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On The Curves $N - T^*N^*$ in $\mathbf{E}^3$

Şeyda Kılıçoğlu[1,*], Süleyman Şenyurt[2]

[1]*Faculty of Education, Department of Mathematics, Başkent University, Ankara.*
[2]*Faculty of Arts and Sciences, Department of Mathematics, Ordu Universty, Ordu.*

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Abstract. In this paper we have defined and examined the new kind curves, with the principal normal vector of the first curve and the vector lying on the osculator plane of the second curve are linearly dependent. As a result we have called these new curves as $N - T^*N^*$ curves. Also similiar to the other offset curves under the spesific condition, we give Frenet apparatus of the second curve based on the Frenet apparatus of the first curve.

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1. Introduction and Preliminaries

The curves which are represented in parametrized form are very important subject in differential geometry. Also to produce a ruled surface we have to know their special properties using frames in spaces. There are special curves. These curves are obtained by assuming a special property based on the original regular curve. Some of them are Bertrand curves, and Mannheim curves and associated curves, etc. Also the evolute and involute curves are the curves whose tangent lines intersect orthogonally, so if the principal normal vector of first curve and tangent vector of second curve are linearly dependent, then first curve is evolute, and the second curve is called involute curve. For more detail see in [2,6].Futher in [5] associated curves are examined by Bishop Frame. In [3], authors study about osculating curve and obtain some results.

Mannheim curve was firstly introduced by A. Mannheim in 1878; a curve is called a Mannheim curve if and only if $\kappa/\left(\kappa^2+\tau^2\right)$ is a nonzero constant, κ is the curvature and τ is the torsion. Also, a new definition of these associated curves was given in [7]; if the principal normal vector of the first curve and binormal vector of the second curve are linearly dependent, then the first curve is called Mannheim curve, and the second curve is called Mannheim partner curve. As a result they called these new curves as Mannheim curves. For more detail see in [7]. Bertrand pair curves are the curves with common principal normal lines.

Further a curve is Bertrand curve, if and only if there exist nonzero real numbers λ and β such that constant $\lambda\kappa+\beta\tau = 1$, κ is the curvature and τ is the torsion for any $s \in I$. For more detail see in [8]. Before in [4] we produce pair partner curve by using similiar way.

In this paper we use a similiar method to produce the new curves based on the other curves with common principal normal vector fields. Before we have examined $N - D^*$ curve with common principal normal vector of first curve

*Corresponding Author
Email addresses: seyda@baskent.edu.tr (Ş. Kılıçoğlu), senyurtsuleyman52@gmail.com (S. Şenyurt)

and Darboux vector D^* of the second curve. The Darboux vector field of any arclengthed curve α has symmetrical properties [1]: $D \times T = T'$; $D \times N = N'$; $D \times B = B'$.

Let $\{T, N, B, \kappa, \tau\}$ are collectively Frenet-Serret apparatus of a curve α, Frenet vector fields be T, N, B, of α and the first and second curvatures of the curve α be κ and τ, respectively.

2. $N - T^*N^*$ Curves in 3-Dimensional Euclidean Space

Let α and α^* be the curves with Frenet-Serret apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, where κ, $\kappa^*$and τ, τ^* are the curvature functions the first curve and the second curve, respectively.

Definition 2.1. Let α and α^* be the differentiable curves and the unit vector

$$O^* = \frac{aT^* + bN^*}{\sqrt{a^2 + b^2}}$$

be lies on the osculator plane of curve α^*. If N and O^* are linearly dependent, then the curve α^* is called $N - T^*N^*$ curve. This curve defined by

$$\alpha(s) = \alpha^*(s) + \lambda O^*(s)$$

or

$$\alpha^*(s) = \alpha(s) + \lambda N(s). \tag{2.1}$$

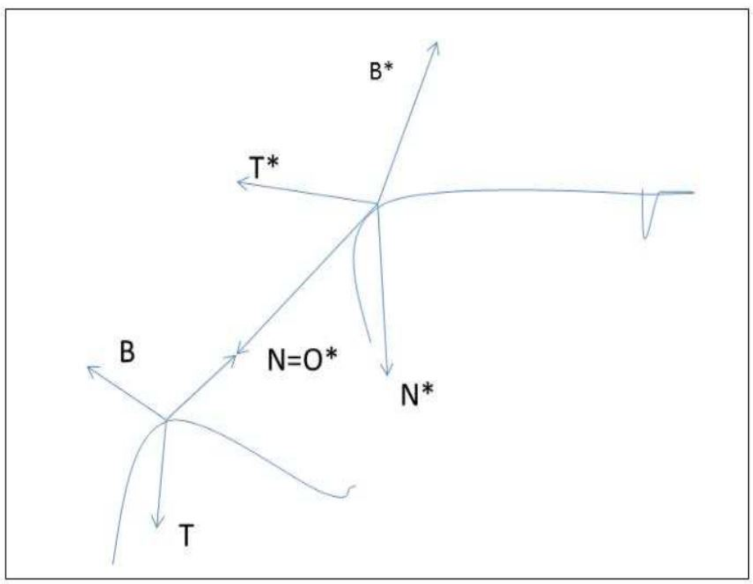

Figure 1. $N - T^*N^*$ curve

From the above figure, we can write

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}, \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}, \ (\measuredangle T^*, O^* = \theta).$$

Theorem 2.2. *Tangent vector of the curve α^* is*

$$T^* = \frac{a}{\sqrt{a^2 + b^2}}\left(\frac{(1 - \lambda\kappa)T + \lambda' N + \tau\lambda B}{\lambda'}\right).$$

Proof. Taking derivative from the (2.1), we have

$$T^*\frac{ds^*}{ds} = (1-\kappa\lambda)\,T + \lambda' N + \lambda\tau B.$$

If the norm is taken from this equation

$$T^* = \frac{(1-\kappa\lambda)\,T + \lambda' N + \tau\lambda B}{\sqrt{(1-\kappa\lambda)^2 + \lambda'^2 + \tau^2\lambda^2}}\,. \tag{2.2}$$

We have already know that $\langle T^*, N\rangle = \langle T^*, O^*\rangle$, so

$$\begin{aligned}\left\langle \frac{(1-\kappa\lambda)\,T + \lambda' N + \tau\lambda B}{\sqrt{(1-\kappa\lambda)^2 + \lambda'^2 + \tau^2\lambda^2}}, N\right\rangle &= \left\langle T^*, \frac{aT^* + bN^*}{\sqrt{a^2+b^2}}\right\rangle \\ \Rightarrow \frac{\lambda'}{\sqrt{(1-\kappa\lambda)^2 + \lambda'^2 + \tau^2\lambda^2}} &= \frac{a}{\sqrt{a^2+b^2}} = c.\end{aligned}$$

If this expression is written into the equation (2.2) and necessary calculations are made, the proof is completed. □

Theorem 2.3. *The first curvature κ^* of the curve α^* is*

$$\kappa^* = \frac{1}{c\sqrt{\delta^3}}\left(\left(-(2\lambda'\kappa + \lambda\kappa')\,\lambda' - (1+\kappa\lambda)\,\lambda''\right)^2 + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right)\right)\lambda'\right)^2 + \left((2\lambda'\tau + \lambda\tau')\,\lambda' - \tau\lambda\lambda''\right)^2\right)^{\frac{1}{2}},$$

where $\delta = (1-\lambda\kappa)^2 + \lambda'^2 + \lambda^2\tau^2$.

Proof. Since $\kappa^* N^* = \frac{dT^*}{ds}\frac{ds}{ds^*}$ and $\frac{ds}{ds^*} = \frac{1}{\sqrt{\delta}}$, it can be calculated as

$$\kappa^* N^* = \left(\frac{\sqrt{\delta}}{\delta}\left(((1-\kappa\lambda)' - \lambda'\kappa)\,T + \left((1-\kappa\lambda)\,\kappa + \lambda'' - \tau^2\lambda\right)N + (\lambda'\tau + (\tau\lambda)')\,B\right) - \frac{\sqrt{\delta}'}{\delta}\left((1-\kappa\lambda)\,T + \lambda' N + \tau\lambda B\right)\right)\frac{ds}{ds^*}\,.$$

Hence we have

$$\begin{aligned}\kappa^* N^* = \frac{1}{\sqrt{\delta^3}}&\left(\left(-(2\lambda'\kappa + \lambda\kappa')\sqrt{\delta} - (1-\kappa\lambda)\sqrt{\delta}'\right)T + \left((\kappa - \lambda(\kappa^2+\tau^2) + \lambda'')\sqrt{\delta} - \lambda'\sqrt{\delta}'\right)N\right.\\ &\left.+\left((2\lambda'\tau + \lambda\tau')\sqrt{\delta} - \tau\lambda\sqrt{\delta}'\right)B\right).\end{aligned}$$

Also since $\kappa^{*2} = \langle \kappa^* N^*, \kappa^* N^*\rangle$, we get

$$\kappa^{*2} = \frac{1}{\delta^3}\left(\left(-(2\lambda'\kappa + \lambda\kappa')\sqrt{\delta} - (1-\kappa\lambda)\sqrt{\delta}'\right)^2 + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right) + \lambda''\right)\sqrt{\delta} - \lambda'\sqrt{\delta}'\right)^2 + \left((2\lambda'\tau + \lambda\tau')\sqrt{\delta} - \tau\lambda\sqrt{\delta}'\right)^2\right).$$

Under the condition $\sqrt{\delta} = \frac{\lambda'}{c}$ and $\sqrt{\delta}' = \frac{\lambda''}{c}$ the proof is completed where $c = \frac{a}{\sqrt{a^2+b^2}}$. □

Theorem 2.4. *If the normal vector field of $N - T^*N^*$ partner curve is N^*, then it can be given based on the Frenet apparatus of the first curve as*

$$N^* = \frac{1}{\Delta}\left(\left(-(2\lambda'\kappa + \lambda\kappa')\,\lambda' - (1-\kappa\lambda)\,\lambda''\right)T + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right)\right)\lambda'\right)N + \left((2\lambda'\tau + \lambda\tau')\,\lambda' - \tau\lambda\lambda''\right)B\right),$$

where

$$\Delta = \frac{1}{c}\left(\left(-(2\lambda'\kappa + \lambda\kappa')\,\lambda' - (1-\kappa\lambda)\,\lambda''\right)^2 + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right)\right)\lambda'\right)^2 + \left((2\lambda'\tau + \lambda\tau')\,\lambda' - \tau\lambda\lambda''\right)^2\right).$$

Proof. Since $\kappa^* N^* = \frac{dT^*}{ds}\frac{ds}{ds^*}$ we have already find out the following result so

$$\begin{aligned}N^* &= \frac{\left(\left(-(2\lambda'\kappa + \lambda\kappa')\sqrt{\delta} - (1-\kappa\lambda)\sqrt{\delta}'\right)T + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right) + \lambda''\right)\sqrt{\delta} - \lambda'\sqrt{\delta}'\right)N + \left((2\lambda'\tau + \lambda\tau')\sqrt{\delta} - \tau\lambda\sqrt{\delta}'\right)B\right)}{\kappa^*\sqrt{\delta^3}}\\ &= \frac{\left(\left(-(2\lambda'\kappa + \lambda\kappa')\sqrt{\delta} - (1-\kappa\lambda)\sqrt{\delta}'\right)T + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right) + \lambda''\right)\sqrt{\delta} - \lambda'\sqrt{\delta}'\right)N + \left((2\lambda'\tau + \lambda\tau')\sqrt{\delta} - \tau\lambda\sqrt{\delta}'\right)B\right)}{\left(\left(((1-\kappa\lambda)' - \lambda'\kappa)\sqrt{\delta} - (1-\kappa\lambda)\sqrt{\delta}'\right)^2 + \left((\kappa - \lambda(\kappa^2+\tau^2) + \lambda'')\sqrt{\delta} - \lambda'\sqrt{\delta}'\right)^2 + \left((\lambda'\tau + (\tau\lambda)')\sqrt{\delta} - \tau\lambda\sqrt{\delta}'\right)^2\right)^{\frac{1}{2}}}\end{aligned}$$

for the simplicity lets take

$$\Delta = \left(\left(-\left(2\lambda'\kappa + \lambda\kappa'\right)\sqrt{\delta} - (1-\kappa\lambda)\sqrt{\delta}'\right)^2 + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right) + \lambda''\right)\sqrt{\delta} - \lambda'\sqrt{\delta}'\right)^2 + \left(\left(2\lambda'\tau + \lambda\tau'\right)\sqrt{\delta} - \tau\lambda\sqrt{\delta}'\right)^2\right)^{\frac{1}{2}}.$$

In generally

$$N^* = \frac{\left(\left(-\left(2\lambda'\kappa+\lambda\kappa'\right)\lambda' - (1-\kappa\lambda)\lambda''\right)T + \left(\left(\kappa - \lambda\left(\kappa^2+\tau^2\right)\right)\lambda'\right)N + \left(\left(2\lambda'\tau+\lambda\tau'\right)\lambda' - \tau\lambda\lambda''\right)B\right)}{\Delta}.$$

For $\sqrt{\delta} = \frac{\lambda'}{c}$, it is trivial. □

Theorem 2.5. *If the binormal vector field of $N - T^*N^*$ partner curve is B^*, then it can be given based on the Frenet apparatus of the first curve as*

$$B^* = \frac{1}{\Delta}\begin{pmatrix} \left(2\lambda'^2\tau + \lambda\lambda'\tau' - \lambda\lambda''\tau + \lambda^2\tau\left(\kappa^2+\tau^2\right) - \lambda\kappa\tau\right)T \\ +\left(2\lambda'\tau + \lambda\tau' - 2\lambda\lambda'\kappa\tau - \lambda^2\kappa\tau' + 2\lambda'\lambda\tau\kappa + \lambda^2\tau\kappa'\right)N \\ +\left((1-\kappa\lambda)\left(\lambda'' - \lambda\left(\kappa^2+\tau^2\right)+\kappa\right) + \left(2\lambda'^2\kappa + \lambda\lambda'\kappa'\right)\right)B \end{pmatrix}.$$

Proof. Since T^* and N^* has been already calculated , it is easy to calculated $B^* = T^*\Lambda N^*$,

$$B^* = \frac{1}{\sqrt{\delta}\Delta}\begin{vmatrix} T & N & B \\ (1-\kappa\lambda) & \lambda' & \lambda\tau \\ -\left(2\lambda'\kappa+\lambda\kappa'\right)\sqrt{\delta} - (1-\lambda\kappa)\sqrt{\delta}' & \left(\lambda'' - \lambda\left(\kappa^2+\tau^2\right)+\kappa\right)\sqrt{\delta} - \lambda'\sqrt{\delta}' & \left(2\lambda'\tau+\lambda\tau'\right)\sqrt{\delta} - \lambda\tau\sqrt{\delta}' \end{vmatrix}.$$

Hence

$$B^* = \frac{1}{\sqrt{\delta}\Delta}\begin{pmatrix} \left(\left(2\lambda'\tau+\lambda\tau'\right)\lambda'\sqrt{\delta} - \left(\lambda'' - \lambda\left(\kappa^2+\tau^2\right)+\kappa\right)\lambda\tau\sqrt{\delta}\right)T \\ +\left((1-\kappa\lambda)\left(2\lambda'\tau+\lambda\tau'\right)\sqrt{\delta}\right) + \left(2\lambda'\kappa+\lambda\kappa'\right)\lambda\tau\sqrt{\delta}N \\ +\left((1-\kappa\lambda)\left(\lambda'' - \lambda\left(\kappa^2+\tau^2\right)+\kappa\right)\sqrt{\delta} + \left(2\lambda'\kappa+\lambda\kappa'\right)\lambda'\sqrt{\delta}\right)B \end{pmatrix}.$$

For $\sqrt{\delta} = \frac{\lambda'}{c}$, it is trivial.

$$B^* = \frac{1}{\Delta}\begin{pmatrix} \left(2\lambda'^2\tau + \lambda\lambda'\tau' - \lambda\lambda''\tau + \lambda^2\tau\left(\kappa^2+\tau^2\right) - \lambda\kappa\tau\right)T \\ +\left(2\lambda'\tau + \lambda\tau' - 2\lambda\lambda'\kappa\tau - \lambda^2\kappa\tau' + 2\lambda'\lambda\tau\kappa + \lambda^2\tau\kappa'\right)N \\ +\left((1-\kappa\lambda)\left(\lambda'' - \lambda\left(\kappa^2+\tau^2\right)+\kappa\right) + \left(2\lambda'^2\kappa + \lambda\lambda'\kappa'\right)\right)B \end{pmatrix}$$

this complete the proof. □

Theorem 2.6. *If the second curvature of $N - T^*N^*$ partner curve is τ^*, then it can be given based on the Frenet apparatus of the first curve as*

$$\tau^* = \frac{\lambda'\lambda\left(\kappa'\tau - \kappa\tau'\right) + \lambda''\tau - \lambda\tau\left(\kappa^2+\tau^2\right) + \kappa\tau}{\Delta\sqrt{\delta}\sin\theta}.$$

Proof. Since the definition of $N - T^*N^*$ *partner curve* and Frenet vector fields, we know that $\langle B^*, T^*\rangle = 0, \langle B^*, N^*\rangle = 0$, so $\langle B^*, O^*\rangle = \langle B^*, N\rangle = 0$. Taking he derivation of the both sides of $\langle B^*, N\rangle = 0$, gives us

$$\left\langle \frac{dB^*}{ds}, N\right\rangle + \langle B^*, N'\rangle = 0,$$

$$-\tau^*\sqrt{\delta}\sin\theta - \kappa\langle B^*, T\rangle + \tau\langle B^*, B\rangle = 0.$$

As a result we have

$$\tau^* = \frac{-\kappa\langle B^*, T\rangle + \tau\langle B^*, B\rangle}{\sqrt{\delta}\sin\theta}.$$

Hence the result of these products completes the proof with the equality, under the condition $O^* = N$. □

Theorem 2.7. *There is the relationship among the curvatures of $N - T^*N^*$ partner curve and λ, based on the Frenet apparatus as in the following way*

$$2\lambda'\tau + \lambda(1-\lambda\kappa)\tau' = 0.$$

Proof. Binormal vector B^* of $N - T^*N^*$ *partner curve* is perpendicular its osculator plane so the binormal vector B^* perpendicular is $O^* = N$ we have $\frac{dB^*}{ds^*} = -\tau^* N^*$ so $N^* = \frac{-1}{\tau^*}\frac{dB^*}{ds}\frac{ds}{ds^*}$. Since the definition of the $N - T^*N^*$ *partner curve*, it is easy to say $\langle B^*, O^*\rangle = \langle B^*, N\rangle = 0$ with $\Delta \neq 0$, we have the proof. □

Theorem 2.8. *There is the relationship among the curvatures of $N - T^*N^*$ partner curve and λ, based on the Frenet apparatus as in the following way*

$$\frac{\lambda(1-\lambda\kappa)}{2\sqrt{\delta}}\frac{\tau'}{\tau} = c \ constant.$$

Proof. We have already find the following equalities;

$$\lambda' = \frac{a\sqrt{\delta}}{\sqrt{a^2+b^2}} = \sqrt{\delta}\cos\theta$$

and

$$\lambda' = \frac{\lambda^2\kappa\tau' - \lambda\tau'}{2\tau}.$$

Hence the equality of the both left side

$$\frac{\lambda^2\kappa\tau' - \lambda\tau'}{2\tau} = \sqrt{\delta}\cos\theta$$

or $\frac{\lambda^2\kappa\tau'-\lambda\tau'}{2\tau} = \frac{a\sqrt{\delta}}{\sqrt{a^2+b^2}}$. We can say that $\lambda' = \frac{(1-\lambda\kappa)}{2\tau}\lambda\tau' = c\sqrt{\delta}$, or $\frac{\lambda(1-\lambda\kappa)\tau'}{2\tau\frac{\lambda'}{c}} = c \ constant$. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] Gray, A., Modern Differential Geometry of Curves and Surfaces with Mathematica, 2nd ed. Boca Raton, FL: CRC Press, p. 205, 1997. 1
[2] Hacisalihoğlu, H.H., Diferensiyel Geometri, Cilt 1, Ínönü Üniversitesi Yayinlari, Malatya 1994. 1
[3] İlarslan, K., Nesovic, E., *Some characterizations of osculating curves in the Euclidean spaces*, Demonstratio Mathematica, **16**(4)(2008), 931–939. 1
[4] Kılıçoğlu, Ş., Şenyut, S., *An examination on NP* curves in E^3*, Turk. J. Math. Comput. Sci, **12**(1)(2020), 26–30. 1
[5] Körpınar, T., Sarıaydın, M.T., Turhan, E., *Associated curves according to Bishop frame in Euclidean 3-space*, AMO, **15**(2015), 71. 1
[6] Lipschutz, M.M., Diferential Geometry, Schaum's Outlines. 1
[7] Liu, H., Wang, F., *Mannheim partner curves in 3-space*, Journal of Geometry, **88**(1)(2008), 120–126. 1
[8] Schief, W.K., *On the integrability of Bertrand curves and Razzaboni surfaces*, Journal of Geometry and Physics, **45**(1-2)(2003), 130–150. 1