_1+\delta\varepsilon+1}\left\|(I-P_n)q\left[\frac{|x|^2}{k^2}\right]q(x,t)\right\|^2\\
&\quad+\left(c(\frac{3}{2}\varepsilon+\beta_2)|z(\theta_t\omega)|+\frac{1}{2}c^2|z(\theta_t\omega)|^2\right)\left(\|\tilde{u}_{n,2}\|^2+\|\tilde{v}_{n,2}\|^2\right)\\
&\quad+\frac{2\beta_2-\alpha}{2}\left((\frac{8\phi_2+2\phi_1}{k^2})^2\|v\|^2+\frac{32\phi_1^2}{k^2}\|\nabla v\|^2\right)\\
&\quad+\frac{\alpha c|z(\theta_t\omega)|+\alpha-(\varepsilon\alpha-1)}{2}\left((\frac{8\phi_2+2\phi_1}{k^2})^2\|u\|^2+\frac{32\phi_1^2}{k^2}\|\nabla u\|^2\right)\\
&\quad+M_4\|\nabla\tilde{u}_{n,2}\|+M_6\lambda_{n+1}^{-1}\left(\|u\|^2+\|v\|^2+\|u\|^4+|z(\theta_t\omega)|^4\right)\\
&\quad+M_6\lambda_{n+1}^{\frac{p-2}{2}}\left(\|u\|^6_{H^1(\mathbb{R}^n)}+\|v\|^6+|z(\theta_t\omega)|^{\frac{12}{3-p}}\right).
\end{aligned}\tag{4.64}$$

Recalling the norm $\|\cdot\|_E$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|\psi_{n,2}\|_E^2+\left(2(I-P_n)q\left[\frac{|x|^2}{k^2}\right]f(u),\tilde{u}_{n,2}\right)\right)\\
&\le-2\left(\gamma-\kappa c|z(\theta_t\omega)|-c^2|z(\theta_t\omega)|^2\right)\left(\|\psi_{n,2}\|_E^2+\left(2(I-P_n)q\left[\frac{|x|^2}{k^2}\right]f(u),\tilde{u}_{n,2}\right)\right)\\
&\quad+\frac{2\lambda_1}{2\alpha\lambda_1+\delta\varepsilon+1}\left\|(I-P_n)q\left[\frac{|x|^2}{k^2}\right]q(x,t)\right\|^2+\frac{2\beta_2-\alpha}{2}\left((\frac{8\phi_2+2\phi_1}{k^2})^2\|v\|^2+\frac{32\phi_1^2}{k^2}\|\nabla v\|^2\right)\\
&\quad+\frac{\alpha c|z(\theta_t\omega)|+\alpha+(1-\varepsilon\alpha)}{2}\left((\frac{8\phi_2+2\phi_1}{k^2})^2\|u\|^2+\frac{32\phi_1^2}{k^2}\|\nabla u\|^2\right)\\
&\quad+M_6\lambda_{n+1}^{\frac{p-2}{2}}\left(\|u\|^6_{H^1(\mathbb{R}^n)}+\|v\|^6+|z(\theta_t\omega)|^{\frac{12}{3-p}}\right)\\
&\quad+M_6\lambda_{n+1}^{-1}\left(\|u\|^2+\|v\|^2+\|u\|^4+|z(\theta_t\omega)|^4\right),
\end{aligned}\tag{4.65}$$

where $\gamma = \min[\varepsilon c_3, \varepsilon]$, note that $0 \le p < 3, \lambda_n \to +\infty$ when $(n \to +\infty)$. Therefore, given $\eta > 0$, there exists $\bar{N} = \bar{N}(\eta) > 1$ and $\bar{K}_1 = \bar{K}_1(\eta) > 0$ such that for all $n \ge \bar{N}$, $k \ge \bar{K}_1$

$$\begin{aligned}
&\frac{d}{dt}\left(\|\psi_{n,2}\|_E^2 + \left(2(I-P_n)q\left[\frac{|x|^2}{k^2}\right]f(u), \tilde{u}_{n,2}\right)\right)\\
&\le 2\Gamma(t,\omega)\left(\|\psi_{n,2}\|_E^2 + \left(2(I-P_n)q\left[\tfrac{|x|^2}{k^2}\right]f(u), \tilde{u}_{n,2}\right)\right)\\
&\quad +\mu\left(\|(I-P_n)q\left[\tfrac{|x|^2}{k^2}\right]q(x,t)\|^2 + \tfrac{c}{k^4}\|u\|^2 + \tfrac{c}{k^2}\|\nabla u\|^2 + \tfrac{c}{k^4}\|v\|^2 + \tfrac{c}{k^2}\|\nabla v\|^2\right)\\
&\quad +\lambda\left(+\|u\|^4 + |z(\theta_t\omega)|^4 + \|u\|^6_{H^1(\mathbb{R}^n)} + \|v\|^6 + |z(\theta_t\omega)|^{\frac{12}{3-p}}\right),
\end{aligned} \tag{4.66}$$

where μ, λ, c are positive constants, $\Gamma(t,\omega) = \gamma - \kappa c|z(\theta_t\omega)| - c^2|z(\theta_t\omega)|^2$. Applying Gronwall's lemma over $[\tau - t\ ,\ \tau]$ and replacing ω by $\theta_{-\tau}\omega$ on (4.66), for all t≥ 0, then we estimate every term on the right-hand side of the results.

For the first terms, by using condition (1.4), (4.59), (4.66) and $\varphi_\tau \in B(\tau - t_n, \theta_{-t_n}\omega)$, $B \in D$, there exists $\bar{T}_1 = \bar{T}_1(\tau, B, \omega, \eta) > 0$ and $\bar{K}_1 = \bar{K}_1(\tau, \omega, \eta) \ge 1$ such that $t \ge \bar{T}_1$, $k \ge \bar{K}_1$,

$$e^{2\int_\tau^{\tau-t}\Gamma(r,\omega)dr}\left(\|\psi_\tau\|_E^2 + \left(2(I-P_n)q\left[\frac{|x|^2}{k^2}\right]f(u_\tau), \tilde{u}_{n,\tau}\right)\right) \le 2\eta, \tag{4.67}$$

by conditions (1.7) and (1.8), $q(x,t) \in L^2(\mathbb{R}^n)$, there is $\bar{N}_2 = \bar{N}_2(\tau, \omega, \eta) \ge \bar{N}_1 > 0$, such that for all $n \ge \bar{N}_1$,

$$c\int_{-\infty}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\|(I-P_n)\left[\frac{|x|^2}{k^2}\right]q(x,s)\|^2 ds \le \eta. \tag{4.68}$$

Due to Lemma 4.1, Lemma 4.2 and (4.66), there exists $\bar{T}_2 = T_2(\tau, B, \omega, \eta) > 0$ and $\bar{K}_2 = \bar{K}_2(\tau, \omega, \eta) \ge 1$, such that for all $t \ge \bar{T}_2$ and $k \ge \bar{k}_2$

$$\begin{aligned}
&\textstyle\int_{\tau-t}^{\tau} e^{2\int_s^\tau(\Gamma(r-\tau,\omega))dr}\left(\frac{c}{k^4}\|u(s,\tau-t,\theta_{-\tau}\omega,\tilde{u}_\tau)\|^2 + \frac{c}{k^2}\|\nabla u(s,\tau-t,\theta_{-t}\omega,\tilde{u}_\tau)\|^2\right)ds\\
&+\textstyle\int_{\tau-t}^{\tau} e^{2\int_s^\tau(\Gamma(r-\tau,\omega))dr}\left(\frac{c}{k^4}\|v(s,\tau-t,\theta_{-\tau}\omega,\tilde{v}_\tau)\|^2 + \frac{c}{k^2}\|\nabla v(s,\tau-t,\theta_{-t}\omega,\tilde{v}_\tau)\|^2\right)ds \le \eta.
\end{aligned} \tag{4.69}$$

For the fourth term on the right-hand side of results, by Lemmas 3.1, 4.1, 4.2, ([19] Lemma 4.3) and (4.59), there exists $\bar{T}_3 = \bar{T}_3(\tau, B, \omega, \eta) > 0$, such that for all $t \ge \bar{T}_3$,

$$\begin{aligned}
&c\eta\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\lambda\left(\|u(\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_\tau)\|^4 + |z(\theta_t\omega)|^4\right.\\
&\left.+\|u(\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_\tau)\|^6_{H^1(\mathbb{R}^n)} + \|v(\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_\tau)\|^6 + |z(\theta_t\omega)|^{\frac{12}{3-p}}\right)ds\\
&\quad\le \eta\mathbb{R}(\tau,\omega).
\end{aligned} \tag{4.70}$$

Let

$$\begin{aligned}
\bar{T} &= \max\{\bar{T}_1, \bar{T}_2, \bar{T}_3\}\\
\bar{K} &= \{\bar{K}_1, \bar{K}_2\},
\end{aligned} \tag{4.71}$$

then by (4.67)-(4.70) and (4.66), there is $\bar{N}_3 = \bar{N}_3(\tau, \omega, \eta) \ge \bar{N}_2 > 0$ such that for all $n \ge \bar{N}_3$, we have for all $t > \bar{T}$ and $n > \bar{N}$

$$\|\psi_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\psi_\tau)\|_E^2 + \left(2(I-P_n)\left[\frac{|x|^2}{k^2}\right]f(u_\tau), \tilde{u}_{n,\tau}\right) \le \eta(4 + \mathbb{R}(\tau,\omega)) \tag{4.72}$$

which implies formula (4.72) holds. Therefore, we conclude

$$\|(I-P_n)\psi_{n,2}(t,\theta_{-t}\omega,\psi_\tau(\theta_{-t}\omega))\|^2_{E(H_{2k})} \le \eta(4 + \mathbb{R}(\tau,\omega)).$$

Then we complete the proof. □

5. Random attractors

In this section, we prove the existence of a $\mathcal{D}$-random attractor for the random dynamical system Φ associated with the stochastic wave equation (3.7) on $\mathbb{R}^n$. We are now ready to apply the lemmas in Section 4 to prove the asymptotic compactness of solutions on $\mathbb{R}^n$. It follows from Lemma 4.1, that it is Φ has a closed random absorbing set in $\mathcal{D}$, which along with the $\mathcal{D}$-pullback asymptotic compactness will imply the existence of a unique $\mathcal{D}$-random attractor. The $\mathcal{D}$-pullback asymptotic compactness of Φ is given below and will be proved by using the uniform estimates on the tails of solutions.

Lemma 5.1. *We assume that (1.3)-(1.8) hold. Then the random dynamical system ϕ of problem (3.4) is $\mathcal{D}$-pullback asymptotically compact in $E(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ $\omega \in \Omega$, the sequence $\{\varphi(\tau, \tau - t_n, \theta_{-\tau}\omega, \varphi_{\tau,n})\}$ has a convergent subsequence in $E(\mathbb{R}^n)$ provided $t_n \longrightarrow \infty$ and $\varphi_{\tau,n} \in B(\tau - t_n, \theta_{-t_n}\omega)$*

Proof. Let $t_n \longrightarrow \infty$, $B \in \mathcal{D}$ and $\varphi_{\tau,n} \in B(\tau - t_n, \theta_{-t_n}\omega)$. Then by Lemma 4.1, for P-a.e $\omega \in \Omega$, we have that $\{\varphi(\tau, \tau - t_n, \theta_{-\tau}\omega, \varphi_{\tau,n})\}$ is bounded in $E(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$, there exists $L_1 = L_1(\tau, \omega, B) > 0$, such that for all $\tilde{L} \geq L_1$,

$$\|\varphi(\tau, \tau - t_n, \theta_{-\tau}\omega, \varphi_{\tau,n})\|^2_{E(\mathbb{R}^n)} \leq \rho^2(\tau, \omega); \tag{5.1}$$

moreover, it follows from Lemma 4.3 that there exist $k_1 = k_1(\tau, \omega, \eta) > 0$ and $L_2 = L_2(\tau, B, \omega, \eta) > 0$, such that for all $\tilde{L} \geq L_2$,

$$\|\psi(\tau, \tau - t_n, \theta_{-\tau}\omega, \varphi_{\tau,n})\|^2_{E(\mathbb{R}^n \backslash H_{k_1})} \leq \eta. \tag{5.2}$$

Next, by using Lemma 4.4, there are $N = N(\tau, \omega, \eta) > 0$, $k_2 = k_2(\tau, \omega, \eta) > k_1$ and $L_3 = L_3(\tau, B, \omega, \eta) > 0$, such that for all $\tilde{L} \geq L_3$,

$$(I - P_n)\tilde{\|}\varphi(\tau, \tau - t_n, \theta_{-\tau}\omega, \varphi_{\tau,n})\|^2_{E(H_{2k_2})} \leq \eta. \tag{5.3}$$

By means of (4.49) and (5.1), we find that $\{P_n\tilde{\varphi}(\tau, \tau - t_n, \theta_{-\tau}\omega, \varphi_{\tau,n})\}$ is bounded infinite-dimensional space $P_nE(H_{2k_2})$, which associated with (5.3) implies $\{\psi(\tau, \tau - t_n, \theta_{-\tau}\omega, \psi_{\tau,n})\}$ is precompact in $H_0^1(H_{2k_2}) \times L^2(H_{2k_2})$.

Note that $q\left[\frac{|x|^2}{k^2}\right] = 1$ for $\{x \in \mathbb{R}^n : |x| \leq k_2\}$, recalling (4.49), we find that $\{\varphi(\tau, \tau - t_n, \theta_{-\tau}\omega, \varphi_{\tau,n})\}$ is pre-compact in $E(H_{2k_2})$, which along with (5.2) show that the pre-compactness of this sequence in $E(\mathbb{R}^n)$, this completes the proof. The main result of this section can now be stated as follows. □

Theorem 5.2. *We assume that (1.3)-(1.8) hold. Then the continuous cocycle Φ associated with problem (3.4) has a unique $\mathcal{D}$-pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $\mathbb{R}^n$.*

Proof. Notice that the continuous cocycle Φ has a closed random absorbing set $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ in $\mathcal{D}$ by Lemma 4.2. On the other hand, by (3.9) and Lemma 5.1, the continuous cocycle Φ is $\mathcal{D}$-pullback asymptotically compact in the $\mathbb{R}^n$. Hence the existence of a unique $\mathcal{D}$-random attractor for Φ follows from Lemma 2.8 immediately. □

Acknowledgment. The authors are extremely grateful to the reviewers for making valuable comments and suggestions leading to a better presentation of the paper.

References

[1] A.D. Abdelmajid, E.M. Ahmed and Q. Ma, *Random attractors for Stochastic strongly damped non-autonomous wave equations with memory and multiplicative noise*, Open J. Math. Anal. **3**, 50–70, 2019.

[2] L. Arnold and H. Crauel, *Random dynamical systems lyapunov exponents proceedings oberwolfach*, in: Lecture Notes in Mathematics, **1486**, 1–22, 1990.

[3] P.W. Bates, K. Lu and B. Wang, *Random attractors for stochastic reaction diffusion equations on unbounded domains*, J. Differ. Equ. **246**, 845–869, 2009.

[4] T. Caraballo and J. Langa, *On the upper semi-continuity of cocycle attractors for non-autonomous and random dynamical systems*, Discr. Contin. Dyn. Impuls. Syst. Ser. Math. Anal. **10** 491–513, 2003.

[5] T. Caraballo, J. Langa and J. Robinson, *Upper semi-continuity of attractors for small random perturbations of dynamical systems*, Commun. Part. Differ. Equ. **23**, 1557–1581, 1998.

[6] F. Chen, B. Guo and P. Wang, *Long time behavior of strongly damped nonlinear wave equations*, J. Differ. Equ. **147**, 231–241,1998.

[7] J.W. Cholewa and T. Dlotko, *Strongly damped wave equation in uniform spaces*, Nonlinear Anal. **64** 174–187, 2006.

[8] I. Chueshov, *Monotone Random Systems Theory and Applications*, Springer-Verlag, New York, 2002.

[9] H. Crauel, A. Debussche, and F. Flandoli, *Random attractors*, J. Dyn. Differ. Equ. **9**, 307–341, 1997.

[10] H. Crauel and F. Fladli Crauel, *Attractors for a random dynamical systems*, Probab. Theory Related Fields **100**, 365–393, 1994.

[11] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[12] J. Duan, K. Lu and B. Schmalfuss, *Invariant manifolds for stochastic partial differential equations*, Ann. Probab. **31**, 2109–2135, 2003.

[13] M.M. Fadlallah, A.D. Abdelmajid, E.M. Ahmed, M.Y.A. Bakhet and Q. Ma, *Random attractors for semilinear reaction diffusion equation with distribution derivatives and multiplicative noise*, Open J. Math. Sci. **4**, 126–141, 2020.

[14] X. Fan, *Random attractors for damped stochastic wave equations with multiplicative noise*, Int. J. Math. **19**, 421-437, 2008.

[15] F. Flandoli and B. Schmalfuss, *Random attractors for the 3-D stochastic Navier-Stokes equation with multiplicative noise*, Stoch. Rep. **59**, 21–45, 1996.

[16] J.M. Ghidaglia and A. Marzocchi, *Longtime behaviour of strongly damped wave equations, global attractors and their dimension*, SIAM J. Math. Anal. **22**, 879–895, 1991.

[17] J.K. Hale and G. Raugel, *Upper semi-continuity of the attractor for a singularly perturbed hyperbolic equation*, J. Differ. Equ. **73** (2), 197–214, 1988.

[18] V. Kalantarov and S. Zelik, *Finite-dimensional attractors for the quasi-linear strongly-damped wave equation*, J. Differ. Equ. **247**, 1120–1155, 2009.

[19] H. Li, Y. You and J. Tu, *Random attractors and averaging for nonautonomous stochastic wave equations with nonlinear damping*, J. Differ. Equ. **258**, 148–190, 2015.

[20] H. Li and S. Zhou, *One-dimensional global attractor for strongly damped wave equations*, Commun. Nonlinear Sci. Numer. Simul. **12**, 784–793, 2007.

[21] H. Li and S. Zhou, *On non-autonomous strongly damped wave equations with a uniform attractor and Some averaging*, J. Math. Anal. Appl. **341**, 791–802, 2008.

[22] P. Massatt, *Limiting behavior for strongly damped nonlinear wave equations*, J. Differ. Equ. **48**, 334–349, 1983.

[23] H. Morimoto, *Attractors of probability measures for semi-linear stochastic evolution equations*, Stoch. Anal. Appl. **10**, 205–212, 1992.

[24] V. Pata and S. Zelik, *Smooth attractors for strongly damped wave equations*, Nonlinearity, **19**, 1495–1506, 2006.

[25] A. Pazy, *Semigroup of linear operators and applications to partial differential equations*, Appl. Math. Sci., Springer, New York, 1983.

[26] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1997.

[27] B. Wang, *Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems*, J. Part. Differ. Equ. **253**, 1544–1583, 2012.

[28] B. Wang, *Random attractors for non-autonomous stochastic wave equations with multiplicative noise*, Discrete Contin. Dyn. Syst. **34**, 269–300, 2014.

[29] B. Wang, *Existence and upper semi-continuity of attractors for stochastic equations with deterministic non-autonomous terms*, Stoch. Dyn. **14** (4), 1450009, 2014.

[30] B. Wang and X. Gao, *Random attractors for wave equations on unbounded domains*, Discrete Contin. Dyn. Syst. Spec. **2009**, 800–809, 2009.

[31] Z. Wang and S. Zhou, *Asymptotic behavior of stochastic strongly wave equation on unbounded domains*, J. Appl. Math. Phys. **3**, 338-357, 2015.

[32] M. Yang and C. Sun, *Dynamics of strongly damped wave equations in locally uniform spaces: attractors and asymptotic regularity*, Trans. Amer. Math. Soc. **361**, 1069–1101, 2009.

[33] M. Yang and C. Sun, *Exponential attractors for the strongly damped wave equations*, Nonlinear. Anal: Real World Appl. **11**, 913–919, 2010.

[34] F. Yin and L. Liu, *D-pullback attractor for a non-autonomous wave equation with additive noise on unbounded domains*, J. Comput. Math. Appl. **68**, 424–438, 2014.

[35] S. Zhou and X. Fan, *Kernel sections for non-autonomous strongly damped wave equations*, J. Math. Anal. Appl. **275**, 850–869, 2002.