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Research Article

Slant submersions in paracontact geometry

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Abstract

In this paper, we investigate some geometric properties of three types of slant submersions whose total space is an almost paracontact metric manifold.

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1. Introduction

Given a $C^\infty-$submersion ψ from a (semi)-Riemannian manifold (N, g_N) onto a (semi)-Riemannian manifold (B, g_B), according to the circumstances on the map $\psi : (N, g_N) \to (B, g_B)$, we get the following: a (semi)-Riemannian submersion ([3, 8, 14, 20]), an almost Hermitian submersion ([27]), a paracontact submersion ([9]), a paracontact paracomplex submersion ([10]), a (para) quaternionic submersion ([6, 17]), a slant submersion ([12, 19, 22, 23]), an anti-invariant submersion ([11, 24]), a conformal semi-slant submersion ([1, 13]), a conformal anti-invariant submersion ([2]), a hemi-slant submersion ([25]), etc. As we know, Riemannian submersions were severally introduced by B. O'Neill ([20]) and A. Gray ([14]) in 1960s. In particular, by using the concept of almost Hermitian submersions, B. Watson ([27]) gave some differential geometric properties among fibers, base manifolds, and total manifolds. After that, there are lots of results on this issue. It is well-known that Riemannian submersions are associated with physics and have their applications in the Yang-Mills theory ([5]), Kaluza-Klein theory ([4, 15]), supergravity and superstring theories ([16]), etc.

The paper is organized as follows. In Section 2, we remind some concepts, which are needed in the following part. In Section 3, we study some geometric properties of three types of proper slant submersions from an almost paracontact metric manifold onto a semi-Riemannian manifold. We present examples, investigate the geometry of leaves of distributions. We obtain a necessary and sufficient circumstance for such submersions to be totally geodesic map, as well.

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2. Preliminaries

2.1. Semi-Riemannian submersions

A $C^\infty-$ submersion $\psi : N \to B$ between two pseudo-Riemannian manifolds (N, g_N) and (B, g_B) is called a semi-Riemannian submersion if it satisfies circumstances:
(i) the fibers $\psi^{-1}(b)$, $b \in B$, are $r-$ dimensional pseudo-Riemannian submanifolds of N, where $r = dim(N) - dim(B)$.
(ii) ψ_* preserves scalar products of vectors normal to fibres.
The tangent bundle TN of the total space N has an orthogonal decomposition

$$TN = ker\psi_* \oplus (ker\psi_*)^\perp,$$

where $ker\psi_*$ is the vertical distribution while $(ker\psi_*)^\perp$ designates the horizontal one. In ([20]), O'Neill has defined two configuration tensors $\mathcal{T}$ and $\mathcal{A}$, of the total space of a semi-Riemannian submersion by setting

$$\mathcal{T}_{X_1}X_2 = h\nabla_{vX_1}vX_2 + v\nabla_{vX_1}hX_2 \tag{2.1}$$

and

$$\mathcal{A}_{X_1}X_2 = v\nabla_{hX_1}hX_2 + h\nabla_{hX_1}vX_2 \tag{2.2}$$

for any $X_1, X_2 \in \chi(N)$, here v and h are the vertical and horizontal projections respectively.
Using (2.1) and (2.2), we get

$$\nabla_{X_1}X_2 = \mathcal{T}_{X_1}X_2 + \hat{\nabla}_{X_1}X_2; \tag{2.3}$$

$$\nabla_{X_1}X_3 = \mathcal{T}_{X_1}X_3 + h(\nabla_{X_1}X_3); \tag{2.4}$$

$$\nabla_{X_3}X_1 = \mathcal{A}_{X_3}X_1 + v(\nabla_{X_3}X_1), \tag{2.5}$$

$$\nabla_{X_3}X_4 = \mathcal{A}_{X_3}X_4 + h(\nabla_{X_3}X_4), \tag{2.6}$$

for any $X_3, X_4 \in \Gamma((ker\psi_*)^\perp)$, $X_1, X_2 \in \Gamma(ker\psi_*)$. In addition, if X_3 is basic then $h(\nabla_{X_1}X_3) = h(\nabla_{X_3}X_1) = \mathcal{A}_{X_3}X_1$.

The fundamental tensor fields $\mathcal{T}, \mathcal{A}$ satisfy:

$$\mathcal{T}_{X_1}X_2 = \mathcal{T}_{X_2}X_1, \quad X_1, X_2 \in \Gamma(ker\psi_*); \tag{2.7}$$

$$\mathcal{A}_{X_3}X_4 = -\mathcal{A}_{X_4}X_3 = \frac{1}{2}v[X_3, X_4], \quad X_3, X_4 \in \Gamma((ker\psi_*)^\perp). \tag{2.8}$$

Lemma 2.1. *If $\psi : (N, g_N) \to (B, g_B)$ is a (semi-)Riemannian submersion and X_3, X_4 fundamental vector fields on N, $\psi-$related to X_{*3} and X_{*4} vector fields on base manifold B, at that time we obtain the following features*

1. *$h[X_3, X_4]$ is a fundamental vector field and $\psi_* h[X_3, X_4] = [X_{3*}, X_{*4}] \circ \psi$;*
2. *$h(\nabla_{X_3}X_4)$ is a fundamental vector field $\psi-$related to $(\nabla^*_{X_{*3}}X_{*4})$, here ∇ and ∇^* are the Riemannian connection on N and B;*
3. *$[E, X_1] \in \Gamma(ker\psi_*)$, for any $X_1 \in \Gamma(ker\psi_*)$ and for any fundamental vector field E ([8, 21]).*

Let (N, g_N) and (B, g_B) be (semi-)Riemannian manifolds and $\psi : (N, g_N) \to (B, g_B)$ is a differentiable map. At that time, the second fundamental form of ψ is given by

$$(\nabla\psi_*)(X_1, X_2) = \nabla^{\psi}_{X_1}\psi_* X_2 - \psi_*(\nabla_{X_1}X_2) \tag{2.9}$$

for $X_1, X_2 \in \Gamma(N)$, here we show conveniently by ∇ the Riemannian connections of the metrics g_N and g_B. Recall that ψ is said to be *harmonic* if *trace*$(\nabla\psi_*) = 0$ and ψ is called a *totally geodesic* map if $(\nabla\psi_*)(X_1, X_2) = 0$ for $X_1, X_2 \in \Gamma(TN)$, [18].

2.2. Almost paracontact metric manifolds

Let N be a differentiable manifold of dimensional $(2n+1)$. An almost paracontact structure on N is a triple (φ, ξ, η), where:
(1) ξ is a Reeb vector field,
(2) η is a one-form such that $\eta(\xi) = 1$, and
(3) φ is a tensor field of type $(1,1)$ satisfying

$$\varphi^2 = Id - \eta \otimes \xi, \;\; \eta \circ \varphi = 0, \;\; \varphi(\xi) = 0. \tag{2.10}$$

If N is equipped with a pseudo-Riemannian metric g_N such that

$$g_N(\varphi X_1, \varphi X_2) = -g_N(X_1, X_2) + \eta(X_1)\eta(X_2), \;\; X_1, X_2 \in \chi(N), \tag{2.11}$$

then $(\varphi, \xi, \eta, g_N)$ is an almost paracontact metric structure. So, the quintuple $(N^{2n+1}, \varphi, \xi, \eta, g_N)$ is an almost paracontact metric manifold ([26, 28]).
Observe that, since (2.11) holds, any compatible with metric g_N has got sign $(n+1, n)$ and by (2.10) and (2.11) we have $\eta(X_1) = g_N(\xi, X_1)$. Furthermore, we can determine an anti-symmetric two-form Φ by $\Phi(X_1, X_2) = g_N(X_1, \varphi X_2)$, which is called the fundamental 2-form corresponding to the structure.

An almost paracontact metric structure $(\varphi, \xi, \eta, g_N)$ is said to be paracosymplectic, if $\nabla\eta = 0$ and $\nabla\Phi = 0$ are closed ([7]), and the structure equation of a paracosymplectic manifold is given by

$$(\nabla_{X_1}\varphi)X_2 = 0, \quad X_1, X_2 \in \chi(N), \tag{2.12}$$

where ∇ denotes the Riemannian connection of the metric g_N on N. Moreover, for a paracosymplectic manifold, we know that

$$\nabla_{X_1}\xi = 0. \tag{2.13}$$

3. Proper slant submersions

Let ψ be a semi-Riemannian submersion from an almost paracontact metric manifold N with the structure $(\varphi, \xi, \eta, g_N)$ onto a semi-Riemannian manifold (B, g_B). Then for $X_1 \in \Gamma(ker\psi_*)$, we write

$$\varphi X_1 = \alpha X_1 + \beta X_1, \tag{3.1}$$

where αX_1 and βX_1 are vertical and horizontal parts of φX_1.

In addition to for $X_2 \in \Gamma((ker\psi_*)^\perp)$, we get

$$\varphi X_2 = tX_2 + rX_2, \tag{3.2}$$

where tX_2 and rX_2 are vertical and horizontal components of φX_2.

If for any spacelike or timelike vertical vector field $X_1 \in ker\psi_* - \{\xi\}$, the quotient $\frac{g_N(\alpha X_1, \alpha X_1)}{g_N(\varphi X_1, \varphi X_1)}$ is constant, i.e. it is independent of the choice of the point $p \in N$ and choice of the spacelike or timelike vertical vector field X_1 in $ker\psi_* - \{\xi\}$, at that time we call that ψ is a slant submersion. In this case, the angle ω is called the slant angle of the slant submersion.
We note that Reeb vector field ξ is a spacelike vertical vector field.

Let $\{E_1, E_2, \xi\}$ be a local orthonormal frame of vertical vector fields with $g_N(E_1, E_1) = 1$, i.e., such that E_1 is spacelike (if both E_1 and E_2 are timelike, the situation would be similar). From (2.11) and (3.1), we have

$$-1 = g_N(\varphi E_1, \varphi E_1) = g_N(\alpha E_1, \alpha E_1) + g_N(\beta E_1, \beta E_1).$$

On the other hand, $\alpha E_1 = \rho E_2$. Let us suppose $\rho \neq 0, \pm 1$; these conditions would correspond to invariant ([9]) and anti-invariant submersions . Clearly, αE_1 and E_2 have the same causal character. Depending on it and the value of ρ, we can separate the following three conditions:
(1) If αE_1 is a timelike and $\|\rho\| > 1$, at that time $g_N(\beta E_1, \beta E_1) = -1 + \rho^2$ and so βE_1 is spacelike.
(2) If αE_1 is a timelike and $\|\rho\| < 1$, at that time $g_N(\beta E_1, \beta E_1) = -1 + \rho^2$ and so βE_1 is timelike.
(3) If αE_1 is a spacelike, $g_N(\beta E_1, \beta E_1) = -1 - \rho^2$, and βE_1 is a timelike vector field.

These three conditions will correspond to three different types of proper slant submersions.

Definition 3.1. Let ψ be a proper slant submersion from an almost paracontact manifold N with the structure $(\varphi, \xi, \eta, g_N)$ onto a semi-Riemannian manifold (B, g_B). We say that it is of
type 1 if for any spacelike (timelike) vertical vector field $X_1 \in \Gamma(ker\psi_*)$, αX_1 is timelike (spacelike), and $\frac{\|\alpha X_1\|}{\|\varphi X_1\|} > 1$,
type 2 if for any spacelike (timelike) vertical vector field $X_1 \in \Gamma(ker\psi_*)$, αX_1 is timelike (spacelike), and $\frac{\|\alpha X_1\|}{\|\varphi X_1\|} < 1$,
type 3 if for any spacelike (timelike) vertical vector field $X_1 \in \Gamma(ker\psi_*)$, αX_1 is timelike (spacelike).

It is known that the distribution $(ker\psi_*)$ is integrable for a semi-Riemannian submersion between semi-Riemannian manifolds. In fact, its leaves are $\psi^{-1}(b), b \in B$, i.e., fibres. Thus it follows from above definition that the fibers of a slant submersion are slant submanifolds of N.

Theorem 3.2. *Let ψ be a proper slant submersion from an almost paracontact manifold N with the structure $(\varphi, \xi, \eta, g_N)$ onto a semi-Riemannian manifold (B, g_B). Then,*
(i) ψ is slant submersion of type 1 if and only if for any spacelike (timelike) vector field $X_1 \in \Gamma(ker\psi_*)$, αX_1 is timelike (spacelike), and there exists a constant $\mu \in (1, \infty)$ such that

$$\alpha^2 X_1 = \mu(X_1 - \eta(X_1)\xi). \tag{3.3}$$

If ψ is a proper slant submersion of type 1, then $\mu = \cosh^2 \omega$, with $\omega > 0$.
(ii) ψ is a proper slant submersion of type 2 if and only if for any spacelike (timelike) vector field $X_1 \in \Gamma(ker\psi_*)$, αX_1 is timelike (spacelike), and there exists a constant $\mu \in (0, 1)$ such that

$$\alpha^2 X_1 = \mu(X_1 - \eta(X_1)\xi). \tag{3.4}$$

If ψ is a proper slant submersion of type 2, then $\mu = \cos^2 \omega$, with $0 < \omega < 2\pi$.
(iii) ψ is slant submersion of type 3 if and only if for any spacelike (timelike) vector field $X_1 \in \Gamma(ker\psi_*)$, αX_1 is timelike (spacelike), and there exists a constant $\mu \in (-\infty, 0)$ such that

$$\alpha^2 X_1 = \mu(X_1 - \eta(X_1)\xi). \tag{3.5}$$

If ψ is a proper slant submersion of type 3, then $\mu = -\sinh^2 \omega$, with $\omega > 0$.
In every case, the angle ω is called the slant angle of the slant submersion.

Proof. (i) If ψ is slant submersion of type 1, for any spacelike vertical vector field $X_1 \in \Gamma(ker\psi_*)$, αX_1 is timelike, and, by virtue of (2.11), φX_1 is timelike. Furthermore, they satisfy $\frac{\|\alpha X_1\|}{\|\varphi X_1\|} > 1$. So, there exists $\omega > 0$ such that

$$\cosh\omega = \frac{\|\alpha X_1\|}{\|\varphi X_1\|} = \frac{\sqrt{-g_N(\alpha X_1, \alpha X_1)}}{\sqrt{-g_N(\varphi X_1, \varphi X_1)}}. \tag{3.6}$$

By using (2.10), (2.11, (3.1)) and (3.6) we obtain

$$\begin{aligned} g_N(\alpha^2 X_1, X_1) &= -g_N(\alpha X_1, \alpha X_1) \\ &= -\cosh^2\omega g_N(\varphi X_1, \varphi X_1) \\ &= \cosh^2\omega g_N(\varphi^2 X_1, X_1) \\ &= \cosh^2\omega g_N(X_1 - \eta(X_1)\xi, X_1) \end{aligned} \tag{3.7}$$

for all $X_1 \in \Gamma(ker\psi_*)$. Since g_N is a semi-Riemannian metric, from (3.7) we get

$$\alpha^2 X_1 = \cosh^2\omega(X_1 - \eta(X_1)\xi), \;\; X_1 \in \Gamma(ker\psi_*). \tag{3.8}$$

Let $\mu = \cosh^2\omega$. Then it is obvious that $\mu \in (1, \infty)$ and $\alpha^2 = \mu(I - \eta \otimes \xi)$.
Everything works in a similar way for any timelike vector field $X_2 \in \Gamma(ker\psi_*)$, but now, αX_2 and φX_2 are spacelike and hence, instead of (3.6) we can write:

$$\cosh\omega = \frac{\|\alpha X_2\|}{\|\varphi X_2\|} = \frac{\sqrt{g_N(\alpha X_2, \alpha X_2)}}{\sqrt{g_N(\varphi X_1, \varphi X_1)}}.$$

Since $\alpha^2 X_1 = \mu(X_1 - \eta(X_1)\xi)$, for any spacelike or timelike X_1 we have that $\alpha^2 = \mu(I - \eta \otimes \xi)$. The converse is just a easy computation.
(ii) is obtained in a similar way.

(iii) If ψ is proper slant submersion of type 3, for any spacelike vector field $X_1 \in \Gamma(ker\psi_*)$, αX_1 is spacelike,as well and hence, there exists $\omega > 0$ such that

$$\sinh\omega = \frac{\|\alpha X_1\|}{\|\varphi X_1\|} = \frac{\sqrt{g_N(\alpha X_1, \alpha X_1)}}{\sqrt{-g_N(\varphi X_1, \varphi X_1)}}.$$

Once more, we can demonstrate that $g_N(\alpha^2 X_1, X_1) = -\sinh^2\omega g_N(X_1 - \eta(X_1)\xi, X_1)$. Let $\mu = -\sinh^2\omega$. At that time it is clear that $\mu \in (-\infty, 0)$ and $\alpha^2 = \mu(I - \eta \otimes \xi)$.
The converse is just a easy computation. □

For slant submersion of type 2, the slant angle coincides with the Wirtinger angle, i.e., the slant angle between φX_1 and αX_1.

Theorem 3.3. *Let ψ be a proper slant submersion from an almost paracontact manifold N with the structure $(\varphi, \xi, \eta, g_N)$ onto a semi-Riemannian manifold (B, g_B). Then,*
(i) ψ is slant submersion of type 1 if and only if $\alpha^2 X_1 = \cosh^2\omega(X_1 - \eta(X_1)\xi)$ for every spacelike vector field $X_1 \in \Gamma(ker\psi_*)$.
(ii) ψ is slant submersion of type 2 if and only if $\alpha^2 X_1 = \cos^2\omega(X_1 - \eta(X_1)\xi)$ for every spacelike vector field $X_1 \in \Gamma(ker\psi_*)$.

Proof. (i) For every timelike vector field $X_2 \in \Gamma(ker\psi_*)$, there exists a spacelike vector field $X_1 \in \Gamma(ker\psi_*)$ such as $\alpha X_1 = X_2$. Then:

$$\alpha^2 X_2 = \alpha^2\alpha X_1 = \alpha\alpha^2 X_1 = \cosh^2\omega(\alpha X_1 - \eta(\alpha X_1)\xi) = \cosh^2\omega(X_2 - \eta(X_2)\xi).$$

The same proof is valid for (ii), but $\alpha^2 X_1 = \cos^2\omega(X_1 - \eta(X_1)\xi)$. □

Theorem 3.4. *Let ψ be a proper slant submersion from an almost paracontact manifold N with the structure $(\varphi, \xi, \eta, g_N)$ onto a semi-Riemannian manifold (B, g_B). Then ψ is slant submersion of*
type 1 if and only if $t\beta X_1 = -\sinh^2\omega(X_1 - \eta(X_1)\xi)$ for every spacelike (timelike) vertical

vector field $X_1 \in \Gamma(ker\psi_*)$.
type 2 *if and only if* $t\beta X_1 = \sin^2\omega(X_1 - \eta(X_1)\xi)$ *for every spacelike (timelike) vertical vector field* $X_1 \in \Gamma(ker\psi_*)$.
type 3 *if and only if* $t\beta X_1 = \cosh^2\omega(X_1 - \eta(X_1)\xi)$ *for every spacelike (timelike) vertical vector field* $X_1 \in \Gamma(ker\psi_*)$.

Proof. For any vertical vector field $X_1 \in \Gamma(ker\psi_*)$, it holds

$$X_1 - \eta(X_1)\xi = \varphi^2 X_1 = \alpha^2 X_1 + \beta\alpha X_1 + t\beta X_1 + r\beta X_1.$$

Equalizing the vertical and the horizontal parts of the above equation, we obtain:

$$\alpha^2 X_1 + t\beta X_1 = X_1 - \eta(X_1)\xi, \quad \beta\alpha X_1 + r\beta X_1 = 0. \tag{3.9}$$

Hence, for a slant submersion of type 1,

$$t\beta X_1 = X_1 - \eta(X_1)\xi - \alpha^2 X_1 = (1 - \cosh^2\omega)(X_1 - \eta(X_1)\xi) = -\sinh^2\omega(X_1 - \eta(X_1)\xi),$$

while for a slant submersion of type 2,

$$t\beta X_1 = X_1 - \eta(X_1)\xi - \alpha^2 X_1 = (1 - \cos^2\omega)(X_1 - \eta(X_1)\xi) = \sin^2\omega(X_1 - \eta(X_1)\xi),$$

and, for a slant submersion of type 3,

$$t\beta X_1 = X_1 - \eta(X_1)\xi - \alpha^2 X_1 = (1 + \sinh^2\omega)(X_1 - \eta(X_1)\xi) = \cosh^2\omega(X_1 - \eta(X_1)\xi).$$

The converse results are deduced from the same equations. □

Theorem 3.5. *Let* ψ *be a semi-Riemannian submersion from an almost paracontact metric manifold* $(N_{2n}^{4n+1}, \varphi, \eta, \xi, g_N)$ *onto a semi-Riemannian manifold* (B_n^{2n}, g_B). *Then* ψ *is a slant submersion of*
type 1 *if and only if* $r^2 X_2 = \cosh^2\omega X_2$ *for every spacelike (timelike) horizontal vector field* $X_2 \in \Gamma((ker\psi_*)^\perp)$.
type 2 *if and only if* $r^2 X_2 = \cos^2\omega X_2$ *for every spacelike (timelike) horizontal vector field* $X_2 \in \Gamma((ker\psi_*)^\perp)$.

Proof. In the case of a slant submersion of
type 1, for every horizontal timelike (spacelike) vector field $X_2 \in \Gamma((ker\psi_*)^\perp)$, there exists a spacelike (timelike) vertical vector field $X_1 \in \Gamma(ker\psi_*)$ such as $\beta X_1 = X_2$. From (3.9), we obtain

$$r^2 X_2 = r^2\beta X_1 = -r\beta\alpha X_1 = \beta\alpha^2 X_1 = \beta(\cosh^2\omega(X_1 - \eta(X_1)\xi)). \tag{3.10}$$

From (3.10), we get $r^2 X_2 = \cosh^2\omega(\beta X_1 - \eta(\beta X_1)\xi)$. Since $\beta X_1 \perp \xi$, we obtain $\eta(\beta X_1) = 0$ and thus $r^2 X_2 == \cosh^2\omega X_2$.

In the case of a slant submersion of
type 2, in a similar way, we get

$$r^2 X_2 = \cos^2\omega X_2.$$

The converse results follow from the fact that $t((ker\psi_*)^\perp) = (ker\psi_*) \oplus < \xi >$. □

Theorem 3.6. *Let* ψ *be a semi-Riemannian submersion from an almost paracontact metric manifold* $(N_{2n}^{4n+1}, \varphi, \eta, \xi, g_N)$ *onto a semi-Riemannian manifold* $(B_{2j}^{2n}, g_B)(0 < j < n)$. *At that time,* ψ *is a slant submersion of type* 3 *if and only if* $r^2 X_2 = -\sinh^2\omega X_2$ *for every horizontal vector field* $X_2 \in \Gamma((ker\psi_*)^\perp)$.

Proof. If X_1 is a spacelike (timelike) vertical vector field, αX_1 is also spacelike (timelike) and βX_1 is timelike (spacelike). Therefore, given that the dimension of B is half the dimension of N, if X_2 is a timelike (spacelike) horizontal vector field, then there exists a vertical vector field $X_1 \in \Gamma(ker\psi_*)$ such that $\beta X_1 = X_2$. Then, from (3.10) we have $r^2 X_2 = \beta(-\sinh^2\omega(X_1 - \eta(X_1)\xi)) = -\sinh^2\omega X_2$. The converse results follow from the fact that $t((ker\psi_*)^\perp) = (ker\psi_*) \oplus < \xi >$. □

From Theorem 3.2, (2.11) and (3.1) we obtain the following result.

Lemma 3.7. *Let ψ be a semi-Riemannian submersion from an almost paracontact metric manifold (N,φ,η,ξ,g_N) onto a semi-Riemannian manifold (B,g_B).*
If ψ is a proper slant submersion of type 1, then, for any spacelike (timelike) vector fields $X_1,X_2\in\Gamma(ker\psi_*)$, we have

$$g_N(\alpha X_1,\alpha X_2) = \cosh^2\omega(-g_N(X_1,X_2)+\eta(X_1)\eta(X_2)) \tag{3.11}$$
$$g_N(\beta X_1,\beta X_2) = -\sinh^2\omega(-g_N(X_1,X_2)+\eta(X_1)\eta(X_2)) \tag{3.12}$$

If ψ is a proper slant submersion of type 2, then, for any spacelike (timelike) vector fields $X_1,X_2\in\Gamma(ker\psi_*)$, we have

$$g_N(\alpha X_1,\alpha X_2) = \cos^2\omega(-g_N(X_1,X_2)+\eta(X_1)\eta(X_2)) \tag{3.13}$$
$$g_N(\beta X_1,\beta X_2) = \sin^2\omega(-g_N(X_1,X_2)+\eta(X_1)\eta(X_2)). \tag{3.14}$$

If ψ is a proper slant submersion of type 3, then, for any spacelike (timelike) vector fields $X_1,X_2\in\Gamma(ker\psi_*)$, we have

$$g_N(\alpha X_1,\alpha X_2) = -\sinh^2\omega(-g_N(X_1,X_2)+\eta(X_1)\eta(X_2)) \tag{3.15}$$
$$g_N(\beta X_1,\beta X_2) = \cosh^2\omega(-g_N(X_1,X_2)+\eta(X_1)\eta(X_2)). \tag{3.16}$$

Note that given a semi-Euclidean space R^{2n+1}_n with coordinates $(x_1,...,x_{2n},z)$ on R^{2n+1}_n, we can naturally choose an almost paracontact structure (φ,ξ,η) on R^{2n+1}_n as follows:

$$\eta=dz,\;\; \xi=\frac{\partial}{\partial z},\;\; \varphi(\frac{\partial}{\partial x_{2i}})=\frac{\partial}{\partial x_{2i-1}},\;\; \varphi(\frac{\partial}{\partial x_{2i-1}})=\frac{\partial}{\partial x_{2i}},\;\; \varphi(\xi)=0$$

where $i=1,...,n$. Let R^{2n+1}_n be a semi-Euclidean space of signature (+,-,+,-,...,+) with respect to the canonical basis $(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_{2n}},\frac{\partial}{\partial z})$.
Now, we can present four examples of proper slant submersions.

Example 3.8. Determine a map $\psi:R^5_2\to R^2_1$ by

$$\psi(x_1,x_2,x_3,x_4,z)=(\frac{x_1-x_3}{\sqrt{2}},x_2).$$

At that time, by direct calculations we obtain

$$ker\psi_*=span\{U_1=\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3},\; U_2=\frac{\partial}{\partial x_4},\; U_3=\xi=\frac{\partial}{\partial z}\}$$

and

$$(ker\psi_*)^\perp=span\{X_1=\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_3},\; X_2=\frac{\partial}{\partial x_2}\}.$$

Thus, the map ψ is a slant submersion of type 2 with the slant angle ω with $\cos^{-1}(\frac{1}{\sqrt{2}})$.

Example 3.9. Define a map $\psi:R^5_2\to R^2_1$ by

$$\psi(x_1,x_2,x_3,x_4,z)=(x_2\sinh x+x_3\cosh x, x_1\sinh y+x_4\cosh y),$$

any for $x,y\in R$. Then, by direct calculations we get

$$ker\psi_*=span\{U_1=\cosh x\frac{\partial}{\partial x_2}-\sinh x\frac{\partial}{\partial x_3},\; U_2=\cosh y\frac{\partial}{\partial x_1}-\sinh y\frac{\partial}{\partial x_4},\; U_3=\xi=\frac{\partial}{\partial z}\}$$

and

$$(ker\psi_*)^\perp=span\{X_1=-\sinh x\frac{\partial}{\partial x_2}+\cosh x\frac{\partial}{\partial x_3},\; X_2=\sinh y\frac{\partial}{\partial x_1}-\cosh y\frac{\partial}{\partial x_4}\}.$$

Thus,the map ψ is a slant submersion of type 1 with the slant angle $\cosh\omega=\cosh(x-y)$.

Example 3.10. Define a map $\psi : R^5_2 \to R^2_1$ by

$$\psi(x_1, x_2, x_3, x_4, z) = (x_1 \sin x + x_3 \cos x, x_2 \sin y + x_4 \cos y),$$

for any $x, y \in R$. The map ψ is a slant submersion of type 2 with the slant angle $\cos\omega = \cos(x - y)$.

Example 3.11. Define a map $\psi : R^5_2 \to R^2_1$ by

$$\psi(x_1, x_2, x_3, x_4, z) = (x_2 \cosh x + x_3 \sinh x, x_4),$$

for any $x \in R^+$. The map ψ is a slant submersion of type 3 with the slant angle $\alpha^2 = -\sinh^2 x$.

Let ψ be proper slant submersions of type 1,2 and 3 from a paracosymplectic manifold N with the structure $(g_N, \varphi, \eta, \xi)$ onto a semi-Riemannian manifold (B, g_B). From (2.11),(3.1) and (3.2), one can simply see that

$$g_N(X_1, \alpha X2) = -g_N(\alpha X_1, X_2) \tag{3.17}$$

and

$$g_N(\beta X_1, X_3) = -g_N(X_1, tX_3), \tag{3.18}$$

for spacelike (timelike) vector fields $X_1, X_2 \in \Gamma(ker\psi_*)$, $X_3 \in \Gamma((ker\psi_*)^\perp)$.

Using (2.3),(2.5) and (2.13) we have

$$\mathcal{T}_{X_1}\xi = 0,\ \mathcal{A}_{X_3}\xi = 0 \tag{3.19}$$

for spacelike (timelike) vector fields $X_1 \in \Gamma(ker\psi_*), X_3 \in \Gamma(ker\psi_*)^\perp$.

We determine the covariant derivatives of α and β as follows

$$(\nabla_{X_1}\alpha)X_2 = \hat{\nabla}_{X_1}\alpha X_2 - \alpha\hat{\nabla}_{X_1}X_2 \tag{3.20}$$

and

$$(\nabla_{X_1}\beta)Y = h\nabla_{X_1}\beta X_2 - \beta\hat{\nabla}_{X_1}X_2 \tag{3.21}$$

for spacelike (timelike) vector fields $X_1, X_2 \in \Gamma(kerF_*)$, where $\hat{\nabla}_{X_1}X_2 = v\nabla_{X_1}X_2$. Then we easily have

Lemma 3.12. *Let $(N, g_N, \varphi, \eta, \xi)$ be a paracosymplectic manifold and (B, g_B) a semi-Riemannian manifold. Let $\psi : N \to B$ be proper slant submersions of type* 1, 2 *and* 3. *Then, we have*

$$\begin{aligned} \hat{\nabla}_{X_1}\alpha X_2 + \mathcal{T}_{X_1}\beta X_2 &= \alpha\hat{\nabla}_{X_1}X_2 + t\mathcal{T}_{X_1}X_2 \\ \mathcal{T}_{X_1}\alpha X_2 + h\nabla_{X_1}\beta X_2 &= \beta\hat{\nabla}_{X_1}X_2 + r\mathcal{T}_{X_1}X_2 \end{aligned}$$

for any spacelike(timelike)vector fields $X_1, X_2 \in \Gamma(ker\psi_*)$.

We say that β is parallel with respect to the Riemannian connection ∇ on $(ker\psi_*)$ if its covariant derivative with respect to ∇ vanishes, i.e., we get

$$(\nabla_{X_1}\beta)X_2 = h\nabla_{X_1}\beta X_2 - \beta\hat{\nabla}_{X_1}X_2 = 0 \tag{3.22}$$

for any spacelike (timelike) vertical vector fields $X_1, X_2 \in \Gamma(ker\psi_*)$.

Theorem 3.13. *Let ψ be a proper slant submersions of type* $1, 2$ *and* 3 *from a paracosymplectic manifold $(N, g_N, \varphi, \eta, \xi)$ onto a semi-Riemannian manifold (B, g_B). At that time, the fibres are not proper totally umbilical.*

Proof. See [19]. □

We now indicate the orthogonal complementary distribution to $\beta(ker\psi_*)$ in $(ker\psi_*)^\perp$ by τ. At that time, we obtain the following.

Theorem 3.14. *Let $\psi : (N, g_N, \varphi, \eta, \xi) \to (B, g_B)$ be a proper slant submersions of type of $1, 2$ and 3. If N is a paracosymplectic manifold, then τ is an invariant distribution of $(ker\psi_*)^\perp$, with respect to φ.*

Proof. For $X_2 \in \Gamma(\tau)$, using (2.11) and (3.1), we get

$$\begin{aligned} g_N(\varphi X_2, \beta X_1) &= -g_N(X_2, X_1) - g_N(\varphi X_2, \alpha X_1) + \eta(X_1)\eta(X_2) \\ &= -g_N(\varphi X_2, \alpha X_1) \\ &= g_N(X_2, \varphi\alpha X_1) = 0 \end{aligned}$$

for any spacelike (timelike) vector field $X_1 \in \Gamma(ker\psi_*)$.
In a similar way, we have $g_N(\varphi X_2, X_3) = -g_N(X_2, \varphi X_3) = 0$ due to $\varphi X_3 \in \Gamma((ker\psi_*) \oplus \beta(ker\psi_*))$ for any spacelike (timelike) vector field $X_2 \in \Gamma(\tau)$ and $X_3 \in \Gamma(ker\psi_*)$. Hence, proof is complete. □

Corollary 3.15. *Let $\psi : (N^{4n+1}_{2n}, g_N, \varphi, \eta, \xi) \to (B^{2n}_{2j}, g_B)(0 < j < n)$ be a proper slant submersion of type 3. If N is a paracosymplectic manifold and $\{E_1, ..., E_{2n}, \xi\}$ is a local orthonormal basis of $(ker\psi_*)$, at that time $\{\frac{1}{\cosh\omega}\beta E_1, ..., \frac{1}{\cosh\omega}\beta E_{2n}\}$ is a local orthonormal basis of $\beta(ker\psi_*)$.*

Proof. It will be enough to demonstrate that $g_N(\frac{1}{\cosh\omega}\beta E_i, \frac{1}{\cosh\omega}\beta E_j) = \epsilon_i\delta_{ij}$, for any $i, j \in \{1, ..., n\}$, where $\epsilon_i = sgn(g_N(E_1, E_1)) = \pm 1$. By using (3.16), we get

$$g_N(\frac{1}{\cosh\omega}\beta E_i, \frac{1}{\cosh\omega}\beta E_j) = (\frac{1}{\cosh\omega})^2 \cosh^2\omega g_N(E_i, E_j) = \epsilon_i\delta_{ij},$$

which proves the assertion. □

In a similar way, we get the following.

Lemma 3.16. *Let $\psi : (N^{4n+1}_{2n}, g_N, \varphi, \eta, \xi) \to (B^{2n}_{2j}, g_B)(0 < j < n)$ be a proper slant submersion of type 3. If N is a paracosymplectic manifold and $E_1, ..., E_n, \xi$ are orthogonal unit vector fields in $(ker\psi_*)$, then*

$$\{E_1, \frac{1}{\sinh\omega}\alpha E_1, E_2, \frac{1}{\sinh\omega}\alpha E_2, ..., E_n, \frac{1}{\sinh\omega}\alpha E_n, \xi\}$$

is a local orthonormal basis of $(ker\psi_*)$.

Let ψ be a proper slant submersion of type 3 from a paracosymplectic manifold $(N^{4n+1}, g_N, \varphi, \eta, \xi)$ onto a semi-Riemannian manifold (B^{2n}, g_B). We call such an orthonormal frame

$$\{E_1, \frac{1}{\sinh\omega}\alpha E_1, E_2, \frac{1}{\sinh\omega}\alpha E_2, ..., E_{2n}, \frac{1}{\sinh\omega}\alpha E_n, \frac{1}{\cosh\omega}\beta E_1, ..., \frac{1}{\cosh\omega}\beta E_{2n}\}$$

an adapted slant frame for proper slant submersion of type 3.

Proposition 3.17. *Let $\psi : (N, g_N, \varphi, \eta, \xi) \to (B, g_B)$ be a proper slant submersion of type 1. If N is a paracosymplectic manifold and β is parallel with respect to ∇ on $(ker\psi_*)$, then we have*

$$\mathcal{T}_{\alpha X_1}\alpha X_1 = \cosh^2\omega \mathcal{T}_{X_1} X_1 \tag{3.23}$$

for any spacelike (timelike) vector field $X_1 \in \Gamma(ker\psi_*)$.

Proof. If β is parallel, at that time from Lemma 3.12 we get $r\mathcal{T}_{X_1}X_2 = \mathcal{T}_{X_1}\alpha X_2$ for any spacelike (timelike) vector fields $X_1, X_2 \in \Gamma(ker\psi_*)$. Interchanging the role of X_1 and X_2, we get $r\mathcal{T}_{X_2}X_1 = \mathcal{T}_{X_2}\alpha X_1$. Thus we have

$$r\mathcal{T}_{X_1}X_2 - r\mathcal{T}_{X_2}X_1 = \mathcal{T}_{X_1}\alpha X_2 - \mathcal{T}_{X_2}\alpha X_1$$

Since $\mathcal{T}$ is symmetric, we derive $\mathcal{T}_{X_1}\alpha X_2 = \mathcal{T}_{X_2}\alpha X_1$. Then substituting X_2 by αX_1 we get $\mathcal{T}_{X_1}\alpha^2 X_1 = \mathcal{T}_{\alpha X_1}\alpha X_1$. Using (3.3) and (3.19) we obtain (3.23). □

In a similar way, we obtain the following.

Corollary 3.18. *Let $\psi : (N, g_N, \varphi, \eta, \xi) \to (B, g_B)$ be a proper slant submersion of type 2. If N is a paracosymplectic manifold and β is parallel with respect to ∇ on $(ker\psi_*)$, then we have*

$$\mathcal{T}_{\alpha X_1}\alpha X_1 = \cos^2\omega \mathcal{T}_{X_1}X_1 \tag{3.24}$$

for any spacelike (timelike) vector field $X_1 \in \Gamma(ker\psi_*)$.

Corollary 3.19. *Let $\psi : (N, g_N, \varphi, \eta, \xi) \to (B, g_B)$ be a proper slant submersion of type 3. If N is a paracosymplectic manifold and β is parallel with respect to ∇ on $(ker\psi_*)$, then we have*

$$\mathcal{T}_{\alpha X_1}\alpha X_1 = -\sinh^2\omega \mathcal{T}_{X_1}X_1 \tag{3.25}$$

for any spacelike (timelike) vector field $X_1 \in \Gamma(ker\psi_*)$.

Theorem 3.20. *Let $\psi : (N^{4n+1}_{2n}, g_N, \varphi, \eta, \xi) \to (B^{2n}_{2j}, g_B)(0 < j < n)$ be a proper slant submersion of type 3. If N is a paracosymplectic manifold and β is parallel with respect to ∇ on $(ker\psi_*)$, at that time ψ is a harmonic map.*

Proof. Using (2.9), we obtain

$$(\nabla\psi_*)(X_3, X_4) = 0$$

for any spacelike (timelike) vector fields $X_3, X_4 \in (ker\psi_*)^\perp$. A proper slant submersion of type 3 ψ is harmonic if and only if $\sum_{i=1}^{2n}(\nabla\psi_*)(E_i^*, E_i^*) = \sum_{i=1}^{2n}(\nabla\psi_*)(\mathcal{T}_{E_i^*}E_i^*) = 0$, here $\{E_i^*\}_{i=1}^{2n}$ is an orthonormal basis of $(ker\psi_*)$. Hence, using Lemma 3.16 we should write

$$\kappa = -\sum_{i=1}^{n}\psi_*(\mathcal{T}_{E_i}E_i + \mathcal{T}_{\frac{1}{\sinh\omega}\alpha E_i}\frac{1}{\sinh\omega}\alpha E_i) - \mathcal{T}_\xi\xi.$$

From (3.19), we have

$$\kappa = -\sum_{i=1}^{n}\psi_*(\mathcal{T}_{E_i}E_i + \frac{1}{\sinh^2\varphi}\mathcal{T}_{\alpha E_i}\alpha E_i).$$

Then using (3.25) we have

$$\kappa = -\sum_{i=1}^{n}\psi_*(\mathcal{T}_{E_i}E_i - \mathcal{T}_{E_i}E_i) = 0$$

which shows that ψ is harmonic. □

Putting $\theta = \alpha^2$, we define $\nabla\theta$ by

$$(\nabla_{X_1}\theta)X_2 = v\nabla_{X_1}\theta X_2 - \theta\hat{\nabla}_{X_1}X_2 \tag{3.26}$$

for any spacelike(timelike) vector fields $X_1, X_2 \in \Gamma(ker\psi_*)$.

Theorem 3.21. *Let $\psi : (N, g_N, \varphi, \eta, \xi) \to (B, g_B)$ be a proper slant submersion of type 1. If N is a paracosymplectic manifold, then $\nabla\theta = 0$.*

Proof. Using (3.3),we have

$$\theta\hat{\nabla}_{X_1}X_2 = \cosh^2\omega(\hat{\nabla}_{X_1}X_2 - \eta(\hat{\nabla}_{X_1}X_2)\xi) \tag{3.27}$$

for all spacelike(timelike) vector fields $X_1, X_2 \in \Gamma(ker\psi_*)$. On the other hand, from (3.3) and (2.13) we obtain

$$\begin{aligned} v(\hat{\nabla}_{X_1}\theta X_2) &= \cosh^2\omega(\hat{\nabla}_{X_1}X_2 - (\hat{\nabla}_{X_1}\eta(X_2))\xi) \\ &= \cosh^2\omega(\hat{\nabla}_{X_1}X_2 - \eta(\hat{\nabla}_{X_1}X_2) - g_N(X_2, \nabla_{X_1}\xi)) \\ &= \cosh^2\omega(\hat{\nabla}_{X_1}X_2 - \eta(\hat{\nabla}_{X_1}X_2). \end{aligned} \tag{3.28}$$

Using (3.27) and (3.28), we get $(\nabla_{X_1}\theta)X_2 = 0$. □

Now, we examine the geometry of the leaves of the distributions $(ker\psi_*)$ and $(ker\psi_*)^\perp$.

Theorem 3.22. *Let $\psi : (N, g_N, \varphi, \eta, \xi) \to (B, g_B)$ be a proper slant submersion of type 1. If N is a paracosymplectic manifold, then the distribution $(ker\psi_*)$ defines a totally geodesic foliation on N if and only if*

$$g_N(h\nabla_{X_1}\beta\alpha X_2, X_3) = g_N(h\nabla_{X_1}\beta X_2, rX_3) + g_N(\mathcal{T}_{X_1}\beta X_2, tX_3) \tag{3.29}$$

for spacelike (timelike) vector fields $X_1, X_2 \in \Gamma(ker\psi_*)$ and $X_3 \in \Gamma((ker\psi_*)^\perp)$.

Proof. For spacelike (timelike) vector fields $X_1, X_2 \in \Gamma(ker\psi_*)$ and $X_3 \in \Gamma((ker\psi_*)^\perp)$, since (2.10) and (2.12) we obtain

$$g_N(\nabla_{X_1}X_2, X_3) = g_N(\varphi\nabla_{X_1}\varphi X_2, X_3) + g_N(\nabla_{X_1}X_2, \xi)\eta(X_3)$$

Using (3.1) and (3.2) we get

$$\begin{aligned} g_N(\nabla_{X_1}X_2, X_3) &= g_N(\nabla_{X_1}\alpha^2 X_2, X3) + g_N(\nabla_{X_1}\beta\alpha X_2, X_3) \\ &- g_N(\nabla_{X_1}\beta X_2, tX_3) - g_N(\nabla_{X_1}\beta X_2, rX_3). \end{aligned}$$

Then from (2.4),(2.13) and (3.3) we obtain

$$\begin{aligned} g_N(\nabla_{X_1}X_2, X_3) &= \cosh^2\omega g_N(\nabla_{X_1}X_2, X_3) + g_N(h\nabla_{X_1}\beta\alpha X_2, X_3) \\ &- g_N(\mathcal{T}_{X_1}\beta X_2, tX_3) - g_N(h\nabla_{X_1}\beta X_2, rX_3). \end{aligned}$$

Hence we have

$$\begin{aligned} -\sinh^2\omega g_N(\nabla_{X_1}X_2, X_3) &= g_N(h\nabla_{X_1}\beta\alpha X_2, X_3) \\ &- g_N(\mathcal{T}_{X_1}\beta X_2, tX_3) - g_N(h\nabla_{X_1}\beta X_2, rX_3) \end{aligned}$$

which proves assertion. □

Theorem 3.23. *Let $\psi : (N, g_N, \varphi, \eta, \xi) \to (B, g_B)$ be a proper slant submersion of type 1. If N is a paracosymplectic manifold, then the distribution $(ker\psi_*)^\perp$ defines a totally geodesic foliation on N if and only if*

$$g_N(h\nabla_{X_1}X_2, \beta\alpha X_3) = g_N(\mathcal{A}_{X_1}tX_2 + h\nabla_{X_1}rX_2, \beta X_3) \tag{3.30}$$

for any spacelike (timelike)vector fields $X_3 \in \Gamma(ker\psi_*)$ and $X_1, X_2 \in \Gamma((ker\psi_*)^\perp)$.

Proof. For $X_3 \in \Gamma(ker\psi_*)$ and $X_1, X_2 \in \Gamma((ker\psi_*)^\perp)$, from (2.12) and (3.1) we obtain

$$\begin{aligned} g_N(\nabla_{X_1}X_2, X_3) &= -g_N(\nabla_{X_1}\varphi X_2, \varphi X_3) + g_N(\nabla_{X_1}X_2, \xi)\eta(X_3) \\ &= -g_N(\varphi\nabla_{X_1}X_2, \alpha X_3) - g_N(\nabla_{X_1}\varphi X_2, \beta X_3) \\ &+ g_N(\nabla_{X_1}X_2, \xi)\eta(X_3). \end{aligned} \tag{3.31}$$

Using (3.1) in (3.31), we get

$$\begin{aligned} g_N(\nabla_{X_1}X_2, X_3) &= g_N(\nabla_{X_1}X_2, \alpha^2 X_3) + g_N(\nabla_{X_1}X_2, \beta\alpha X_3) \\ &- g_N(\nabla_{X_1}\varphi X_2, \beta X_3) + g_N(\nabla_{X_1}X_2, \xi)\eta(X_3). \end{aligned} \tag{3.32}$$

Using (3.2) and (3.3) we get

$$\begin{aligned} g_N(\nabla_{X_1}X_2, X_3) &= \cosh^2\omega g_N(\nabla_{X_1}X_2, X_3) - \cosh^2\omega\eta(\nabla_{X_1}X_2)\eta(X_3) \\ &+ g_N(\nabla_{X_1}X_2, \beta\alpha X_3) - g_N(\nabla_{X_1}tX_2, \beta X_3) \\ &- g_N(\nabla_{X_1}rX_2, \beta X_3) + g_N(\nabla_{X_1}X_2, \xi)\eta(X_3). \end{aligned} \tag{3.33}$$

Using (2.5),(2.6) and (2.13) in (3.33), we get

$$-\sinh^2\omega g_N(\nabla_{X_1}X_2, X_3) = g_N(h\nabla_{X_1}X_2, \beta\alpha X_3) - g_N(\mathcal{A}_{X_1}tX_2 + h\nabla_{X_1}rX_2, \beta X_3).$$

Thus, we have (3.30). □

Now, we show necessary and sufficient conditions for a proper slant submersion of type 1 to be totally geodesic. Recall that a smooth map ψ between (semi-) Riemannian manifolds (N, g_N) and (B, g_B) is called a totally geodesic map if $(\nabla\psi_*)(X_1, X_2) = 0$ for all $X_1, X_2 \in \Gamma(TN)$.

Theorem 3.24. *Let $\psi : (N, g_N, \varphi, \eta, \xi) \to (B, g_B)$ be a proper slant submersion of type* 1. *If N is a paracosymplectic manifold, at that time ψ is totally geodesic if and only if*

$$g_N(h\nabla_{X_1}\beta\alpha X_2, X_3) = g_N(h\nabla_{X_1}\beta X_2, rX_3) + g_N(\mathcal{T}_{X_1}\beta X_2, tX_3) \tag{3.34}$$

and

$$g_N(h\nabla_{X_4}\beta\alpha X_1, X_5) = -g_N(\mathcal{A}_{X_4}tX_5 + h\nabla_{X_4}rX_5, \beta X_1) \tag{3.35}$$

for any spacelike (timelike) vector fields $X_3, X_4, X_5 \in \Gamma((ker\psi_*)^\perp)$ and $X_1, X_2 \in \Gamma(ker\psi_*)$.

Proof. First of all, since ψ is a semi-Riemannian submersion we get

$$(\nabla F_*)(X_4, X_5) = 0$$

for sapacelike (timelike) vector fields $X_4, X_4 \in \Gamma((ker\psi_*)^\perp)$.
For sapacelike (timelike) vector fields $X_1, X_2 \in \Gamma(ker\psi_*)$ and $X_3, X_4, X_5 \in \Gamma((ker\psi_*)^\perp)$, from (2.10) and (2.12) we have

$$\nabla_{X_1}X_2 = \varphi\nabla_{X_1}\varphi X_2 + \eta(\nabla_{X_1}X_2)\xi. \tag{3.36}$$

Using (2.9),(3.1) and (3.36) we get

$$g_B((\nabla\psi_*)(X_1, X_2), \psi_*X_3) = -g_N(\nabla_{X_1}\varphi\alpha X_2, X_3) + g_N(\nabla_{X_1}\beta X_2, \varphi X_3).$$

Using (3.1) and (3.2) we get

$$\begin{aligned} g_B((\nabla\psi_*)(X_1, X_2), \psi_*X_3) &= -g_N(\nabla_{X_1}\alpha^2 X_2, X_3) - g_N(\nabla_{X_1}\beta\alpha X_2, X_3) \\ &+ g_N(\nabla_{X_1}\beta X_2, tX_3) + g_N(\nabla_{X_1}\beta X_2, rX_3). \end{aligned}$$

Using (2.3), (2.4) and (3.3) we have

$$\begin{aligned} g_B((\nabla\psi_*)(X_1, X_2), \psi_*X_3) &= -\cosh^2\omega g_N(\nabla_{X_1}X_2, X_3) - g_N(h\nabla_{X_1}\beta\alpha X_2, X_3) \\ &+ g_N(\mathcal{T}_{X_1}\beta X_2, tX_3) + g_N(h\nabla_{X_1}\beta X_2, rX_3). \end{aligned}$$

Hence we obtain

$$\begin{aligned} -\sinh^2\omega g_B((\nabla\psi_*)(X_1, X_2), \psi_*X_3) &= -g_N(h\nabla_{X_1}\beta\alpha X_2, X_3) + g_N(\mathcal{T}_{X_1}\beta X_2, tX_3) \\ &+ g_N(h\nabla_{X_1}\beta X_2, rX_3). \end{aligned} \tag{3.37}$$

Similarly, we get

$$\begin{aligned} -\sinh^2\omega g_B((\nabla\psi_*)(X_1, X_4), \psi_*X_5) &= -g_N(\mathcal{A}_{X_4}tX_5 + h\nabla_{X_4}rX_5, \beta X_1) \\ &- g_N(h\nabla_{X_4}\beta\alpha X_1, X_5). \end{aligned} \tag{3.38}$$

Thus from (3.37) and (3.38), we get (3.34) and (3.35). □

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References

[1] M.A. Akyol, *Conformal semi-slant submersions*, Int. J. Geom. Methods Mod. Phys. **14** (7), 2017.

[2] M.A. Akyol, *Conformal anti-invariant submersions from cosymplectic manifolds,* Hacet. J. Math. Stat. **46** (2), 177-192, 2017.

[3] G. Baditoiu and S. Ianus, *Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces,* Diff. Geom. and appl. **16**, 79-84, 2002.

[4] J.P. Bourguignon and H.B. Lawson, *A mathematician's visit to Kaluza- Klein theory,* Rend. Sem. Mat. Univ. Politec. Torino, Special Issue, 143-163, 1989.

[5] J.P. Bourguignon and H.B. Lawson, Stability and isolation phenomena for Yang-Mills fields, Comm. Math. Phys. **79**, 189-230, 1981.

[6] A.V. Caldarella, *On para-quaternionic submersions between para-quaternionic Kähler manifolds,* Acta Applicandae Mathematicae **112**, 1-14, 2010.

[7] P. Dacko, *On almost para-cosymplectic manifolds,* Tsukuba J. Math. **28** (1), 193-213, 2004.
[8] M. Falcitelli, S. Ianus and A. M. Pastore, *Riemannian Submersions and Related Topics,* World Scientific, 2004.
[9] Y. Gündüzalp and B. Şahin, *Paracontact semi-Riemannian submersions,* Turk. J.Math. **37** (1), 114-128, 2013.
[10] Y. Gündüzalp, B. Şahin, *Para-contact para-complex semi-Riemannian submersions,* Bull. Malays. Math. Sci. Soc. **37** (1), 139-152, 2014.
[11] Y. Gündüzalp, *Anti-invariant semi-Riemannian submersions from almost para-Hermitian manifolds,* J. Funct. Spaces **2013**, ID 720623, 2013.
[12] Y. Gündüzalp, *Slant submersions from almost product Riemannian manifolds,* Turk. J. Math. **37**, 863-873, 2013.
[13] Y. Gündüzalp and M.A. Akyol, *Conformal slant submersions from cosymplectic manifolds*, Turk. J. Math. **42** (5), 2672-2689, 2018.
[14] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions,* J. Math. Mech. **16**, 715-737, 1967.
[15] S. Ianus and M. Visinescu, *Kaluza-Klein theory with scalar fields and generalised Hopf manifolds*, Classical Quantum Gravity **4**, 1317-1352, 1987.
[16] S. Ianus, and M. Visinescu, *Space-time compactification and Riemannian submersions,* The mathematical heritage of C.F. Gauss, World Sci. Publ., River Edge, NJ, 358-371, 1991.
[17] S. Ianus, R. Mazzocco and G. E. Vilcu, *Riemannian submersions from quaternionic manifolds,* Acta Appl. Math. **104**, 83-89, 2008.
[18] S. Ianus, G.E. Vilcu and R.C. Voicu, *Harmonic maps and Riemannian submersions between manifolds endowed with special structures,* Banach Center Publications **93**, 277-288, 2011.
[19] I.K. Erken and C. Murathan, *On slant Riemannian submersions for cosymplectic manifolds,* Bull. Korean Math. Soc. **51** (6), 1749-1771, 2014.
[20] B. O'Neill, *The fundamental equations of a submersion,* Michigan Math. J. **13**, 459-469, 1966.
[21] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity,* Academic Press, New York, 1983.
[22] K.S.Park, *H-slant submersions,* Bull. Korean Math. Soc. **49**, 329-338, 2012.
[23] B. Şahin, *Slant submersions from almost Hermitian manifolds,* Bull. Math. Soc.Sci. Math. Roumanie Tome. **54**, 93-105, 2011.
[24] B. Şahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds,* Central European J.Math **8** (3), 437-447, 2010.
[25] H.M. Taştan , B. Şahin and Ş. Yanan , *Hemi-slant submersions,* Mediterr. J. Math. **13**, 2171-2184, 2016.
[26] J. Welyczko, *On Legendre curves in 3-dimensional normal almost paracontact metric manifolds,* Results Math. **54**, 377-387, 2009.
[27] B. Watson, *Almost Hermitian submersions,* J. Differential Geom. **11**, 147-165, 1976.
[28] S. Zamkovoy, *Canonical connections on paracontact manifolds,* Ann. Global Anal. Geometry **36**, 37-60, 2009.