International Journal of
Applied Mathematics,
Electronics and Computers
ISSN: 2147-8228 www.atscience.org/IJAMEC
Advanced Technology and Science
Original Research Paper

B-closed Spaces and Fuzzy b-closed Spaces

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Accepted 30th December 2016

Abstract: The purpose of this paper is to establish and project the theorems which exhibit the characterization of b-closed spaces and obtain some of interesting properties of b-closed spaces. Moreover, fuzzy b-closed spaces are introduced, and some characterization of their properties are obtained.

Keywords: *Topological Spaces; b-Closed Spaces; Fuzzy Spaces; Fuzzy b-Closed Spaces.*

1. Introduction

In [4], the authors introduced the notion of b-closed spaces and investigated its fundamental properties. The concept of b-open sets in fuzzy settings was introduced by Benchalli and Karnel [1]. In this paper, we investigate a class of sets called b- closed sets. We study some of its basic properties. Afterward, we introduce the concept of fuzzy b-closed spaces.

In particular, the notion of generalized b-closed spaces and its various characterizations are given (see Section 2). In Section 3, we study various forms of fuzzy b-closed spaces.

Now, we recall the following definitions which are useful in the sequel.

Proposition 1.1. A subset A of a space X is b-open if and only if $A = B \bigcup C$, where B is semi-open and C is preopen.

Proposition 1.2.

(i) Let A and B be subsets of a space X such that $A \subset B$. If $A \in bo(X)$, then $A \in bo(B)$.

(ii) If $A \in bo(B)$, $B \in \alpha o(X)$, then $A \in bo(X)$.

Proposition 1.3. A space X is extremally disconnected if and only if every b-open subset of X is preopen.

Proposition 1.4. A space X is strongly irresolvable if and only if every b-open subset of X is semi-open.

Proposition 1.5. For a space X, the following are equivalent:

(i) X is locally indiscrete,

(ii) Every b-open subset of X is preclosed.

2. b-closed Spaces

Definition 2.1. A space X is called b-closed if any b-open cover of X has a finite subfamily, the union of the preclosures of whose members covers X.

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Remark 2.2. Since $so(X) \bigcup po(X) \subset bo(X)$, and since $pclA = \overline{A}$ whenever A is semi-open, it is clear that every b-closed space is both S-closed and p-closed. However, the author asks about the existence of a space that is both S-closed and p-closed but not b-closed.

The following two propositions follows from Propositions 1.3 and 1.4 and from the fact that $pclA = \overline{A}$ whenever A is semi-open.

Proposition 2.3. For an extremally disconnected space X, the following are equivalent:

(i) X is b-closed.

(ii) X is p-closed.

Proposition 2.4. For a strongly irresolvable space X, the following are equivalent:

(i) X is b-closed.

(ii) X is S-closed.

The following result is an immediate consequence of Proposition 1.1 and from the fact that $so(X) \bigcup po(X) \subset bo(X)$.

Proposition 2.5. A space X is b-closed if and only if any cover of X whose members are semi-open or preopen has a finite subfamily, the union of the preclosures of whose members covers X.

Lemma 2.6. A subset A of a space X is b-open if and only if there exists a preopen subset U of X such that $U \subset A \subset pclU$.

Theorem 2.7. For a space X, the following are equivalent:

(i) X is b-closed.

(ii) Any regular p-open cover of X has a finite subfamily, the union of the preclosures of whose members covers X.

(iii) Any pre-regular p-closed cover of X has a finite subcover.

Proof. (i) to (ii): Follows since every regular p-open set is b-open.

(ii) to (iii): Follows since every pre-regular p-closed set is regular p-open and preclosed.

(iii) to (i): Let $u = \{U_\alpha : \alpha \in \Lambda\}$ be a b-open cover of X. Then by Lemma 2.6, for each $\alpha \in \Lambda$, there exists a preopen subset V_α of X such that $V_\alpha \subset U_\alpha \subset pclV_\alpha$. Now $v = \{pclV_\alpha : \alpha \in \Lambda\}$ is pre-regular p-closed cover of X and thus by (ii), there exists $\alpha_1, \alpha_2, ..., \alpha_n \in \Lambda$ such that $X = \bigcup_{i=1}^{n} pclV_{\alpha_i} = \bigcup_{i=1}^{n} pclU_{\alpha_i}$. Hence, X is b-closed.

The following result follows from the the definition of a b-closed space and from Propositions 2.5 and Theorem 2.7, the straightforward proof is omitted.

Proposition 2.8. For a space X, the following are equivalent:

(i) X is B-closed.

(ii) For any family $u = \{U_\alpha : \alpha \in \Lambda\}$ of b-closed subsets of X such that $\bigcap = \{U_\alpha : \alpha \in \Lambda\} = \varnothing$, there exists a finite subset Λ_0 of Λ such that $\bigcap = \{\text{pint}U_\alpha : \alpha \in \Lambda_0\} = \varnothing$.

(iii) For any family $u = \{U_\alpha : \alpha \in \Lambda\}$ each of whose members is semi-closed or preclosed in X such that $\bigcap = \{U_\alpha : \alpha \in \Lambda\} = \varnothing$, there exists a finite subset Λ_0 of Λ such that $\bigcap = \{\text{pint}U_\alpha : \alpha \in \Lambda_0\} = \varnothing$.

(iv) For any family $u = \{U_\alpha : \alpha \in \Lambda\}$ of regular p-closed subsets of X such that $\bigcap = \{U_\alpha : \alpha \in \Lambda\} = \varnothing$, there exists a finite subset Λ_0 of Λ such that $\bigcap = \{\text{pint}U_\alpha : \alpha \in \Lambda_0\} = \varnothing$.

(v) For any family $u = \{U_\alpha : \alpha \in \Lambda\}$ of pre-regular p-open subsets of X such that $\bigcap = \{U_\alpha : \alpha \in \Lambda\} = \varnothing$, there exists a finite subset Λ_0 of Λ such that $\bigcap = \{U_\alpha : \alpha \in \Lambda_0\} = \varnothing$.

Definition 2.9. Let A be a subset of a space X. A point $x \in X$ is said to be a b-pre-θ-accumulation point of A if $pcl(U) \bigcap A \neq \varnothing$ for every b-open subset U of X that contains x. The set of all b-θ-accumulation points of A is called the b-pre-θ-closure of A and is denoted by b-$pcl_\theta(A)$. A is said to be b-pre-θ-closed if b-$pcl_\theta(A) = A$. The complement of a b–pre-θ-closed set is called b-pre-θ-open.

It is clear that A is b-pre-θ-open if and only if for each $x \in A$, there exists a b-open set U such that $x \in U \subset pclU \subset A$, thus, every b-pre-$\theta$-open set is b-open.

Definition 2.10.

(i) A space X is called b-regular if for each b-open subset U of X and for each $x \in U$ there exists a b-open subset V of X and a b-closed subset F of X such that $x \in V \subset F \subset U$.

(ii) A space X is called strongly b-regular if for each b-open subset U of X and for each $x \in U$ there exists a b-open subset V of X and a preclosed subset F of X such that $x \in V \subset F \subset U$.

The following lemma can be easily established.

Lemma 2.11.

(i) A space X is strongly b-regular if and only if every b-open subset of X is b-pre-θ-open.

(ii) If A is pre-regular p-open, then A is b-pre-θ-closed.

(iii) $bclA \subset bcl_\theta A$.

(iv) If A is preopen, then $bcl_\theta A = bclA$.

Remark 2.12.

(i) The converse of Lemma 2.11 (ii) is not true, e.g. if X is an infinite set and τ_{cof} is the cofinite topology on X, then in (X, τ_{cof}), every cofinite subset of X is b-pre-θ-open but not pre-regular p-closed as it is not preclosed (observe that the nonempty b-open (preopen) subsets of (X, τ_{cof}) are the infinite subsets of X).

It follows also from Proposition 1.5 that every locally indiscrete space is strongly b-regular. The converse is, howere, not true, e.g. if X is an infinite set and τ_{cof} is the cofinite topology on X, then in (X, τ_{cof}), every b-open subset of X is b-pre-θ-open. Thus by Proposition 2.11 (i), X is strongly b-regular. Howere, (X, τ_{cof}) is not locally indiscrete.

Theorem 2.13. A space X is b-closed if and only if every b-pre-θ-open cover of X has a finite subcover.

Proof. Suppose that X is b-closed and let $u = \{U_\alpha : \alpha \in \Lambda\}$ be a b-pre-θ-open cover of X. Then for each $x \in X$, there exists $\alpha_x \in \Lambda$ such that $x \in U_{\alpha x}$. Since $U_{\alpha x}$ is b-pre-θ-open, there exists a b-open set V_x such that $x \in V_x \subset pclV_x \subset U_{\alpha x}$, but X is b-closed, so there exists $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup_{i=1}^{n} U_{\alpha x_i}$.

Sufficiency. Follows from Theorem 2.7 and Lemma 2.11 (ii).

Proposition 2.14. let X be a b-closed, strongly b-regular space. Then X is finite.

Proof. It follows from Lemma 2.11 (i) and Theorem 2.13, that if X is a B-closed, strongly b-regular space, then every b-open cover of X has a finite subcover. Since

$so(X) \cup po(X) \subset bo(X)$, X is both semi-compact and strongly compact. Hence, X is finite.

Definition 2.15. A filter base Γ on a space X is said b-pre-θ-converge to a point $x \in X$ if for each b-open subset U of X such that $x \in U$, there exists $F \in \Gamma$ such that $F \subset pclU$. Γ is said to b-pre-θ-accumulate at $x \in U$ if $(pclU) \cap F \neq \varnothing$ for every $F \in \Gamma$ and for every b-open subset U of X such that $x \in U$.

Observe that if a filter base Γ b-pre-θ-converges to a point $x \in U$, then Γ b-pre-θ-accumulate at x. On the other hand, it is easy to see that a maximal filter base Γ b-pre-θ-converges to a point $x \in X$ if and only if Γ b-pre-θ-accumulate at x.

Theorem 2.16. For a space X, the following are equivalent:

(i) X is b-closed.

(ii) Every maximal filter base on X b-pre-θ-converges to some point of X.

(iii) Every filter base on X b-pre-θ-accumulate at some point of X.

Proof. (i) to (ii): Let Γ be a maximal filter base on X such that Γ does not b-pre-θ-converge to any point of X. Since Γ is maximal, Γ does not b-pre-θ-accumulate at any point of X. Thus, for each $x \in X$ exists $F_x \in \Gamma$ and a b-open subset U_x of X such that $x \in U_x$ and $(pclU_x) \cap F_x = \varnothing$, but X is B-closed, so there exists $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup_{i=1}^{n} pclU_{x_i}$. Since Γ is a filter base on X, there exists $F \in \Gamma$ such that $F \subset \bigcap_{i=1}^{n} F_{x_i}$, but $(pclU_{x_i}) \cap F_{x_i} = \varnothing$ for eaech $i \in \{1, 2, ..., n\}$, so $(pclU_{x_i}) \cap F = \varnothing$ for eaech $i \in \{1, 2, ..., n\}$, i.e. $\left(\bigcup_{i=1}^{n} pclU_{x_i}\right) \cap F = X \cap F = F = \varnothing$, a contradiction.

(ii) to (iii): Let Γ be a filter base on X. Then Γ is contained in a maximal filter base Υ on X.

By (ii), Υ b-pre-θ-converges to some point x of X, thus Υ b-pre-θ-accumulates at x, but $\Gamma \subset \Upsilon$, so Γ b-pre-θ-accumulate at x.

(iii) to (ii): Suppose that X is not B-closed. Then by Proposition 2.8, there exists a b-open cover $u = \{U_\alpha : \alpha \in \Lambda\}$ of X such that for any finite subset Λ_0 of Λ, $\bigcap\{p\operatorname{int}(X \setminus U_\alpha) : \alpha \in \Lambda_0\} \neq \varnothing$. For each finite subset Λ_0 of Λ, let $F_{\Lambda_0} = \bigcap\{p\operatorname{int}(X \setminus U_\alpha) : \alpha \in \Lambda_0\}$. Then $\Gamma = \{F_{\Lambda_0} : \Lambda_0 \text{ is a finite subset of } \Lambda\}$ is a filter base on X. Since u is a b-open cover of X, there exists $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$, but Γ b-pre-θ-accumulates at x, so $(pclU_{x_i}) \cap F \neq \varnothing$ for every $F \in \Gamma$. Let $F = p\operatorname{int}(X \setminus U_{\alpha_0})$. Then $F \in \Gamma$ and thus $(pclU_{x_i}) \cap (p\operatorname{int}(X \setminus U_{\alpha_0})) \neq \varnothing$ a contradiction.

3. Fuzzy b-close Spases

Definition 3.1. [7] For two fuzzy subsets μ_1 and μ_2 of X, the fuzzy subset $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \vee\{\mu_1(x_1) \wedge \mu_2(x_2) \mid x = x_1 + x_2\}.$$

And for a scalar t of K and a fuzzy subset μ of X, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu\left(x/t\right) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \text{ and } x \neq 0 \\ \vee\{\mu(y) \mid y \in X\} & \text{if } t = 0 \text{ and } x = 0 \end{cases}.$$

Definition 3.2. [5] $\mu \in I^X$ is said to be,

1. convex if $t\mu + (1-t)\mu \subseteq \mu$ for each $t \in [0,1]$
2. balanced if $t\mu \subseteq \mu$ for each $t \in K$ with $|t| \leq 1$
3. absorbing if $\vee\{t\mu(x) \mid t > 0\} = 1$ for all $x \in X$.

Definition 3.3. [5] Let (X, τ) be a topological space and $\omega(\tau) = \{f : (X, \tau) \to [0,1] \mid \text{ f is lower semicontinuous}\}$, then $\omega(\tau)$ is a fuzzy topology on X. This topology is called the fuzzy topology generated by τ on X. The fuzzy usual topology on K means the fuzzy topology generated by the usual topology of K.

$$n \geq M \text{ implies } \frac{t}{2}\rho(x_n - x) > 1 - \varepsilon$$

therefore

$$n \geq M \text{ implies } P_{1-\varepsilon}(x_n - x) \leq \frac{t}{2} < t.$$

Definition 3.4. [5] A fuzzy linear topology on a vector space X over K is a fuzzy topology on X such that the two mappings

$$+ : \mathrm{X} \times \mathrm{X} \to \mathrm{X}, \quad (x, y) \to x + y$$
$$. : K \times X \to X, \quad (t, x) \to tx$$

Are continuous when K has the fuzzy usual topology and $K \times X$ and $X \times X$ have the corresponding product fuzzy topologies. A linear space with a fuzzy linear topology is called a fuzzy topological linear space or a fuzzy topological vector space.

Definition 3.5. [5] Let x be a point in a fuzzy topological space X. A family F of neighborhood of x is called a base for the system of all neighborhoods of x if for each neighborhood μ of x and each $0 < \theta < \mu(x)$, there exists $\mu_1 \in F$ with $\mu_1 \leq \mu$ and $\mu_1(x) > \theta$.

Definition 3.6. [6] A fuzzy semi norm on X is a fuzzy set ρ in X which is convex, balanced and absorbing. If in addition $\wedge\{(t_\rho)(x) \mid t > 0\}$ for $x \neq 0$, then ρ is called a fuzzy norm.

Definition 3.7. [6] If ρ is a fuzzy semi norm on X, then the family $B_\rho = \{\theta \wedge (t_\rho) \mid 0 < \theta \le 1, t > 0\}$ is a base at zero for a fuzzy linear topology τ_ρ. The fuzzy topology τ_ρ is called the fuzzy topology induced by the fuzzy semi norm ρ. And a linear space equipped with a fuzzy semi norm is called a fuzzy semi normed linear space.

Definition 3.8. [8] Let ρ be a fuzzy semi norm on X. $P_\varepsilon : X \to R_+$ Is defined by

$$P_\varepsilon(x) = \wedge\{t > 0 \mid t\,\rho(x) > \varepsilon\}$$

For each $\varepsilon \in (0,1)$.

Theorem 3.9. [8] The P_ε is a semi norm on X for each $\varepsilon \in (0,1)$. Further P_ε is norm on X for each $\varepsilon \in (0,1)$ if and only if ρ is a fuzzy norm on X.

Definition 3.10. A fts X is said to be fuzzy b-closed iff for every family λ of fuzzy b-open set such that $\underset{A\in\lambda}{\vee} A = 1_X$ there is a finite subfamily $\delta \subseteq \lambda$ such that that $\left(\underset{A\in\delta}{\vee} bCl(A)\right)(x) = 1_X$, for every $x \in X$.

Definition 3.11. A fuzzy set U in a fts X is said to be fuzzy b-closed relative to X iff for every family λ of fuzzy b-open set such that $\underset{A\in\lambda}{\vee} A = 1_X$ there is a finite subfamily $\delta \subseteq \lambda$ such that that $\underset{A\in\delta}{\vee} bCl(A)(x) = U(x)$, for every $x \in S(U)$.

Remark 3.12. Every fuzzy b-compact space is fuzzy b-closed, but the converse is not true.

Theorem 3.13. A fts X is fuzzy b-closed iff for every fuzzy filterbases Γ in X, $\left(\underset{G\in\Gamma}{\wedge} bCl(G)\right) \ne 0_X$.

Proof. Let μ be a fuzzy b-open set cover of X and let for every finite family of μ, $\underset{A\in\partial}{\vee} bCl(A)(x) < 1_X$ for some $x \in X$. Then $\left(\underset{A\in\partial}{\wedge} \overline{bCl(G)}\right)(x) > 0_X$ for some. $x \in X$

Thus $\left\{\left(\overline{bCl(A)} : A \in \mu\right)\right\} = \Gamma$ forms a fuzzy b-open filterbases in X. Since μ is a fuzzy b-open set cover of X, then $\left(\underset{A\in\mu}{\wedge} A\right) = 0_X$, which implies $\left(\underset{A\in\mu}{\wedge} bCl\left(\overline{bCl(G)}\right)\right)(x) = 0_X$, which is a contradiction. Then every fuzzy b-open μ of X has a finite subfamily ∂ such that $\left(\underset{A\in\partial}{\vee} bCl(A)(x)\right) = 1_X$ for every $x \in X$.

Hence X is a fuzzy b-closed.

Conversely, suppose there exists a fuzzy b-open filterbases Γ in X such that $\left(\underset{G\in\Gamma}{\wedge} bCl(G)\right) = 0_X$. That implies $\left(\underset{G\in\Gamma}{\vee}\left(\overline{bCl(G)}\right)\right)(x) = 1_X$ for $x \in X$ and hence $\mu = \left\{\left(\overline{bCl(G)}\right) : G \in \Gamma\right\}$ is a fuzzy b-open set cover of X. Since X is fuzzy b-closed, by definition μ has a finite subfamily ∂ such that $\left(\underset{G\in\partial}{\vee} bCl\left(\overline{bCl(G)}\right)\right)(x) = 1_X$ for every $x \in X$, and hence $\underset{\lambda\in\partial}{\wedge}\left(\overline{bCl\left(\overline{bCl(G)}\right)}\right) = 0_X$. Thus $\underset{G\in\partial}{\wedge} G = 0_X$ is a contradiction. Hence $\underset{G\in\Gamma}{\wedge} bCl(G) \ne 0_X$.

Theorem 3.14. Let $f : (X,\tau) \to (Y,\sigma)$ be a fuzzy b*–continuous surjection. If X is fuzzy b-closed space, then Y is fuzzy b-closed space.

Proof. Let $\{A_\lambda : \lambda \in \Lambda\}$ be a fuzzy b-open cover of Y. Since f is fuzzy b*–continuous, $\{f^{-1}(A_\lambda) : \lambda \in \Lambda\}$ is fuzzy b-open cover of X. By hypothesis, there exists a finite subset Δ of Γ such that $\underset{\lambda\in\Delta}{\vee} bCl\left(f^{-1}(A_\lambda)\right) = 1_X$. Since f is surjection and by theorem

$$1_Y = f(1_X) = f\left(\underset{\lambda\in\Delta}{\vee} bCl\left(f^{-1}(A_\lambda)\right)\right)$$
$$\le \underset{\lambda\in\Delta}{\vee} bCl\left(f\left(f^{-1}(A_\lambda) = \underset{\lambda\in\Delta}{\vee} bCl(A_\lambda)\right)\right).$$

Hence Y is fuzzy b-closed space.

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