Journal of
New Theory
ISSN: 2149-1402

28 (2019) 62-73
Journal of New Theory
http://www.newtheory.org
Open Access

Direct Product of Fuzzy Multigroups

Paul Augustine Ejegwa[1]

Article History
Received: 22.10.2018
Accepted: 04.09.2019
Published: 08.09.2019
Original Article

Abstract − The paper introduces direct product in fuzzy multigroup setting as an extension of direct product of fuzzy subgroups. Some properties of direct product of fuzzy multigroups are explicated. It is established that the direct product of fuzzy multigroups is a fuzzy multigroup. The notion of homomorphism and some of its properties in the context of direct product of fuzzy multigroups are introduced.

Keywords − *Fuzzy multisets, Fuzzy multigroups, Direct product of fuzzy multigroups*

1. Introduction

The concept of set theory put forward by a German mathematician George Cantor (1845-1918) is a linchpin for the whole of mathematics. Notwithstanding, an element of a set must be distinct and definite in a collection, which is not consistent with real-life issues. The "famous" fuzzy set proposed by Zadeh [1] is a veritable tool for handling uncertainty and/or imprecision in real-life problems. Fuzzy set theory posited that there are cases where an element would not be definite in a collection. The theory of fuzzy set has grown tremendously over time giving birth to some algebraic structures like fuzzy group introduced by Rosenfeld [2]. Some properties of the fuzzy groups have been discussed in details in [3, 4], etc. In the same vein, multiset theory [5–7] violated the rule that an element must be distinct in a collection, i.e., the idea of multisets allows repetition of elements.

Motivated by Zadeh [1], fuzzy multiset was proposed by Yager [8] as a generalization of fuzzy set. The idea of fuzzy multisets allows the repetition of membership function of an element in multiset framework, unlike the case in fuzzy set where membership function of an element does not allow to repeat. Some details of the notion of fuzzy multisets can be found in [9–11]. Subsequently, Shinoj *et al.* [12] followed the footsteps of Rosenfeld [2] and introduced a non-classical group called fuzzy multigroup, which constitutes an application of fuzzy multiset to the theory of group. The idea of abelian fuzzy multigroups was proposed and studied in [13, 14]. Ejegwa [15] introduced fuzzy multigroupoids, the ideas of center and centralizer in fuzzy multigroup context with some related results. The notions of fuzzy submultigroups and normal fuzzy submultigroups were explicated in [15, 16] with a number of results. Also, the concept of homomorphism of fuzzy multigroups and its properties have been explored with some results [17].

This paper is motivated by the work of Ray [18] on product of fuzzy subgroups, which was extended from group theory but presented in the light of fuzzy groups. In the same vein, we are spurred to propose direct product in fuzzy multigroup structure as an extension of the work in [18], and explicate

[1]ocholohi@gmail.com, ejegwa.augustine@uam.edu.ng (Corresponding Author)
[1]Department of Mathematics/Statistics/Computer Science, College of Science, University of Agriculture, P.M.B. 2373, Makurdi, Nigeria

some of it properties in details. The rest of the paper are thus outline: Section 2 provides some preliminaries while Section 3 proposes direct product in fuzzy multigroup setting, discusses some of its properties and outline some related results. Finally, Section 4 contains the conclusion and recommendations for future studies.

2. Preliminaries

This section presents some foundational concepts which are germane to the subject under consideration.

Definition 2.1. [8] Suppose X is a nonempty set. Then, a fuzzy bag/multiset A drawn from X can be characterized by a count membership function CM_A where

$$CM_A : X \to Q$$

and Q is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.

A fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

By [19], it follows that $CM_A(x)$ for $x \in X$ is given as

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ...\},$$

where $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ... \in [0, 1]$ where $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x) \geq ...$, whereas in a finite case, we write

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x)\},$$

for $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented by

$$A = \{\langle \frac{CM_A(x)}{x} \rangle \mid x \in X\} \text{ or } A = \{\langle x, CM_A(x) \rangle \mid x \in X\}.$$

In short, a fuzzy multiset A of X is characterized by the count membership function $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0, 1]$ [20, 21].

We denote the set of all fuzzy multisets by $FMS(X)$.

Definition 2.2. [10] Suppose $A, B \in FMS(X)$. Then, A is called a fuzzy submultiset of B denoted by $A \subseteq B$ if $CM_A(x) \leq CM_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.3. [11] Suppose A and B are fuzzy multisets of a set X. Then, the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, respectively, are defined by the rules that for any object $x \in X$,

(i) $CM_{A \cap B}(x) = CM_A(x) \wedge CM_B(x)$,

(ii) $CM_{A \cup B}(x) = CM_A(x) \vee CM_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum respectively.

Definition 2.4. [10] Let $A, B \in FMS(X)$. Then, A and B are comparable to each other if and only if $A \subseteq B$ or $B \subseteq A$, and $A = B \Leftrightarrow CM_A(x) = CM_B(x) \forall x \in X$.

Definition 2.5. [12] Suppose X is a group. Then, a fuzzy multiset A of X is a fuzzy multigroup of X if

(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \forall x, y \in X$,

(ii) $CM_A(x^{-1}) \geq CM_A(x) \forall x \in X$.

It follows immediately that,

$$CM_A(x^{-1}) = CM_A(x), \forall x \in X$$

since

$$CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1}).$$

Also,

$$CM_A(e) \geq CM_A(x) \forall x \in X$$

because

$$CM_A(e) = CM_A(xx^{-1}) \geq CM_A(x) \wedge CM_A(x) = CM_A(x)$$

and

$$CM_A(x^n) \geq CM_A(x) \forall x \in X$$

since

$$\begin{aligned} CM_A(x^n) = CM_A(x^{n-1}x) &\geq CM_A(x^{n-1}) \wedge CM_A(x) \\ &\geq CM_A(x) \wedge ... \wedge CM_A(x) \\ &= CM_A(x). \end{aligned}$$

Every fuzzy multigroup is a fuzzy multiset but the converse is not true. We denote the set of all fuzzy multigroups of X by $FMG(X)$.

Definition 2.6. [15] Suppose $A \in FMG(X)$. Then, a fuzzy submultiset B of A is a fuzzy submultigroup of A denoted by $B \sqsubseteq A$ if B a fuzzy multigroup. A fuzzy submultigroup B of A is a proper denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Remark 2.7. [15] If $A \in FMG(X)$ and $B \sqsubseteq A$, then $B \in FMG(X)$. Again, suppose $C \in FMS(X)$ and $C \subseteq B$. Then $C \sqsubseteq A \Leftrightarrow C \sqsubseteq B$.

Definition 2.8. [13] A fuzzy multiset A of a set X is commutative if $CM_A(xy) = CM_A(yx)$ for all $x, y \in X$.

Definition 2.9. [12,15] Suppose $A \in FMG(X)$. Then, A_* and A^* are defined by

(i) $A_* = \{x \in X \mid CM_A(x) > 0\}$ and

(ii) $A^* = \{x \in X \mid CM_A(x) = CM_A(e)\}$, where e is the identity element of X.

Proposition 2.10. [12,15] Suppose $A \in FMG(X)$, then A_* and A^* are subgroups of X.

Definition 2.11. [16] Let $A, B \in FMG(X)$ such that $A \subseteq B$. Then, A is a normal fuzzy submultigroup of B if for all $x, y \in X$,

$$CM_A(xyx^{-1}) \geq CM_A(y).$$

Proposition 2.12. [16] Let $A, B \in FMG(X)$. Then, the following statements are equivalent.

(i) A is a normal fuzzy submultigroup of B.

(ii) $CM_A(xyx^{-1}) = CM_A(y) \forall x, y \in X$.

(iii) $CM_A(xy) = CM_A(yx) \forall x, y \in X$.

Definition 2.13. [16] Let $A, B \in FMG(X)$. We say A and B are conjugate to each other if for all $x, y \in X$,

$$CM_A(x) = CM_B(yxy^{-1}) \text{ and } CM_B(y) = CM_A(xyx^{-1}).$$

Definition 2.14. Suppose $A \in FMG(X)$. Then, $A_{[\alpha]}$ and $A_{(\alpha)}$ defined by

(i) $A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$ and

(ii) $A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$

are called strong upper alpha-cut and weak upper alpha-cut of A, where $\alpha \in [0, 1]$.

Definition 2.15. Let $A \in FMG(X)$. Then, $A^{[\alpha]}$ and $A^{(\alpha)}$ defined by

(i) $A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$ and

(ii) $A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$

are called strong lower alpha-cut and weak lower alpha-cut of A, where $\alpha \in [0,1]$.

Theorem 2.16. Suppose $A \in FMG(X)$. Then $A_{[\alpha]}$ is a subgroup of X if $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a subgroup of X if $\alpha \geq CM_A(e)$, where e is the identity element of X and $\alpha \in [0,1]$.

Definition 2.17. Suppose $A, B \in FMG(X)$ such that $A \subseteq B$. Then, A is a characteristic (fully invariant) fuzzy submultigroup of B if

$$CM_{A^\theta}(x) = CM_A(x) \; \forall x \in X$$

for every automorphism, θ of X. That is, $\theta(A) \subseteq A$ for every $\theta \in Aut(X)$.

Proposition 2.18. Suppose X is a group. Every characteristic fuzzy submultigroup of a fuzzy multigroup B of X is normal.

Definition 2.19. [17] Suppose X and Y are groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are fuzzy multigroups of X and Y respectively, then f induces a homomorphism from A to B which satisfies

(i) $CM_A(f^{-1}(y_1y_2)) \geq CM_A(f^{-1}(y_1)) \wedge CM_A(f^{-1}(y_2)) \; \forall y_1, y_2 \in Y$,

(ii) $CM_B(f(x_1x_2)) \geq CM_B(f(x_1)) \wedge CM_B(f(x_2)) \; \forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a fuzzy multiset over Y defined by

$$CM_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a fuzzy multiset over X defined by

$$CM_{f^{-1}(B)}(x) = CM_B(f(x)) \forall x \in X.$$

Theorem 2.20. [17] Suppose X and Y are groups and $f : X \to Y$ is an isomorphism. Then

(i) $A \in FMG(X) \Leftrightarrow f(A) \in FMG(Y)$.

(ii) $B \in FMG(Y) \Leftrightarrow f^{-1}(B) \in FMG(X)$.

3. Main results

Suppose X and Y are two groups. Then, the direct product, $X \times Y$ is the Cartesian product of ordered pair (x, y) such that $x \in X$ and $y \in Y$, and the group operation is component-wise, so $(x_1, y_1) \times (x_2, y_2) = (x_1x_2, y_1y_2)$. The resulting algebraic structure satisfies the axioms for a group. Since the ordered pair (x, y) such that $x \in X$ and $y \in Y$ is an element of $X \times Y$, we simply write $(x, y) \in X \times Y$. In this section, we discuss the notion of direct product of two fuzzy multigroups defined over X and Y, respectively.

Definition 3.1. Suppose $A \in FMG(X)$ and $B \in FMG(Y)$ where X and Y are groups. The direct product of A and B depicted by $A \times B$ is a function

$$CM_{A\times B} : X \times Y \to Q$$

defined by

$$CM_{A\times B}((x, y)) = CM_A(x) \wedge CM_B(y) \forall x \in X, \forall y \in Y,$$

where Q is the set of all multisets from the unit interval $I = [0, 1]$.

Example 3.2. Let $X = \{1, x\}$ be a group, where $x^2 = 1$ and $Y = \{e, a, b, c\}$ be a Klein 4-group, where $a^2 = b^2 = c^2 = e$. Suppose

$$A = \{\langle \frac{1, 0.8}{1} \rangle, \langle \frac{0.8, 0.5}{x} \rangle\}$$

and

$$B = \{\langle \frac{1, 0.9}{e} \rangle, \langle \frac{0.6, 0.5}{a} \rangle, \langle \frac{0.7, 0.6}{b} \rangle, \langle \frac{0.6, 0.5}{c} \rangle\}$$

are fuzzy multigroups of X and Y by Definition 2.5. Now

$$X \times Y = \{(1, e), (1, a), (1, b), (1, c), (x, e), (x, a), (x, b), (x, c)\}$$

is a group from the classical sense. By Definition 3.1, we get

$$A \times B = \{\langle \frac{1, 0.8}{(1, e)} \rangle, \langle \frac{0.6, 0.5}{(1, a)} \rangle, \langle \frac{0.7, 0.6}{(1, b)} \rangle, \langle \frac{0.6, 0.5}{(1, c)} \rangle, \langle \frac{0.8, 0.5}{(x, e)} \rangle, \langle \frac{0.6, 0.5}{(x, a)} \rangle, \langle \frac{0.7, 0.5}{(x, b)} \rangle, \langle \frac{0.6, 0.5}{(x, c)} \rangle\}.$$

Certainly, $A \times B$ is a fuzzy multigroup of $X \times Y$ in accordance to Definition 2.5.

Next, we consider an example to investigate what happens of the direct product of a fuzzy multigroup of a group X and a fuzzy multiset of a group Y.

Example 3.3. Let X and Y be groups as in Example 3.2. Suppose we have a fuzzy multigroup of X given as

$$A = \{\langle \frac{1, 0.5}{1} \rangle, \langle \frac{0.7, 0.4}{x} \rangle\},$$

and a fuzzy multiset of Y as

$$B = \{\langle \frac{0.7, 0.5}{e} \rangle, \langle \frac{0.6, 0.4}{a} \rangle, \langle \frac{0.7, 0.6}{b} \rangle, \langle \frac{0.6, 0.4}{c} \rangle\}.$$

Synthesizing Definitions 2.5 and 3.1, we get

$$A \times B = \{\langle \frac{0.7, 0.5}{(1, e)} \rangle, \langle \frac{0.6, 0.4}{(1, a)} \rangle, \langle \frac{0.7, 0.5}{(1, b)} \rangle, \langle \frac{0.6, 0.4}{(1, c)} \rangle, \langle \frac{0.7, 0.4}{(x, e)} \rangle, \langle \frac{0.6, 0.4}{(x, a)} \rangle, \langle \frac{0.7, 0.4}{(x, b)} \rangle, \langle \frac{0.6, 0.4}{(x, c)} \rangle\},$$

and it follows that $A \times B$ is a fuzzy multigroup of $X \times Y$ although B is not a fuzzy multigroup of Y.

Theorem 3.4. Let $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then for all $\alpha \in [0, 1]$,

(i) $A \times B)_{[\alpha]} = A_{[\alpha]} \times B_{[\alpha]}$.

(ii) $(A \times B)^{[\alpha]} = A^{[\alpha]} \times B^{[\alpha]}$.

Proof. (i) Let $(x, y) \in (A \times B)_{[\alpha]}$. Using Definition 2.14, we have

$$CM_{A \times B}((x, y)) = (CM_A(x) \wedge CM_B(y)) \geq \alpha.$$

This implies that $CM_A(x) \geq \alpha$ and $CM_B(y) \geq \alpha$, then $x \in A_{[\alpha]}$ and $y \in B_{[\alpha]}$. Thus,

$$(x, y) \in A_{[\alpha]} \times B_{[\alpha]}.$$

Also, let $(x, y) \in A_{[\alpha]} \times B_{[\alpha]}$. Then $CM_A(x) \geq \alpha$ and $CM_B(y) \geq \alpha$. That is,

$$(CM_A(x) \wedge CM_B(y)) \geq \alpha.$$

This yields us $(x, y) \in (A \times B)_{[\alpha]}$. Therefore, $(A \times B)_{[\alpha]} = A_{[\alpha]} \times B_{[\alpha]}$ $\forall \alpha \in [0, 1]$.

(ii) Similar to (i). □

Corollary 3.5. Suppose $A \in FMG(X)$ and $B \in FMG(Y)$, then

(i) $(A \times B)_* = A_* \times B_*$,

(ii) $(A \times B)^* = A^* \times B^*$.

Proof. Similar to Theorem 3.4. □

Theorem 3.6. Suppose $A \in FMG(X)$ and $B \in FMG(Y)$. Then $A \times B$ is a fuzzy multigroup of $X \times Y$.

PROOF. Let $(x, y) \in X \times Y$ and let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We have

$$\begin{aligned} CM_{A\times B}(xy) &= CM_{A\times B}((x_1,x_2)(y_1,y_2)) \\ &= CM_{A\times B}((x_1y_1, x_2y_2)) \\ &= CM_A(x_1y_1) \wedge CM_B(x_2y_2) \\ &\geq \wedge(CM_A(x_1) \wedge CM_A(y_1), CM_B(x_2) \wedge CM_B(y_2)) \\ &= \wedge(CM_A(x_1) \wedge CM_B(x_2), CM_A(y_1) \wedge CM_B(y_2)) \\ &= CM_{A\times B}((x_1,x_2)) \wedge CM_{A\times B}((y_1,y_2)) \\ &= CM_{A\times B}(x) \wedge CM_{A\times B}(y). \end{aligned}$$

Also,

$$\begin{aligned} CM_{A\times B}(x^{-1}) &= CM_{A\times B}((x_1,x_2)^{-1}) = CM_{A\times B}((x_1^{-1}, x_2^{-1})) \\ &= CM_A(x_1^{-1}) \wedge CM_B(x_2^{-1}) = CM_A(x_1) \wedge CM_B(x_2) \\ &= CM_{A\times B}((x_1,x_2)) = CM_{A\times B}(x). \end{aligned}$$

Hence, $A \times B \in FMG(X \times Y)$. □

Corollary 3.7. Let $A_1, B_1 \in FMG(X_1)$ and $A_2, B_2 \in FMG(X_2)$, respectively such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. If A_1 and A_2 are normal fuzzy submultigroups of B_1 and B_2, then $A_1 \times A_2$ is a normal fuzzy submultigroup of $B_1 \times B_2$.

PROOF. By Theorem 3.6, $A_1 \times A_2$ is a fuzzy multigroup of $X_1 \times X_2$. Also, $B_1 \times B_2$ is a fuzzy multigroup of $X_1 \times X_2$. We show that $A_1 \times A_2$ is a normal fuzzy submultigroup of $B_1 \times B_2$. Let $(x, y) \in X_1 \times X_2$ such that $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then we get

$$\begin{aligned} CM_{A_1\times A_2}(xy) &= CM_{A_1\times A_2}((x_1,x_2)(y_1,y_2)) \\ &= CM_{A_1\times A_2}((x_1y_1, x_2y_2)) \\ &= CM_{A_1}(x_1y_1) \wedge CM_{A_2}(x_2y_2) \\ &= CM_{A_1}(y_1x_1) \wedge CM_{A_2}(y_2x_2) \\ &= CM_{A_1\times A_2}((y_1x_1, y_2x_2)) \\ &= CM_{A_1\times A_2}((y_1,y_2)(x_1,x_2)) \\ &= CM_{A_1\times A_2}(yx). \end{aligned}$$

Hence, the result follows by Proposition 2.12. □

Theorem 3.8. Suppose A and B are fuzzy multigroups of X and Y, respectively. Then

(i) $(A \times B)_*$ is a subgroup of $X \times Y$,

(ii) $(A \times B)^*$ is a subgroup of $X \times Y$,

(iii) $(A \times B)_{[\alpha]}$ is a subgroup of $X \times Y$, $\forall\, \alpha \leq CM_{A\times B}(e, e')$ and $\alpha \in [0, 1]$,

(iv) $(A \times B)^{[\alpha]}$ is a subgroup of $X \times Y$, $\forall\, \alpha \geq CM_{A\times B}(e, e')$ and $\alpha \in [0, 1]$.

PROOF. Combining Proposition 2.10, Theorems 2.16 and 3.6, the results follow. □

Corollary 3.9. Suppose $A, C \in FMG(X)$ such that $A \subseteq C$ and $B, D \in FMG(Y)$ such that $B \subseteq D$, respectively. If A and B are normal, then

(i) $(A \times B)_*$ is a normal subgroup of $(C \times D)_*$,

(ii) $(A \times B)^*$ is a normal subgroup of $(C \times D)^*$,

(iii) $(A \times B)_{[\alpha]}$ is a normal subgroup of $(C \times D)_{[\alpha]}$, $\forall\, \alpha \leq CM_{A\times B}(e, e')$ and $\alpha \in [0, 1]$,

(iv) $(A \times B)^{[\alpha]}$ is a normal subgroup of $(C \times D)^{[\alpha]}$, $\forall\, \alpha \geq CM_{A\times B}(e, e')$ and $\alpha \in [0, 1]$.

PROOF. Combining Proposition 2.10, Theorems 2.16, 3.6 and Corollary 3.7, the results follow. □

Proposition 3.10. Let $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then $\forall (x, y) \in X \times Y$, we have

(i) $CM_{A\times B}((x^{-1}, y^{-1})) = CM_{A\times B}((x, y))$,

(ii) $CM_{A\times B}((e, e')) \geq CM_{A\times B}((x, y))$,

(iii) $CM_{A\times B}((x, y)^n) \geq CM_{A\times B}((x, y))$,

where e and e' are the identity elements of X and Y, respectively and $n \in \mathbb{N}$.

PROOF. Let $x \in X,\ y \in Y$ and $(x, y) \in X \times Y$. By Theorem 3.6, it follows that $A \times B \in FMG(X \times Y)$. Now,
(i)

$$\begin{aligned} CM_{A\times B}((x^{-1}, y^{-1})) &= CM_A(x^{-1}) \wedge CM_B(y^{-1}) \\ &= CM_A(x) \wedge CM_B(y) \\ &= CM_{A\times B}((x, y)). \end{aligned}$$

Clearly, $CM_{A\times B}((x^{-1}, y^{-1})) = CM_{A\times B}((x, y))\ \forall (x, y) \in X \times Y$.

(ii)

$$\begin{aligned} CM_{A\times B}((e, e')) &= CM_{A\times B}((x, y)(x^{-1}, y^{-1})) \\ &\geq CM_{A\times B}((x, y)) \wedge CM_{A\times B}((x^{-1}, y^{-1})) \\ &= CM_{A\times B}((x, y)) \wedge CM_{A\times B}((x, y)) \\ &= CM_{A\times B}((x, y))\ \forall (x, y) \in X \times Y. \end{aligned}$$

Hence, $CM_{A\times B}((e, e')) \geq CM_{A\times B}((x, y))$.

(iii)

$$\begin{aligned} CM_{A\times B}((x, y)^n) &= CM_{A\times B}((x^n, y^n)) \\ &= CM_{A\times B}((x^{n-1}, y^{n-1})(x, y)) \\ &\geq CM_{A\times B}((x^{n-1}, y^{n-1})) \wedge CM_{A\times B}((x, y)) \\ &\geq CM_{A\times B}((x^{n-2}, y^{n-2})) \wedge CM_{A\times B}((x, y)) \wedge CM_{A\times B}((x, y)) \\ &\geq CM_{A\times B}((x, y)) \wedge CM_{A\times B}((x, y)) \wedge ... \wedge CM_{A\times B}((x, y)) \\ &= CM_{A\times B}((x, y)), \end{aligned}$$

$\Rightarrow CM_{A\times B}((x, y)^n) = CM_{A\times B}((x^n, y^n)) \geq CM_{A\times B}((x, y))\ \forall (x, y) \in X \times Y$. □

Theorem 3.11. Let A and B be fuzzy multisets of groups X and Y, respectively. Suppose that e and e' are the identity elements of X and Y, respectively. If $A \times B$ is a fuzzy multigroup of $X \times Y$, then at least one of the following statements hold.

(i) $CM_B(e') \geq CM_A(x)\ \forall x \in X$,

(ii) $CM_A(e) \geq CM_B(y)\ \forall y \in Y$.

PROOF. Let $A \times B \in FMG(X \times Y)$. By contrapositive, suppose that none of the statements holds. Then suppose we can find a in X and b in Y such that

$$CM_A(a) > CM_B(e') \text{ and } CM_B(b) > CM_A(e).$$

From these we have

$$\begin{aligned} CM_{A\times B}((a,b)) &= CM_A(a)\wedge CM_B(b) \\ &> CM_A(e)\wedge CM_B(e') \\ &= CM_{A\times B}((e,e')). \end{aligned}$$

Thus, $A\times B$ is not a fuzzy multigroup of $X\times Y$ by Proposition 3.10. Hence, either $CM_B(e')\geq CM_A(x)\,\forall x\in X$ or $CM_A(e)\geq CM_B(y)\,\forall y\in Y$. This completes the proof. □

Theorem 3.12. Let A and B be fuzzy multisets of groups X and Y, respectively, such that $CM_A(x)\leq CM_B(e')\,\forall x\in X$, e' being the identity element of Y. If $A\times B$ is a fuzzy multigroup of $X\times Y$, then A is a fuzzy multigroup of X.

Proof. Let $A\times B$ be a fuzzy multigroup of $X\times Y$ and $x,y\in X$. Then $(x,e'),(y,e')\in X\times Y$. Now, using the property $CM_A(x)\leq CM_B(e')\,\forall x\in X$, we get

$$\begin{aligned} CM_A(xy) &= CM_A(xy)\wedge CM_B(e'e') \\ &= CM_{A\times B}((xy,e'e')) \\ &= CM_{A\times B}((x,e')(y,e')) \\ &\geq CM_{A\times B}((x,e'))\wedge CM_{A\times B}((y,e')) \\ &= \wedge(CM_A(x)\wedge CM_B(e'),CM_A(y)\wedge CM_B(e')) \\ &= CM_A(x)\wedge CM_A(y). \end{aligned}$$

Also,

$$\begin{aligned} CM_A(x^{-1}) &= CM_A(x^{-1})\wedge CM_B(e'^{-1}) = CM_{A\times B}((x^{-1},e'^{-1})) \\ &= CM_{A\times B}((x,e')^{-1}) = CM_{A\times B}((x,e')) \\ &= CM_A(x)\wedge CM_B(e') = CM_A(x). \end{aligned}$$

Hence, A is a fuzzy multigroup of X. This completes the proof. □

Theorem 3.13. Let A and B be fuzzy multisets of groups X and Y, respectively, such that $CM_B(x)\leq CM_A(e)\,\forall x\in Y$, e being the identity element of X. If $A\times B$ is a fuzzy multigroup of $X\times Y$, then B is a fuzzy multigroup of Y.

Proof. Similar to Theorem 3.12. □

Corollary 3.14. Let A and B be fuzzy multisets of groups X and Y, respectively. If $A\times B$ is a fuzzy multigroup of $X\times Y$, then either A is a fuzzy multigroup of X or B is a fuzzy multigroup of Y.

Proof. Combining Theorems 3.11, 3.12 and 3.13, the result follows. □

Theorem 3.15. If A and C are conjugate fuzzy multigroups of a group X, and B and D are conjugate fuzzy multigroups of a group Y. Then $A\times B$ is a conjugate of $C\times D$.

Proof. Since A and C are conjugate, it implies that for $g_1\in X$, we have

$$CM_A(x)=CM_C(g_1^{-1}xg_1)\,\forall x\in X.$$

Also, since B and D are conjugate, for $g_2\in Y$, we get

$$CM_B(y)=CM_D(g_2^{-1}yg_2)\,\forall y\in Y.$$

Now,

$$\begin{aligned} CM_{A\times B}((x,y))=CM_A(x)\wedge CM_B(y) &= CM_C(g_1^{-1}xg_1)\wedge CM_D(g_2^{-1}yg_2) \\ &= CM_{C\times D}((g_1^{-1}xg_1),(g_2^{-1}yg_2)) \\ &= CM_{C\times D}((g_1^{-1},g_2^{-1})(x,y)(g_1,g_2)) \\ &= CM_{C\times D}((g_1,g_2)^{-1}(x,y)(g_1,g_2)). \end{aligned}$$

This completes the proof. □

Theorem 3.16. Let $A \in FMG(X)$ and $B \in FMG(Y)$. Suppose C and D are two fuzzy submultisets of A and B, respectively. Then $C \times D$ is a fuzzy submultigroup of $A \times B$ if and only if both C and D are fuzzy submultigroups of A and B, respectively.

PROOF. Suppose C and D are two fuzzy submultigroups of A and B, respectively. Then $C \in FMG(X)$ and $D \in FMG(Y)$ by Remark 2.7. It then follows that $C \times D \in FMG(X \times Y)$ by Theorem 3.6. Since $A \times B$ is a fuzzy multigroup of $X \times Y$ by the same reason, and $C \sqsubseteq A$ and $D \sqsubseteq B$, thus, $C \times D$ is a fuzzy submultigroup of $A \times B$.

Conversely, If $C \times D$ is a fuzzy submultigroup of $A \times B$. Then, it follows that $C \sqsubseteq A$ and $D \sqsubseteq B$. These complete the proof. □

Corollary 3.17. Let $A \in FMG(X)$ and $B \in FMG(Y)$. Suppose C and D are two fuzzy submultigroups of A and B, respectively. Then $C \times D$ is a normal fuzzy submultigroup of $A \times B$ if and only if both C and D are normal fuzzy submultigroups of A and B, respectively.

PROOF. Combining both Definition 2.11, Theorems 3.6 and 3.16, the proof follows. □

Corollary 3.18. Let $A \in FMG(X)$ and $B \in FMG(Y)$. Suppose C and D are two fuzzy submultigroups of A and B, respectively. Then $C \times D$ is a characteristic fuzzy submultigroup of $A \times B$ if and only if both C and D are characteristic fuzzy submultigroups of A and B, respectively.

PROOF. Combining both Theorems 3.6 and 3.16, the proof follows. □

Remark 3.19. With the same hypothesis as in Corollary 3.18, it follows that $C \times D$ is a normal fuzzy submultigroup of $A \times B$ if both C and D are characteristic fuzzy submultigroups of A and B, respectively.

Corollary 3.20. Let $A \in FMG(X)$ and C be a fuzzy submultiset of A. Then $C \times C$ is a fuzzy submultigroup of $A \times A$ if and only if C is a fuzzy submultigroup of A.

PROOF. The proof is straightforward from Theorem 3.16. □

Remark 3.21. Let $A \in FMG(X)$ and C be a fuzzy submultigroup of A. Then

(i) $C \times C$ is a normal fuzzy submultigroup of $A \times A$ if and only if C is a normal fuzzy submultigroup of A.

(ii) $C \times C$ is a characteristic fuzzy submultigroup of $A \times A$ if and only if C is a characteristic fuzzy submultigroup of A.

(iii) $C \times C$ is a normal fuzzy submultigroup of $A \times A$ if C is a characteristic fuzzy submultigroup of A.

Theorem 3.22. Let A and B be fuzzy multigroups of groups X and Y, respectively. Then A and B are commutative if and only if $A \times B$ is a commutative fuzzy multigroup of $X \times Y$.

PROOF. Suppose A and B are commutative. We show that $A \times B$ is a commutative fuzzy multigroup of $X \times Y$. It is a known fact that $A \times B \in FMG(X \times Y)$ by Theorem 3.6. Let $(x, y) \in X_1 \times X_2$ such that $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then we get

$$\begin{aligned} CM_{A\times B}(xy) &= CM_{A\times B}((x_1, x_2)(y_1, y_2)) \\ &= CM_{A\times B}(x_1y_1, x_2y_2) \\ &= CM_A(x_1y_1) \wedge CM_B(x_2y_2) \\ &= CM_A(y_1x_1) \wedge CM_B(y_2x_2) \\ &= CM_{A\times B}(y_1x_1, y_2x_2) \\ &= CM_{A\times B}((y_1, y_2)(x_1, x_2)) \\ &= CM_{A\times B}(yx). \end{aligned}$$

Hence, $A \times B$ is a commutative fuzzy multigroup of $X \times Y$ by Definition 2.8.

Conversely, suppose $A \times B$ is a commutative fuzzy multigroup of $X \times Y$. Then, it is clear that both A and B are commutative fuzzy multigroups of groups X and Y, respectively. □

Now, we present some homomorphic properties of direct product of fuzzy multigroups. This is an extension of the notion of homomorphism in fuzzy multigroup setting (cf. Definition 2.19) to direct product of fuzzy multigroups.

Definition 3.23. Let $W \times X$ and $Y \times Z$ be groups and let $f : W \times X \to Y \times Z$ be a homomorphism. Suppose $A \times B \in FMS(W \times X)$ and $C \times D \in FMS(Y \times Z)$, respectively. Then

(i) the image of $A \times B$ under f, denoted by $f(A \times B)$, is a fuzzy multiset of $Y \times Z$ defined by

$$CM_{f(A\times B)}((y,z)) = \begin{cases} \bigvee_{(w,x)\in f^{-1}((y,z))} CM_{A\times B}((w,x)), & f^{-1}((y,z)) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

for each $(y, z) \in Y \times Z$.

(ii) the inverse image of $C \times D$ under f, denoted by $f^{-1}(C \times D)$, is a fuzzy multiset of $W \times X$ defined by

$$CM_{f^{-1}(C\times D)}((w,x)) = CM_{C\times D}(f((w,x))) \; \forall (w,x) \in W \times X.$$

Theorem 3.24. Let W, X, Y, Z be groups, $A \in FMS(W), B \in FMS(X), C \in FMS(Y)$ and $D \in FMS(Z)$. If $f : W \times X \to Y \times Z$ is an isomorphism, then

(i) $f(A \times B) = f(A) \times f(B)$,

(ii) $f^{-1}(C \times D) = f^{-1}(C) \times f^{-1}(D)$.

Proof. (i) Let $(w, x) \in W \times X$. Suppose $\exists\, (y, z) \in Y \times Z$ such that

$$f((w,x)) = (f(w), f(x)) = (y,z).$$

Then we get

$$\begin{aligned} CM_{f(A\times B)}((y,z)) &= CM_{A\times B}(f^{-1}((y,z))) \\ &= CM_{A\times B}((f^{-1}(y), f^{-1}(z))) \\ &= CM_A(f^{-1}(y)) \wedge CM_B(f^{-1}(z)) \\ &= CM_{f(A)}(y) \wedge CM_{f(B)}(z) \\ &= CM_{f(A)\times f(B)}((y,z)) \end{aligned}$$

Thus, $f(A \times B) \subseteq f(A) \times f(B)$. Hence, the result follows by symmetry.

(ii) For $(w, x) \in W \times X$, we have

$$\begin{aligned} CM_{f^{-1}(C\times D)}((w,x)) &= CM_{C\times D}(f((w,x))) \\ &= CM_{C\times D}((f(w), f(x))) \\ &= CM_C(f(w)) \wedge CM_D(f(x)) \\ &= CM_{f^{-1}(C)}(w) \wedge CM_{f^{-1}(D)}(x) \\ &= CM_{f^{-1}(C)\times f^{-1}(D)}((w,x)). \end{aligned}$$

Hence, $f^{-1}(C \times D) \subseteq f^{-1}(C) \times f^{-1}(D)$.

Similarly,

$$\begin{aligned} CM_{f^{-1}(C)\times f^{-1}(D)}((w,x)) &= CM_{f^{-1}(C)}(w) \wedge CM_{f^{-1}(D)}(x) \\ &= CM_C(f(w)) \wedge CM_D(f(x)) \\ &= CM_{C\times D}((f(w), f(x))) \\ &= CM_{C\times D}(f((w,x))) \\ &= CM_{f^{-1}(C\times D)}((w,x)). \end{aligned}$$

Again, $f^{-1}(C) \times f^{-1}(D) \subseteq f^{-1}(C \times D)$. Therefore, the result follows. □

Theorem 3.25. Suppose $f : W \times X \to Y \times Z$ is an isomorphism, A, B, C and D be fuzzy multigroups of W, X, Y and Z, respectively. Then, the following statements hold.

(i) $f(A \times B) \in FMG(Y \times Z)$.

(ii) $f^{-1}(C) \times f^{-1}(D) \in FMG(W \times X)$.

Proof. (i) Since $A \in FMG(W)$ and $B \in FMG(X)$, then $A \times B \in FMG(W \times X)$ by Theorem 3.6. From Theorem 2.20 and Definition 3.23, it follows that, $f(A \times B) \in FMG(Y \times Z)$.

(ii) Combining Theorems 2.20, 3.6, Definition 3.23 and Theorem 3.24, the result follows. □

Corollary 3.26. Suppose X and Y are groups, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. If

$$f : X \times X \to Y \times Y$$

be homomorphism, then

(i) $f(A \times A) \in FMG(Y \times Y)$,

(ii) $f^{-1}(B \times B) \in FMG(X \times X)$.

Proof. Similar to Theorem 3.25. □

4. Conclusion

The idea of direct product in fuzzy multigroup setting have been successfully established and lucidly exemplified. Some related results were obtained and proved accordingly. Homomorphism and some of its properties were proposed in the context of direct product of fuzzy multigroups. The idea of generalized direct product of fuzzy multigroups could be exploited.

Acknowledgement

The author thank the Editor in-chief for his technical comments and to the anonymous reviewers for their suggestions, which have improved the quality of the paper.

References

[1] L.A. Zadeh, *Fuzzy sets*, Information and Control 8 (1965) 338–353.

[2] A. Rosenfeld, *Fuzzy subgroups*, Journal of Mathematical Analysis and Applications 35 (1971) 512–517.

[3] J.M. Mordeson, K.R. Bhutani and A. Rosenfeld, *Fuzzy group theory*, Springer-Verlag Berlin Heidelberg, 2005.

[4] B. Seselja, A. Tepavcevic, *A note on fuzzy groups*, Yugoslav Journal of Operation Research 7(1) (1997) 49–54.

[5] A. Syropoulos, *Mathematics of multisets*, Springer-Verlag Berlin Heidelberg (2001) 347–358.

[6] N.J. Wildberger, *A new look at multisets*, School of Mathematics, UNSW Sydney 2052, Australia, 2003.

[7] D. Singh, A.M. Ibrahim, T. Yohanna, and J.N. Singh, *An overview of the applications of multisets*, Novi Sad Journal of Mathematics 37(2) (2007) 73–92.

[8] R.R. Yager, *On the theory of bags*, International Journal of General Systems 13 (1986) 23–37.

[9] P.A. Ejegwa, *Correspondence between fuzzy multisets and sequences*, Global Journal Science Frontier Research: F Mathematics and Decision Science 14(7) (2014) 61–66.

[10] S. Miyamoto, *Basic operations of fuzzy multisets*, Journal of Japan Society of Fuzzy Theory and Systems 8(4) (1996) 639–645.

[11] A. Syropoulos, *On generalized fuzzy multisets and their use in computation*, Iranian Journal of Fuzzy Systems 9(2) (2012) 113–125.

[12] T.K. Shinoj, A. Baby and J.J. Sunil, *On some algebraic structures of fuzzy multisets*, Annals of Fuzzy Mathematics and Informatics 9(1) (2015) 77–90.

[13] A. Baby, T.K. Shinoj and J.J. Sunil, *On abelian fuzzy multigroups and order of fuzzy multigroups*, Journal of New Theory 5(2) (2015) 80–93.

[14] P.A. Ejegwa, *On abelian fuzzy multigroups*, Journal of Fuzzy Mathematics 26(3) (2018) 655–668.

[15] P.A. Ejegwa, *On fuzzy multigroups and fuzzy submultigroups*, Journal of Fuzzy Mathematics 26(3) (2018) 641–654.

[16] P.A. Ejegwa, *On normal fuzzy submultigroups of a fuzzy multigroup*, Theory and Applications of Mathematics and Computer Science 8(1) (2018) 64–80.

[17] P.A. Ejegwa, *Homomorphism of fuzzy multigroups and some of its properties*, Applications and Applied Mathematics 13(1) (2018) 114–129.

[18] A.K. Ray, *On product of fuzzy subgroups*, Fuzzy Sets and Systems 105 (1999) 181–183.

[19] S. Miyamoto and K. Mizutani, *Fuzzy multiset model and methods for nonlinear document clustering for information retrieval*, Springer-Verlag Berlin Heidelberg (2004) 273–283.

[20] R. Biswas, *An application of Yager's bag theory in multicriteria based decision making problems*, International Journal of Intelligent Systems 14 (1999) 1231–1238.

[21] K. Mizutani, R. Inokuchi and S. Miyamoto, *Algorithms of nonlinear document clustering based on fuzzy multiset model*, International Journal of Intelligent Systems 23 (2008) 176–198.