

Bernstein Operator Approach for Solving Linear Differential Equations

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Abstract

In this study, an alternative numerical method having regard to the Bernstein operator is generated for approximate solutions of linear differential equations in the most general form under the initial and boundary conditions. Some applications are also revealed to show how the procedure can be performed for the problems.

Keywords: Bernstein polynomials; Linear differential equations; Operator numerical method

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1. Introduction

Sergei Natanovich Bernstein have produced the Bernstein polynomials verified the Weierstrass Theorem [8] in 1912. Then, these polynomials have led the constructive function theory and have become center of the applied analysis. Considering the Bernstein polynomials as a sequence of linear positive operators, these polynomials hold the Korovkin Theorem [23]. For this reason, the Bernstein operator and their derivatives converge to a continuous function and their derivatives on a closed and bounded interval [24].

In ages, the generated numerical methods have been investigated by considering the Bernstein polynomials. Numerical examples of these polynomials have been come on differential equations [7, 10, 18, 27, 36], integral equations [5, 35], integro-differential equations [6], partial differential equations [4, 39], fractional order equations [38] from past to present.

In recent years, there are several searches about reformed numerical methods based on the Bernstein polynomials. Only in four years, collocation method that is one of the numerical method has been developed for numerical solution of many types of equations like integral equations [17], integro-differential equations [11, 37] and differential equations [25]. The Bernstein polynomials have also been modified the Adomian decomposition [31] and Laplace decomposition method [33] for solving linear and nonlinear differential equations. The spectral Petrov-Galerkin method [20] has been applied for numerical solution of fractional partial differential equations that are one of the most popular and remarkable equations in the science world. Discretization method [13] that is other numerical method has been applied to the Volterra-Fredholm integral equations by utilizing the Bernstein polynomials. Improved variational iteration method [30] has been used for approximate solution of partial differential equations.

Moreover, some operational matrix forms of the Bernstein polynomials have been generated for numerical methods. For example, operational matrix of shifted orthonormal Bernstein polynomials has been used for the numerical solution of pantograph equations [21], computational methods based on operational matrix of differentiation and integration via the Bernstein polynomials has been presented for solving differential equations [3, 22, 28, 29, 39], Volterra integral equations [26] and fractional order differential equations [34].

However, in point of fact, numerical methods have not examined by taking the Bernstein operator instead of the Bernstein polynomials for numerical solutions of these types of equations. In this direction, our aim is to converge the solution of m th-order linear differential equation

$$\sum_{k=0}^m a_k(x) y^{(k)}(x) = g(x), \quad a \leq x \leq b \quad (1.1)$$

under the mixed conditions

$$\sum_{k=0}^{m-1} \sum_{j=0}^l \alpha_{ij}^k y^{(k)}(c_j) = \beta_i; \quad i = 0, 1, \dots, m-1, \quad a \leq c_j \leq b. \quad (1.2)$$

Here the solution, coefficient and source functions assume that have Bernstein operator defined on the interval $[a, b]$ as follows:

$$\begin{aligned} y^{(k)}(x) &\cong B_n^{(k)}(y; x) = \sum_{i=0}^n y \left(a + \frac{i(b-a)}{n} \right) p_{i,n}^{(k)}(x), \\ g(x) &\simeq B_n(g, x) = \sum_{i=0}^n g \left(a + \frac{i(b-a)}{n} \right) p_{i,n}(x), \\ a_k(x) &\cong B_n(a_k; x) = \sum_{i=0}^n a_k \left(a + \frac{i(b-a)}{n} \right) p_{i,n}(x). \end{aligned} \quad (1.3)$$

Here $a_k(x), g(x) \in C[a, b]$, and $y(x)$ is unknown function. Moreover, $p_{i,n}(x)$ called the generalized Bernstein polynomials [14].

Theorem 1.1. *The definite integral of the Bernstein basis polynomials on the interval $[a, b]$ has the following equality [11]*

$$\int_a^b p_{i,n}(x) dx = \frac{b-a}{n+1}; \quad 0 \leq i \leq n.$$

Theorem 1.2. *There is a relation between generalized Bernstein basis polynomials matrix and their derivatives in the form*

$$\mathbf{P}^{(k)}(x) = \mathbf{P}(x) \mathbf{N}^k; \quad k = 0, 1, \dots, m.$$

Here $\mathbf{P}(x) = [p_{i,n}(x)]$, $\mathbf{P}^k(x) = [p_{i,n}^{(k)}(x)]$ are $1 \times (n+1)$ matrices, $\mathbf{N} = (d_{ij})$ is $(n+1) \times (n+1)$ matrix such that the elements of \mathbf{N} are defined by

$$d_{ij} = \frac{1}{b-a} \begin{cases} n-i & ; \text{ if } j = i+1 \\ 2i-n & ; \text{ if } j = i \\ -i & ; \text{ if } j = i-1 \\ 0 & ; \text{ otherwise} \end{cases}$$

for $i, j = 0, 1, \dots, n$ and $\mathbf{N}^0 = \mathbf{I}$ is identity matrix [1].

The historical development of the Bernstein polynomials, derivated numerical methods by way of these polynomials, the definition of the differential equation, some main theorems and fundamental matrix relation required for the mentioned work are given in Section 1. The new method is presented, how it is found out by using the Bernstein operator approach, in Section 2. The proposed method is shown how is applied to the differential equations by tables, and some errors used in the problems are defined in Section 3. In this part, four examples are also considered to see convergency of the method, and the numerical results of the proposed method and other methods are compared in the last example. Finally, in Section 4, advantages and disadvantages of the presented method are commented, and given some ideas about future studies.

2. Method of Solution

Theorem 2.1. Let y , a_k and g functions have the Bernstein operator approach as given in (1.3), then the differential equation (1.1) holds the following matrix equation:

$$\sum_{k=0}^m \mathcal{A}_k \mathcal{I} \mathbf{N}^k \mathcal{Y} = \mathcal{G}.$$

Here \mathcal{Y} , \mathcal{A}_k and \mathcal{G} are $(n+1) \times 1$ matrices and \mathcal{I} is $1 \times (n+1)$ matrix. Besides, the elements of these matrices are expressed as

$$\begin{aligned} \mathcal{Y} &= \left[y(a) \quad y\left(a + \frac{b-a}{n}\right) \quad \dots \quad y(b) \right]^T, \\ \mathcal{A}_k &= \left[a_k(a) \quad a_k\left(a + \frac{b-a}{n}\right) \quad \dots \quad a_k(b) \right]^T, \\ \mathcal{G} &= \left[g(a) \quad g\left(a + \frac{b-a}{n}\right) \quad \dots \quad g(b) \right]^T, \\ \mathcal{I} &= \frac{1}{n+1} \left[1 \quad 1 \quad 1 \quad \dots \quad 1 \right]. \end{aligned}$$

Proof. Considering Bernstein operator approach (1.3) and Theorem 1.2, matrix forms of the y , a_k and g functions can be denoted as follows:

$$y(x) \simeq \mathbf{P}(x)\mathcal{Y}, \quad y^{(k)}(x) \simeq \mathbf{P}^{(k)}(x)\mathcal{Y} = \mathbf{P}(x)\mathbf{N}^k\mathcal{Y}, \quad (2.1)$$

$$g(x) \simeq \mathbf{P}(x)\mathcal{G}, \quad a_k(x) \simeq \mathbf{P}(x)\mathcal{A}_k. \quad (2.2)$$

Substituting expressions (2.1) and (2.2) into equation (1.1), we obtain a linear algebraic equation

$$\sum_{k=0}^m \mathbf{P}(x)\mathcal{A}_k\mathbf{P}(x)\mathbf{N}^k\mathcal{Y} = \mathbf{P}(x)\mathcal{G}$$

and making some required simplifications, the equation can be rewritten as

$$\sum_{k=0}^m \mathcal{A}_k\mathbf{P}(x)\mathbf{N}^k\mathcal{Y} = \mathcal{G}. \quad (2.3)$$

We integrate both sides of equation (2.3) on the interval $[a, b]$, then we have the following relations

$$\int_a^b \mathcal{G} dx = (b-a)\mathcal{G}$$

and,

$$\begin{aligned} \int_a^b \mathbf{P}(x) dx &= \left[\int_a^b p_{0,n}(x) \quad \int_a^b p_{1,n}(x) \quad \dots \quad \int_a^b p_{n,n}(x) \right] \\ &= \frac{b-a}{n+1} \left[1 \quad 1 \quad \dots \quad 1 \right] \end{aligned}$$

from Theorem 1.1.

Substituting these relations into equation (2.3), the equation becomes

$$\left(\frac{b-a}{n+1} \right) \sum_{k=0}^m \mathcal{A}_k \left[1 \quad 1 \quad \dots \quad 1 \right] \mathbf{N}^k \mathcal{Y} = (b-a)\mathcal{G}.$$

Denoting the matrix $\mathcal{I} = \frac{1}{n+1} \left[1 \quad 1 \quad 1 \quad \dots \quad 1 \right]$, then the desired matrix relation is obtained as

$$\sum_{k=0}^m \mathcal{A}_k \mathcal{I} \mathbf{N}^k \mathcal{Y} = \mathcal{G}. \quad (2.4)$$

Moreover, equation (2.4) and mixed conditions (1.2) can be written simply as follows:

$$\begin{aligned}\mathcal{W}\mathcal{Y} &= \mathcal{G} \text{ or } [\mathcal{W};\mathcal{G}], \\ \mathbf{M}_i\mathbf{Y} &= \beta_i \text{ or } [\mathbf{M}_i;\beta_i]; \quad i = 0, \dots, m-1\end{aligned}\quad (2.5)$$

so that $\mathcal{W} = \sum_{k=0}^m \mathcal{A}_k \mathcal{I} \mathbf{N}^k$ and $\mathbf{M}_i = \sum_{k=0}^{m-1} \sum_{j=0}^l \alpha_{ij}^k \mathbf{P}(c_j) \mathbf{N}^k$. Here matrix equation (2.5) corresponds to a linear algebraic system. Since \mathcal{W} is a square matrix, this linear algebraic system can be solved easily. So, the values of unknown matrix \mathcal{Y} are approximately found at the points $a + \frac{(b-a)i}{n}$. \square

3. Numerical Results

Linear differential equations with both constant and variable coefficients given respectively in the initial, boundary and mixed conditions are considered in numerical examples. The numerical results obtained by the Bernstein operator method are calculated by using adding and deleting techniques mentioned in [19] on the program MATLAB 7.1. In the tables, computations of the maximum errors and a comparison of the approximate solutions and exact solutions for the proposed method are given. So then, the using formulas can be defined as follows:

$$|e_n(x)| = |y(x) - y_n(x)|, \quad E_{\max} = \max_{x \in [a,b]} |e_n(x)|$$

so that y is exact solution and y_n is the Bernstein operator approximate solution of the problem. Here attention should be paid that the numerical results are computed at the $x_i = a + \frac{(b-a)i}{n}$ points of function taken part in the Bernstein operator, instead of collocation points.

Example 3.1. Consider the following eighth order differential equation under the initial conditions [16]

$$\begin{aligned}y^{(8)} - y &= -8e^x; \quad 0 < x < 1 \\ y(0) &= 1, \quad y'(0) = 0, \quad y^{(2)}(0) = -2, \quad y^{(3)}(0) = -2, \quad y^{(4)}(0) = -3 \\ y^{(5)}(0) &= -4, \quad y^{(6)}(0) = -5, \quad y^{(7)}(0) = -6\end{aligned}$$

that has exact solution $y(x) = (1-x)e^x$.

Table 1. E_{\max} errors of Example 3.1.

n	Adding tech.	Deleting tech.			
	adding without deletion	deleting last rows	deleting first rows	deleting first and last rows	deleting middle rows
10	$8.7e-004$	$8.5e-007$	$5.5e-005$	$3.3e-006$	$6.0e-006$
15	$7.2e-007$	$7.2e-007$	$5.5e-007$	$7.2e-007$	$7.2e-007$
20	$5.4e-007$	$5.1e-007$	$2.3e-004$	$5.3e-007$	$9.2e-006$
25	$4.4e-007$	$2.6e-005$	$6.1e-001$	$4.1e-007$	$3.6e-004$
30	$4.5e-007$	$6.3e-004$	$4.9e-001$	$4.4e-007$	$3.2e-001$
35	$2.6e-007$	$2.3e-002$	$9.6e-001$	$1.0e-006$	$5.4e-001$

In Table 1, the maximum errors of the Bernstein operator method are given. These results are calculated on the points $x_i = \frac{i}{n}; n = 0, 1, \dots, n$. It shows that the values of the error shrinks slowly for increasing values n . Moreover, the numerical results calculated with deleting first and last rows are better than the others. For $n = 35$, the best value is obtained by considering the adding technique.

Example 3.2. Consider the following fourth order differential equation under the boundary conditions [12] that have an exact solution $y(x) = e^x$:

$$\begin{aligned}y^{(4)} - 3y &= -2e^x; \quad x \in [0, 1] \\ y(0) &= 1, \quad y(1) = e, \quad y'(0) = 1, \quad y'(1) = e.\end{aligned}$$

Table 2. A comparison of the y and y_n solutions for Example 3.2.

x	$y(x)$	$y_n(x)$			
		$n = 6$	$n = 8$	$n = 10$	$n = 12$
0.1	1.1051 709	1.1051 698	1.1051 575	1.1051 598	1.1051 617
0.2	1.2214 028	1.2213 906	1.2213 586	1.2213 665	1.2213 725
0.3	1.3498 588	1.3498 251	1.3497 802	1.3497 945	1.3498 052
0.4	1.4918 247	1.4917 660	1.4917 191	1.4917 386	1.4917 530
0.5	1.6487 213	1.6486 438	1.6486 042	1.6486 262	1.6486 420
0.6	1.8221 188	1.8220 368	1.8220 094	1.8220 301	1.8220 449
0.7	2.0137 527	2.0136 831	2.0136 683	2.0136 844	2.0136 958
0.8	2.2255 409	2.2254 973	2.2254 919	2.2255 013	2.2255 079
0.9	2.4596 031	2.4595 885	2.4595 877	2.4595 907	2.4595 928

In Table 2, the Bernstein approximate solution and exact solution of the problem are compared for the different points and values n . The numerical results are calculated by considering the adding technique. The table shows that convergence of the Bernstein operator method are better for $x = 0.1, 0.2$ and $n = 6$. For increasing x and n values, numerical results get worse.

Example 3.3. Let the following boundary value problem [32] be considered:

$$y^{(6)} + y = 6(2x \cos(x) + 5 \sin(x)); \quad x \in [-1, 1]$$

$$y(-1) = y(1) = 0, \quad y'(-1) = y'(1) = 2 \sin(1), \quad y''(-1) = -y''(1) = -4 \cos(1) - 2 \sin(1)$$

An exact solution of the problem is $y(x) = (x^2 - 1) \sin(x)$.

Table 3. E_{\max} errors of Example 3.3.

n	Adding tech.	Deleting tech.	
	adding without deletion	deleting first and last rows	deleting middle rows
9	$2.5e - 002$	$7.4e - 004$	$1.2e - 004$
15	$6.8e - 005$	$6.8e - 005$	$6.8e - 005$
20	$5.2e - 005$	$5.2e - 005$	$5.2e - 005$
25	$4.2e - 005$	$4.2e - 005$	$4.2e - 005$
32	$3.3e - 005$	$4.3e - 005$	$3.3e - 005$
40	$2.7e - 005$	$2.4e - 002$	$2.7e - 005$
50	$2.1e - 005$	$4.2e - 001$	$2.1e - 005$
64	$1.7e - 005$	$3.5e - 001$	$1.7e - 005$
128	$8.5e - 006$	$3.7e - 001$	$8.6e - 006$

The maximum errors of the proposed method are given on the points $x_i = -1 + \frac{2i}{n}$; $n = 0, 1, \dots, n$ in Table 3. The table shows that the numerical results obtained by deleting technique better than the adding technique for $n = 9$. Then, it is seen that the numerical results obtained by both adding and deleting techniques are same for $n = 15, 20, 25$. The best numerical result is obtained by adding technique and for $n = 128$.

Example 3.4. Consider the linear differential equation under the mixed conditions [15]:

$$\begin{aligned} y'' + \log \sqrt[8]{(x+3)} y &= 3x \cos(x^2 - 2) \\ y(-1) - 3y'(-1) &= 0, \quad 2y(1) - 4y'(1) = 0. \end{aligned}$$

Table 4. Comparison of approximate solutions for Example 3.4.

n	Bernstein Operator Method		Nyström Method		Bernstein Collocation Method	
	$x = -0.4$	$x = 0.7$	$x = -0.4$	$x = 0.7$	$x = -0.4$	$x = 0.7$
8	0.343 335	0.252 831	0.222 949	0.121 655	0.220 505	0.120 426
16	0.260 116	0.208 530	0.222 945	0.121 655	0.222 949	0.121 655

The approximate solutions obtained by the Bernstein operator method and the other methods are given for different two points in Table 4. The numerical results of the proposed method are also calculated with the adding technique. By the table, the results of the Bernstein operator method have been compared with the results of the Nyström method [15] and the Bernstein collocation method [2]. However exact solution of the problem is not known, the table shows that the numerical results are decreased for increasing n values.

4. Conclusions

In this paper, a new numerical method is generated by considering the generalized Bernstein polynomials as a sequence of the linear positive operators for finding solutions of the linear differential equations under the mixed conditions on the $C[a, b]$. The mentioned method has a lot of remarkable advantages. One of these, this method doesn't need to use the collocation points for approximating to unknown function of the differential equations. Because, the approximate solutions of the problems are found on the only points defined the generalized Bernstein polynomials by means of new method. This main point supports the Theorem 1.1 [2]. By considering this perspective, to approximate all the functions fallen in the differential equations via the Bernstein polynomials is created appearance of the numerical method worked on. Alternatively, if the Theorem 2.1 is considered, operational matrix form of the Bernstein polynomials is enough to constitute this method. So, second of the advantages is obtained the theory of the proposed method simply. Third of these, this method is applied to calculate the numerical errors in the setting of the programme easily. The adding and deleting techniques used for computations of the numerical results are also compared in the Tables [1, 3]. Looking to these tables, we can say that adding technique is more better than the deleting techniques for calculating the numerical results. In this reason, to use the adding technique is an effective way for computing numerical results. Besides, the numerical results of the method are not better more and more n values from the tables [1-3]. In this case, to calculate the numerical results for smaller n values is sufficient. In line with this developments, many numerical methods can be improved by considering the linear positive operators like Stancu-Chlodowsky operator [9] generalized Bernstein-Chlodowsky operator defined on the interval $[0, \infty)$ instead of the Bernstein operator. New methods derivated with this way can be transferred for approximate solutions of many types of differential, integral, integro-differential, fractional equations and other recallable equations in the future works.

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