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Integral transforms for the new generalized Beta function

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Abstract — In this paper, some representation formulas for the generalized Gamma and Beta functions are obtained. Also, certain integral transforms for the generalized Beta function associated with the Wright hypergeometric function are derived.

***Keywords* —** *Gamma function; Beta function; integral transform*

1. Introduction

Motivated mainly by a variety of applications of Euler's Beta, hypergeometric, and confluent hypergeometric functions together with their extensions in a wide range of research fields such as engineering, chemical, and physical problems, very recently, Al-Gonah and Mohammed [1] introduced and studied a new form of the generalized Gamma and Beta functions denoted by $\Gamma_p^{(\alpha,\beta,\gamma)}(x)$ and $B_p^{(\alpha,\beta,\gamma)}(x,y)$ respectively by talking further advantage from the various existing forms of the Mittag-Leffler function. The generalized Gamma and Beta functions are defined by:

$$\Gamma_p^{(\alpha,\beta,\gamma)}(x) = \int_0^\infty t^{x-1} E_{\alpha,\beta}^{\gamma}\left(-t-\frac{p}{t}\right)\,dt, \tag{1}$$

$$(Re(p)\geq 0, Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, Re(x)>0),$$

and

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right)\,dt, \tag{2}$$

$$(Re(p)\geq 0, Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, Re(x)>0, Re(y)>0),$$

where $E_{\alpha,\beta}^{\gamma}(z)$ denotes the generalized Mittag-Leffler function defined by [2,p.7(1.3)]:

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{3}$$

$$(z,\alpha,\beta,\gamma\in\mathbb{C}; Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0),$$

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which is clearly that

$$\Gamma(\beta)E^{\gamma}_{1,\beta}(z) = {}_1F_1(\gamma;\beta;z), \tag{4}$$

$$E^{1}_{1,1}(z) = e^{z}. \tag{5}$$

Note that by using relations (4) and (5), we get the following special cases:

$$\Gamma_p^{(1,\beta,\gamma)}(x) = \frac{1}{\Gamma(\beta)}\Gamma_p^{(\gamma,\beta)}(x), \tag{6}$$

$$B_p^{(1,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\beta)}B_p^{(\gamma,\beta)}(x,y), \tag{7}$$

$$\Gamma_p^{(1,1,1)}(x) = \Gamma_p(x), \tag{8}$$

$$B_p^{(1,1,1)}(x,y) = B_p(x,y), \tag{9}$$

where $\Gamma_p^{(\gamma,\beta)}(x),\ B_p^{(\gamma,\beta)}(x,y)$ and $\Gamma_p(x),\ B_p(x,y)$ denoted the generalized Gamma and Beta functions given in [3] and [4,5].

Also, we not that

$$B_p^{(\alpha,1,1)}(x,y) = B^{p}_{\alpha}(x,y), \tag{10}$$

$$\Gamma_0^{(\alpha,1,1)}(x) = \Gamma^{\alpha}(x), \tag{11}$$

where $B^{p}_{\alpha}(x,y)$ and $\Gamma^{\alpha}(x)$ denoted the new extended Beta and Gamma functions given recently in [6] and [7] respectively.

In [8], Agrawal gave some interesting integral transforms for the generalized hypergeometric function. This paper is a further attempt in this direction for deriving some integral transforms and representation formulas for the generalized Beta function defined in [1]. For this aim, we recall that the Wright generalized hypergeometric function denoted by ${}_p\Psi_q$ is defined by [9]:

$${}_p\Psi_q\left[\begin{array}{cc}(\alpha_1,A_1),\ldots,(\alpha_p,A_p); & \\ & z\\ (\beta_1,B_1),\ldots,(\beta_q,B_q); & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{l=1}^{p}\Gamma(\alpha_l+A_l n)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j n)}\frac{z^n}{n!}, \tag{12}$$

where the parameters $\alpha_l,\beta_j\in\mathbb{C}$, and $A_l,B_j\in\mathbb{Z}$ $(l=1,2,\ldots,p;\ j=1,2,\ldots,q)$, such that $1+\sum_{j=1}^{q}B_j-\sum_{l=1}^{p}A_l>0$. Also, we note that

$$\begin{aligned}{}_p\Psi_q&\left[\begin{array}{cc}(\alpha_1,A_1),\ldots,(\alpha_p,A_p); & \\ & z\\ (\beta_1,B_1),\ldots,(\beta_q,B_q); & \end{array}\right]\\ &= H^{1,p}_{p,q+1}\left[-z\left|\begin{array}{c}(1-\alpha_1,A_1),\ldots,(1-\alpha_p,A_p)\\ \\ (0,1),(1-\beta_1,B_1),\ldots,(1-\beta_q,B_q)\end{array}\right.\right],\end{aligned} \tag{13}$$

where $H^{m,n}_{p,q}[.]$ denotes the H-function given in [9] (see also [10]).

2. Hypergeometric representations

Here, we establish some representation formulas for the generalized Gamma and Beta functions in form of the following theorems:

Theorem 2.1. For the new extended Beta function, we have the following hypergeometric representation:

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\gamma)}{}_3\Psi_2\left[\begin{array}{cc}(\gamma,1),(x,-1),(y,-1); & \\ & -p\\ (\beta,\alpha),(x+y,-2); & \end{array}\right], \tag{14}$$

$$(\alpha\in\mathbb{Z}^{+};\beta,\gamma,x,y\in\mathbb{C};Re(\beta),Re(\gamma),Re(x),Re(y)>0).$$

PROOF. Using definition (3) in relation (2), we get

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)\, n!} \frac{(-p)^n}{t^n(1-t)^n}\, dt. \tag{15}$$

Interchanging the order of integration and summation in the R.H.S. of equation (15), we get

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\,(-p)^n}{\Gamma(\beta+\alpha n)\, n!} \int_0^1 t^{x-n-1}(1-t)^{y-n-1}\, dt, \tag{16}$$

which on using the following relation [9]:

$$\begin{aligned} \int_0^1 t^{x-1}(1-t)^{y-1}\, dt &= \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \\ &= B(x,y), \end{aligned} \tag{17}$$

gives

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\,\Gamma(x-n)\,\Gamma(y-n)}{\Gamma(\beta+\alpha n)\,\Gamma(x+y-2n)} \frac{(-p)^n}{n!}. \tag{18}$$

Now, in view of definition (12), we get the desired result. □

Remark 2.2. Using relation (13) in assertion (14) of Theorem 2.1, we get the following relation:

Corollary 2.3. For the new extended Beta function, we have the following hypergeometric representation:

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\gamma)} H_{3,3}^{1,3}\left[p \left| \begin{matrix} (1-\gamma,1),(1-x,-1),(1-y,-1) \\ \\ (0,1),(1-\beta,\alpha),(1-x-y,-2) \end{matrix} \right. \right]. \tag{19}$$

.

Theorem 2.4. For the new extended Gamma function, we have the following representation:

$$\Gamma_p^{(\alpha,\beta,\gamma)}(x) = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{x-1}\, {}_1\Psi_1 \left[\begin{matrix} (\gamma,1); \\ \\ (\beta,\alpha); \end{matrix} \quad -t-\tfrac{p}{t} \right] dt, \tag{20}$$

$$\left(\alpha \in \mathbb{Z}^+; \beta,\gamma \in \mathbb{C}; Re(\beta) > 0, Re(\gamma) > 0\right).$$

PROOF. Using the following relation [11,p.810(6.3.2)]:

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} {}_1\Psi_1 \left[\begin{matrix} (\gamma,1); \\ \\ (\beta,\alpha); \end{matrix} \quad z \right], \tag{21}$$

in definition (1), we get the desired result. □

Remark 2.5. Using relation (13) in assertion (20) of Theorem 2.4, we get the following relation:

Corollary 2.6. For the new extended Gamma function, we have the following representation:

$$\Gamma_p^{(\alpha,\beta,\gamma)}(x) = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{x-1} H_{1,2}^{1,1}\left[t+\frac{p}{t} \left| \begin{matrix} (1-\gamma,1) \\ \\ (0,1),(1-\beta,\alpha) \end{matrix} \right. \right] dt. \tag{22}$$

3. Integral transforms

In this section, we derive some integral transforms for the generalized Beta function by applying certain integral transforms (like Beta transform, Laplace transform and Whittaker transform).

Theorem 3.1. The following Beta transform formula holds true:

$$B\left\{B_p^{(\alpha,\beta,\gamma)}(x,y):l,m\right\} = \frac{\Gamma(m)}{\Gamma(\gamma)}\,{}_4\Psi_3\left[\begin{array}{c}(\gamma,1),(l,1),(x,-1),(y,-1);\\ \\ (\beta,\alpha),(l+m,1),(x+y,-2);\end{array}\quad -1\right], \tag{23}$$

$$(\alpha\in\mathbb{Z}^+;\beta,\gamma,x,y,l,m\in\mathbb{C};Re(m),Re(\beta),Re(\gamma),Re(x),Re(y),Re(l)>0).$$

Proof. We know that the Beta transform is defined as (see [12]):

$$B\{f(p):a,b\} = \int_0^1 p^{a-1}(1-p)^{b-1}f(p)\,dp. \tag{24}$$

Using equation (24) and applying definition (2), we get

$$\begin{aligned}B&\left\{B_p^{(\alpha,\beta,\gamma)}(x,y):l,m\right\}\\ &= \int_0^1 p^{l-1}(1-p)^{m-1}\int_0^1 t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right)dt\,dp,\\ &= \int_0^1 p^{l-1}(1-p)^{m-1}\int_0^1 t^{x-1}(1-t)^{y-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,(-1)^n}{\Gamma(\alpha n+\beta)\,n!}\frac{p^n}{t^n(1-t)^n}\,dt\,dp.\end{aligned} \tag{25}$$

Interchanging the order of integration and summation and using relation (17) in the R.H.S. of equation (25), we obtain

$$B\left\{B_p^{(\alpha,\beta,\gamma)}(x,y):l,m\right\} = \frac{\Gamma(m)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)\,\Gamma(l+n)\,\Gamma(x-n)\,\Gamma(y-n)}{\Gamma(\beta+\alpha n)\,\Gamma(l+m+n)\,\Gamma(x+y-2n)}\frac{(-1)^n}{n!}, \tag{26}$$

which on using definition (12), yields the desired result. □

Remark 3.2. Using relation (13) in assertion (23) of Theorem 3.1, we get the following relation:

Corollary 3.3. The following Beta transform formula holds true:

$$\begin{aligned}B&\left\{B_p^{(\alpha,\beta,\gamma)}(x,y):l,m\right\}\\ &= \frac{\Gamma(m)}{\Gamma(\gamma)}H_{4,4}^{1,4}\left[1\left|\begin{array}{c}(1-\gamma,1),(1-l,1),(1-x,-1),(1-y,-1)\\ \\ (0,1),(1-\beta,\alpha),(1-l-m,1),(1-x-y,-2)\end{array}\right.\right].\end{aligned} \tag{27}$$

Theorem 3.4. The following Laplace transform formula holds true:

$$L\left\{p^{l-1}B_p^{(\alpha,\beta,\gamma)}(x,y);s\right\} = \frac{s^{-l}}{\Gamma(\gamma)}\,{}_4\Psi_2\left[\begin{array}{c}(\gamma,1),(l,1),(x,-1),(y,-1);\\ \\ (\beta,\alpha),(x+y,-2);\end{array}\quad -\tfrac{1}{s}\right], \tag{28}$$

$$(\alpha\in\mathbb{Z}^+;\beta,\gamma,x,y,l,s\in\mathbb{C};Re(s)>,Re(\beta),Re(\gamma),Re(x),Re(y),Re(l)>0;\left|\tfrac{-1}{s}\right|<1).$$

Proof. We know that the Laplace transform of $f(p)$ is defined as [12]:

$$L\{f(p);s\} = \int_0^{\infty} e^{-sp}f(p)\,dp, \qquad (Re(s)>0). \tag{29}$$

Using relation (29) and applying definition (2), we get

$$L\left\{p^{l-1}B_p^{(\alpha,\beta,\gamma)}(x,y);s\right\} = \int_0^{\infty} p^{l-1}e^{-sp}\int_0^1 t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right)dt\,dp,$$

$$= \int_0^\infty p^{l-1} e^{-sp} \int_0^1 t^{x-1}(1-t)^{y-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n \, (-1)^n}{\Gamma(\alpha n+\beta)\, n!} \frac{p^n}{t^n(1-t)^n} \, dt \, dp. \tag{30}$$

Interchanging the order of integration and summation and using the following integral formula [9,p.218(3)]:

$$\int_0^\infty t^{\lambda-1} e^{-st} \, dt = \frac{\Gamma(\lambda)}{s^\lambda}, \qquad (min\,\{Re(\lambda), Re(s)\} > 0), \tag{31}$$

and relation (17) in the R.H.S. of equation (30), we obtain

$$L\left\{p^{l-1} B_p^{(\alpha,\beta,\gamma)}(x,y); s\right\} = \frac{s^{-l}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\,\Gamma(l+n)\,\Gamma(x-n)\,\Gamma(y-n)}{\Gamma(\beta+\alpha n)\,\Gamma(x+y-2n)} \frac{\left(-\frac{1}{s}\right)^n}{n!}, \tag{32}$$

which on using definition (12), yields the desired result. □

Remark 3.5. Using relation (13) in assertion (28) of Theorem 3.4, we get the following relation:

Corollary 3.6. The following Laplace transform formula holds true:

$$L\left\{p^{l-1} B_p^{(\alpha,\beta,\gamma)}(x,y); s\right\}$$

$$= \frac{s^{-l}}{\Gamma(\gamma)} H_{4,3}^{1,4}\left[\frac{1}{s} \,\middle|\, \begin{matrix} (1-\gamma,1),(1-l,1),(1-x,-1),(1-y,-1) \\ (0,1),(1-\beta,\alpha),(1-x-y,-2) \end{matrix}\right]. \tag{33}$$

Theorem 3.7. The following Whittaker transform formula holds true:

$$\int_0^\infty p^{q-1} e^{\frac{-\delta p}{2}} W_{\lambda,\mu}(\delta p) B_p^{(\alpha,\beta,\gamma)}(x,y) \, dp$$

$$= \frac{\delta^{-q}}{\Gamma(\gamma)} {}_5\Psi_3 \left[\begin{matrix} (\gamma,1),(x,-1),(y,-1),\left(\frac{1}{2}+\mu+q,1\right),\left(\frac{1}{2}-\mu+q,1\right); \\ (\beta,\alpha),(x+y,-2),(1-\lambda+q,1); \end{matrix} \; -\frac{1}{\delta}\right], \tag{34}$$

$$(\alpha \in \mathbb{Z}^+; \beta,\gamma,x,y,q,\lambda,\mu,\delta \in \mathbb{C}; Re(\delta), Re(q), Re(\beta), Re(\gamma), Re(x), Re(y), Re\left(\tfrac{1}{2} \pm \mu + q\right), Re(1-\lambda+q) > 0).$$

Proof. Denoting the L.H.S. of equation (34) by Δ and setting $\delta p = v$, we get

$$\Delta = \delta^{-q} \int_0^\infty B_{\left(\frac{v}{\delta}\right)}^{(\alpha,\beta,\gamma)}(x,y)\, v^{q-1}\, e^{-\frac{v}{2}}\, W_{\lambda,\mu}(v)\, dv, \tag{35}$$

which on using definition (2), gives

$$\Delta = \delta^{-q} \int_0^\infty \int_0^1 t^{x-1}\,(1-t)^{y-1}\, E_{\alpha,\beta}^{\gamma}\left(\frac{-\frac{v}{\delta}}{t(1-t)}\right) v^{q-1}\, e^{-\frac{v}{2}}\, W_{\lambda,\mu}(v)\, dt\, dv,$$

$$= \delta^{-q} \int_0^\infty \int_0^1 t^{x-1}\,(1-t)^{y-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n\,(-1)^n\, v^n}{\Gamma(\beta+\alpha n)\, n!\, \delta^n\, t^n\,(1-t)^n}\, v^{q-1}\, e^{-\frac{v}{2}}\, W_{\lambda,\mu}(v)\, dt\, dv. \tag{36}$$

Interchanging the order of integration and summation in the R.H.S. of equation (36), we obtain

$$\Delta = \frac{\delta^{-q}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\left(-\frac{1}{\delta}\right)^n}{\Gamma(\beta+\alpha n)\, n!} \int_0^1 t^{x-n-1}\,(1-t)^{y-n-1}\, dt \int_0^\infty v^{q+n-1}\, e^{-\frac{v}{2}}\, W_{\lambda,\mu}(v)\, dv. \tag{37}$$

Now using relation (17) and the following integral formula involving the Whittaker function [13,p.9(2.24)]:

$$\int_0^\infty z^{v-1} e^{\frac{-z}{2}} W_{\lambda,\mu}(z)\, dz = \frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)}, \left(Re(v \pm \mu) > -\frac{1}{2}\right), \tag{38}$$

in the R.H.S. of equation (37) and after little simplification, we get

$$\Delta = \frac{\delta^{-q}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\,\Gamma(x-n)\,\Gamma(y-n)\,\Gamma\left(\frac{1}{2}+\mu+q+n\right)\Gamma\left(\frac{1}{2}-\mu+q+n\right)}{\Gamma(\beta+\alpha n)\,\Gamma(x+y-2n)\,\Gamma(1-\lambda+q+n)} \frac{\left(-\frac{1}{\delta}\right)^n}{n!}, \tag{39}$$

which on using definition (12), yields the R.H.S. of equation (34), then the proof of Theorem 3.7 is completed. □

Remark 3.8. Using relation (13) in assertion (34) of Theorem 3.7, we get the following relation:

Corollary 3.9. The following Whittaker transform formula holds true:

$$\int_0^\infty p^{q-1}\, e^{\frac{-\delta p}{2}}\, W_{\lambda,\mu}(\delta p) B_p^{(\alpha,\beta,\gamma)}(x,y)\, dp$$

$$= \frac{\delta^{-q}}{\Gamma(\gamma)} H^{1,5}_{5,4}\left[\frac{1}{\delta}\left|\begin{array}{r}(1-\gamma,1),(1-x,-1),(1-y,-1),\left(\frac{1}{2}-\mu-q,1\right),\left(\frac{1}{2}+\mu-q,1\right)\\ (0,1),(1-\beta,\alpha),(1-x-y,-2),(\lambda-q,1)\end{array}\right.\right]. \tag{40}$$

4. Special cases

In this section, we derive some results for various forms of the extended Gamma and Beta functions as special cases of the main results derived in the previous sections.

I. Putting $\alpha = 1$ in relations (14) and (19) and using relation (7), we get the following hypergeometric representations for the generalized Beta function $B_p^{(\gamma,\beta)}(x,y)$:

$$B_p^{(\gamma,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\gamma)}\, {}_3\Psi_2\left[\begin{array}{cc}(\gamma,1),(x,-1),(y,-1); & \\ & -p\\ (\beta,1),(x+y,-2); & \end{array}\right], \tag{41}$$

$$B_p^{(\gamma,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\gamma)} H^{1,3}_{3,3}\left[p\left|\begin{array}{r}(1-\gamma,1),(1-x,-1),(1-y,-1)\\ (0,1),(1-\beta,1),(1-x-y,-2)\end{array}\right.\right]. \tag{42}$$

Further, putting $\alpha = 1$ in relations (20) and (22) and using relation (6), we get the following representations for the generalized Gamma function $\Gamma_p^{(\gamma,\beta)}(x)$:

$$\Gamma_p^{(\gamma,\beta)}(x) = \frac{\Gamma(\beta)}{\Gamma(\gamma)} \int_0^\infty t^{x-1}\, {}_1\Psi_1\left[\begin{array}{cc}(\gamma,1); & \\ & -t-\frac{p}{t}\\ (\beta,1); & \end{array}\right] dt, \tag{43}$$

$$\Gamma_p^{(\gamma,\beta)}(x) = \frac{\Gamma(\beta)}{\Gamma(\gamma)} \int_0^\infty t^{x-1} H^{1,1}_{1,2}\left[t+\frac{p}{t}\left|\begin{array}{r}(1-\gamma,1)\\ (0,1),(1-\beta,1)\end{array}\right.\right] dt. \tag{44}$$

Again, putting $\beta = \gamma = 1$ in relations (14) and (19) and using relation (10), we get the following hypergeometric representations for $B_\alpha^p(x,y)$:

$$B_\alpha^p(x,y) = {}_3\Psi_2\left[\begin{array}{cc}(1,1),(x,-1),(y,-1); & \\ & -p\\ (1,\alpha),(x+y,-2); & \end{array}\right], \tag{45}$$

$$B_\alpha^p(x,y) = H^{1,3}_{3,3}\left[p\left|\begin{array}{r}(0,1),(1-x,-1),(1-y,-1)\\ (0,1),(0,\alpha),(1-x-y,-2)\end{array}\right.\right]. \tag{46}$$

Next, putting $\beta = \gamma = 1$ and $p = 0$ in relations (20) and (22) and using relation (11), we get the following representations for $\Gamma^\alpha(x)$:

$$\Gamma^\alpha(x) = \int_0^\infty t^{x-1}\, {}_1\Psi_1\left[\begin{array}{cc}(1,1); & \\ & -t\\ (1,\alpha); & \end{array}\right] dt, \tag{47}$$

$$\Gamma^\alpha(x) = \int_0^\infty t^{x-1} H^{1,1}_{1,2}\left[t\left|\begin{array}{r}(0,1)\\ (0,1),(0,\alpha)\end{array}\right.\right] dt. \tag{48}$$

II. Putting $\alpha = 1$ in relations (23), (28) and (34) and using relation (7), we get the following integral transforms for the generalized Beta function $B_p^{(\gamma,\beta)}(x,y)$:

$$B\left\{B_p^{(\gamma,\beta)}(x,y) : l, m\right\} = \frac{\Gamma(m)\Gamma(\beta)}{\Gamma(\gamma)} {}_4\Psi_3 \left[\begin{array}{cc} (\gamma,1),(l,1),(x,-1),(y,-1); & \\ & -1 \\ (\beta,1),(l+m,1),(x+y,-2); & \end{array}\right], \tag{49}$$

$$L\left\{p^{l-1} B_p^{(\gamma,\beta)}(x,y); s\right\} = \frac{s^{-l}\,\Gamma(\beta)}{\Gamma(\gamma)} {}_4\Psi_2 \left[\begin{array}{cc} (\gamma,1),(l,1),(x,-1),(y,-1); & \\ & -\frac{1}{s} \\ (\beta,1),(x+y,-2); & \end{array}\right], \tag{50}$$

$$\int_0^\infty p^{q-1}\, e^{\frac{-\delta p}{2}}\, W_{\lambda,\mu}(\delta p)\, B_p^{(\gamma,\beta)}(x,y)\, dp$$

$$= \frac{\delta^{-q}\,\Gamma(\beta)}{\Gamma(\gamma)} {}_5\Psi_3 \left[\begin{array}{cc} (\gamma,1),(x,-1),(y,-1),\left(\frac{1}{2}+\mu+q,1\right),\left(\frac{1}{2}-\mu+q,1\right); & \\ & -\frac{1}{\delta} \\ (\beta,1),(x+y,-2),(1-\lambda+q,1); & \end{array}\right]. \tag{51}$$

Further, putting $\alpha = 1$ in relations (27), (33) and (40) and using relation (7), we get the following second form of the integral transforms for the generalized Beta function $B_p^{(\gamma,\beta)}(x,y)$:

$$B\left\{B_p^{(\gamma,\beta)}(x,y) : l, m\right\}$$

$$= \frac{\Gamma(m)\Gamma(\beta)}{\Gamma(\gamma)} H^{1,4}_{4,4}\left[1 \left| \begin{array}{c} (1-\gamma,1),(1-l,1),(1-x,-1),(1-y,-1) \\ (0,1),(1-\beta,1),(1-l-m,1),(1-x-y,-2) \end{array}\right.\right], \tag{52}$$

$$L\left\{p^{l-1}\, B_p^{(\gamma,\beta)}(x,y); s\right\}$$

$$= \frac{s^{-l}\,\Gamma(\beta)}{\Gamma(\gamma)} H^{1,4}_{4,3}\left[\frac{1}{s} \left| \begin{array}{c} (1-\gamma,1),(1-l,1),(1-x,-1),(1-y,-1) \\ (0,1),(1-\beta,1),(1-x-y,-2) \end{array}\right.\right], \tag{53}$$

$$\int_0^\infty p^{q-1}\, e^{\frac{-\delta p}{2}}\, W_{\lambda,\mu}(\delta p)\, B_p^{(\gamma,\beta)}(x,y)\, dp$$

$$= \frac{\delta^{-q}\,\Gamma(\beta)}{\Gamma(\gamma)} H^{1,5}_{5,4}\left[\frac{1}{\delta} \left| \begin{array}{c} (1-\gamma,1),(1-x,-1),(1-y,-1),\left(\frac{1}{2}-\mu-q,1\right),\left(\frac{1}{2}+\mu-q,1\right) \\ (0,1),(1-\beta,1),(1-x-y,-2),(\lambda-q,1) \end{array}\right.\right]. \tag{54}$$

Again, putting $\beta = \gamma = 1$ in relations (23), (28) and (34) and using relation (10), we get the following integral transforms for the generalized Beta function $B_\alpha^p(x,y)$:

$$B\left\{B_\alpha^p(x,y) : l, m\right\} = \Gamma(m) {}_4\Psi_3 \left[\begin{array}{cc} (1,1),(l,1),(x,-1),(y,-1); & \\ & -1 \\ (1,\alpha),(l+m,1),(x+y,-2); & \end{array}\right], \tag{55}$$

$$L\left\{p^{l-1} B_\alpha^p(x,y); s\right\} = s^{-l} {}_4\Psi_2 \left[\begin{array}{cc} (1,1),(l,1),(x,-1),(y,-1); & \\ & -\frac{1}{s} \\ (1,\alpha),(x+y,-2); & \end{array}\right], \tag{56}$$

$$\int_0^\infty p^{q-1}\, e^{\frac{-\delta p}{2}}\, W_{\lambda,\mu}(\delta p)\, B_\alpha^p(x,y)\, dp$$

$$= \delta^{-q}\, {}_5\Psi_3 \left[\begin{array}{cc} (1,1),(x,-1),(y,-1),\left(\frac{1}{2}+\mu+q,1\right),\left(\frac{1}{2}-\mu+q,1\right); & \\ & \frac{1}{\delta} \\ (1,\alpha),(x+y,-2),(1-\lambda+q,1); & \end{array}\right]. \tag{57}$$

Next, putting $\beta = \gamma = 1$ in relations (27), (33) and (40) and using relation (10), we get the following second form of the integral transforms for the generalized Beta function $B_\alpha^p(x,y)$:

$$B\left\{B_\alpha^p(x,y) : l, m\right\}$$

$$= \Gamma(m) H_{4,4}^{1,4}\left[1 \left| \begin{array}{r} (0,1),(1-l,1),(1-x,-1),(1-y,-1) \\ \\ (0,1),(0,\alpha),(1-l-m,1),(1-x-y,-2) \end{array}\right.\right], \tag{58}$$

$$L\left\{p^{l-1} B_\alpha^p(x,y); s\right\}$$

$$= s^{-l} H_{4,3}^{1,4}\left[\frac{1}{s} \left| \begin{array}{r} (0,1),(1-l,1),(1-x,-1),(1-y,-2) \\ \\ (0,1),(0,\alpha),(1-x-y,-1) \end{array}\right.\right], \tag{59}$$

$$\int_0^\infty p^{q-1}\, e^{\frac{-\delta p}{2}}\, W_{\lambda,\mu}(\delta p)\, B_\alpha^p(x,y)\, dp$$

$$= \delta^{-q} H_{5,4}^{1,5}\left[\frac{1}{\delta} \left| \begin{array}{r} (0,1),(1-x,-1),(1-y,-1),\left(\frac{1}{2}-\mu-q,1\right),\left(\frac{1}{2}+\mu-q,1\right) \\ \\ (0,1),(0,\alpha),(1-x-y,-2),(\lambda-q,1) \end{array}\right.\right]. \tag{60}$$

In a forthcoming investigation, the new extension of Beta function given in equation (2) will be used to introduce other extensions of the extended Gauss hypergeometric and the confluent hypergeometric functions. For each of these new extensions we will obtain various properties.

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