Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (2) (2020), 727 – 739
DOI : 10.15672/hujms.621536

Research Article

On total mean curvatures of foliated half-lightlike submanifolds in semi-Riemannian manifolds

Fortuné Massamba*, Samuel Ssekajja

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209 South Africa

Abstract

We derive total mean curvature integration formulas of a three co-dimensional foliation $\mathcal{F}^n$ on a screen integrable half-lightlike submanifold, M^{n+1} in a semi-Riemannian manifold $\overline{M}^{n+3}$. We give generalized differential equations relating to mean curvatures of a totally umbilical half-lightlike submanifold admitting a totally umbilical screen distribution, and show that they are generalizations of those given by [K. L. Duggal and B. Sahin, Differential geometry of lightlike submanifolds, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2010].

Mathematics Subject Classification (2010). 53C25, 53C40, 53C50

Keywords. half-lightlike submanifold, Newton transformation, foliation and mean curvature

1. Introduction

The rapidly growing importance of lightlike submanifolds in semi-Riemannian geometry, particularly Lorentzian geometry, and their applications to mathematical physics–like in general relativity and electromagnetism motivated the study of lightlike geometry in semi-Riemannian manifolds. More precisely, lightlike submanifolds have been shown to represent different black hole horizons (see [4] and [6] for details). Among other motivations for investing in lightlike geometry by many physicists is the idea that the universe we are living in can be viewed as a 4-dimensional hypersurface embedded in $(4+m)$-dimensional spacetime manifold, where m is any arbitrary integer. There are significant differences between lightlike geometry and Riemannian geometry as shown in [4] and [6], and many more references therein. Some of the pioneering work on this topic is due to Duggal-Bejancu [4], Duggal-Sahin [6] and Kupeli [7]. It is upon those books that many other researchers, including but not limited to [3, 5, 8–11], have extended their theories.

Lightlike geometry rests on a number of operators, like shape and algebraic invariants derived from them, such as trace, determinants, and in general the r-th mean curvature S_r. There is a great deal of work so far on the case $r = 1$ (see some in [4, 6] and many more) and as far as we know, very little has been done for the case $r > 1$. This is partly due to the non-linearity of S_r for $r > 1$, and hence very complicated to study. A great

*Corresponding Author.
Email addresses: massfort@yahoo.fr and Massamba@ukzn.ac.za (F. Massamba), ssekajja.samuel.buwaga@aims-senegal.org (S. Ssekajja)
Received: 07.03.2017; Accepted: 18.03.2019

deal of research on higher order mean curvatures S_r in Riemannian geometry has been done with numerous applications, for instance see [2] and [1]. This gap has motivated our introduction of lightlike geometry of S_r for $r > 1$. In this paper we have considered a half-lightlike submanifold admitting an integrable screen distribution, of a semi-Riemannian manifold. On it we have focused on a codimension 3 foliation of its screen distribution and thus derived integral formulas of its total mean curvatures (see Theorems 4.9 and 4.10). Furthermore, we have considered totally umbilical half-lightlike submanifolds, with a totally umbilical screen distribution and generalized Theorem 4.3.7 of [6] (see Theorem 5.2 and its Corollaries). The paper is organized as follows; In Section 2 we summarize the basic notions on lightlike geometry necessary for other sections. In Section 3 we give some basic information on Newton transformations of a foliation $\mathcal{F}$ of the screen distribution. Section 4 focuses on integration formulae of $\mathcal{F}$ and their consequences. In Section 5 we discus screen umbilical half-lightlike submanifolds and generalizations of some well-known results of [6].

2. Preliminaries

Let (M^{n+1}, g) be a two-co-dimensional submanifold of a semi-Riemannian manifold $(\overline{M}^{n+3}, \overline{g})$, where $g = \overline{g}|_{TM}$. The submanifold (M^{n+1}, g) is called a *half-lightlike* if the radical distribution $\mathrm{Rad}\,TM = TM \cap TM^{\perp}$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$ of M , with rank one. Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of $\mathrm{Rad}\,TM$ in TM, and also choose a *screen transversal bundle* $S(TM^{\perp})$, which is semi-Riemannian and complementary to $\mathrm{Rad}\,TM$ in $TM^{\perp}$. Then,

$$TM = \mathrm{Rad}\,TM \perp S(TM), \;\; TM^{\perp} = \mathrm{Rad}\,TM \perp S(TM^{\perp}). \tag{2.1}$$

We will denote by $\Gamma(\Xi)$ the set of smooth sections of the vector bundle Ξ. It is well-known from [4] and [6] that for any null section E of $\mathrm{Rad}\,TM$, there exists a unique null section N of the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that $g(E, N) = 1$, it follows that there exists a lightlike *transversal vector bundle* $l\mathrm{tr}(TM)$ locally spanned by N. Let $W \in \Gamma(S(TM^{\perp}))$ be a unit vector field, then $\overline{g}(N, N) = \overline{g}(N, Z) = \overline{g}(N, W) = 0$, for any $Z \in \Gamma(S(TM))$.

Let $\mathrm{tr}(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}$. Then we have the following decompositions of $\mathrm{tr}(TM)$ and $T\overline{M}$

$$\mathrm{tr}(TM) = l\mathrm{tr}(TM) \perp S(TM^{\perp}), \tag{2.2}$$

$$T\overline{M} = S(TM) \perp S(TM^{\perp}) \perp \{\mathrm{Rad}\,TM \oplus l\mathrm{tr}(TM)\}. \tag{2.3}$$

It is important to note that the distribution $S(TM)$ is not unique, and is canonically isomorphic to the factor vector bundle $TM/\mathrm{Rad}\,TM$ [4]. Let P be the projection of TM on to $S(TM)$. Then the local Gauss-Weingarten equations of M are the following;

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)W, \tag{2.4}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)W, \tag{2.5}$$

$$\overline{\nabla}_X W = -A_W X + \phi(X)N, \tag{2.6}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E, \tag{2.7}$$

$$\nabla_X E = -A^*_E X - \tau(X)E, \tag{2.8}$$

for all $E \in \Gamma(\mathrm{Rad}\,TM)$, $N \in \Gamma(l\mathrm{tr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$, respectively, B and D are called the local second fundamental forms of M, C is the local second fundamental form on $S(TM)$. Furthermore, $\{A_N, A_W\}$ and A^*_E are the shape operators on TM and $S(TM)$ respectively, and τ, ρ, ϕ and δ are differential 1-forms on TM. Notice that ∇^* is a metric connection on $S(TM)$ while ∇ is generally not a metric connection. In fact, ∇ satisfies the following relation

$$(\nabla_X g)(Y,Z) = B(X,Y)\lambda(Z) + B(X,Z)\lambda(Y), \tag{2.9}$$

for all $X,Y,Z \in \Gamma(TM)$, where λ is a 1-form on TM given $\lambda(\cdot) = \overline{g}(\cdot, N)$. It is well-known from [4] and [6] that B and D are independent of the choice of $S(TM)$ and they satisfy

$$B(X,E) = 0, \quad D(X,E) = -\phi(X), \ \ \forall\, X \in \Gamma(TM). \tag{2.10}$$

The local second fundamental forms B, D and C are related to their shape operators by the following equations

$$g(A^*_E X, Y) = B(X,Y), \qquad \overline{g}(A^*_E X, N) = 0, \tag{2.11}$$

$$g(A_W X, Y) = \varepsilon D(X,Y) + \phi(X)\lambda(Y), \tag{2.12}$$

$$g(A_N X, PY) = C(X, PY), \ \ \overline{g}(A_N X, N) = 0, \tag{2.13}$$

$$\overline{g}(A_W X, N) = \varepsilon\rho(X), \ \ \text{where} \ \ \varepsilon = \overline{g}(W,W), \tag{2.14}$$

for all $X, Y \in \Gamma(TM)$. From equations (2.11) we deduce that A^*_E is $S(TM)$-valued, self-adjoint and satisfies $A^*_E E = 0$. Let $\overline{R}$ denote the curvature tensor of $\overline{M}$, then

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)PZ, N) &= g((\nabla_X A_N)Y, PZ) - g((\nabla_Y A_N)X, PZ) \\ &\quad + \tau(Y)C(X,PZ) - \varepsilon\tau(X)C(Y,PZ)\{\rho(Y)D(X,PZ) \\ &\quad - \rho(X)D(Y,PZ)\}, \quad \forall\, X,Y,Z \in \Gamma(TM).\end{aligned} \tag{2.15}$$

A half-lightlike submanifold (M,g) of a semi-Riemannian manifold $\overline{M}$ is said to be totally umbilical [6] if on each coordinate neighborhood $\mathcal{U}$ there exist smooth functions $\mathcal{H}_1$ and $\mathcal{H}_2$ on $l\mathrm{tr}(TM)$ and $S(TM^\perp)$ respect such that

$$B(X,Y) = \mathcal{H}_1 g(X,Y), \quad D(X,Y) = \mathcal{H}_2 g(X,Y), \ \ \forall\, X,Y \in \Gamma(TM). \tag{2.16}$$

Furthermore, when M is totally umbilical then the following relations follows by straightforward calculations

$$A^*_E X = \mathcal{H}_1 PX, \ \ P(A_W X) = \varepsilon\mathcal{H}_2 PX, \ \ D(X,E) = 0, \ \ \rho(E) = 0, \tag{2.17}$$

for all $X, Y \in \Gamma(TM)$.

Next, we suppose that M is a half-lightlike submanifold of $\overline{M}$, with an integrable screen distribution $S(TM)$. Let M' be a leaf of $S(TM)$. Notice that for any screen integrable half-lightlike M, the leaf M' of $S(TM)$ is a co-dimension 3 submanifold of $\overline{M}$ whose normal bundle is $\{\mathrm{Rad}\, TM \oplus l\mathrm{tr}(TM)\} \perp S(TM^\perp)$. Now, using (2.4) and (2.7) we have

$$\overline{\nabla}_X Y = \nabla^*_X Y + C(X,PY)E + B(X,Y)N + D(X,Y)W, \tag{2.18}$$

for all $X, Y \in \Gamma(TM')$. Since $S(TM)$ is integrable, then its leave is semi-Riemannian and hence we have

$$\overline{\nabla}_X Y = \nabla^{*'}_X Y + h'(X,Y), \ \ \forall\, X,Y \in \Gamma(TM'), \tag{2.19}$$

where h' and $\nabla^{*'}$ are second fundamental form and the Levi-Civita connection of M' in $\overline{M}$. From (2.18) and (2.19) we can see that

$$h'(X,Y) = C(X,PY)E + B(X,Y)N + D(X,Y)W, \tag{2.20}$$

for all $X, Y \in \Gamma(TM')$. Since $S(TM)$ is integrable, then it is well-known from [6] that C is symmetric on $S(TM)$ and also A_N is self-adjoint on $S(TM)$ (see Theorem 4.1.2 for details). Thus, h' given by (2.20) is symmetric on TM'.

Let $L \in \Gamma(\{\mathrm{Rad}\, TM \oplus l\mathrm{tr}(TM)\} \perp S(TM^\perp))$, then we can decompose L as

$$L = aE + bN + cW, \tag{2.21}$$

for non-vanishing smooth functions on $\overline{M}$ given by $a = \overline{g}(L, N)$, $b = \overline{g}(L, E)$ and $c = \varepsilon\overline{g}(L, W)$. Suppose that $\overline{g}(L, L) > 0$, then using (2.21) we obtain a unit normal vector $\widehat{W}$ to M' given by

$$\widehat{W} = \frac{1}{\overline{g}(L, L)}(aE + bN + cW) = \frac{1}{\overline{g}(L, L)}L. \tag{2.22}$$

Next we define a (1,1) tensor $\mathcal{A}_{\widehat{W}}$ in terms of the operators A^*_E, A_N and A_W by

$$\mathcal{A}_{\widehat{W}}X = \frac{1}{\overline{g}(L, L)}(aA^*_E X + bA_N X + cA_W X), \tag{2.23}$$

for all $X \in \Gamma(TM)$. Notice that $\mathcal{A}_{\widehat{W}}$ is self-adjoint on $S(TM)$. Applying $\overline{\nabla}_X$ to $\widehat{W}$ and using equations (2.23) (2.4) and (2.11)-(2.13), we have

$$g(\mathcal{A}_{\widehat{W}}X, PY) = -\overline{g}(\overline{\nabla}_X \widehat{W}, PY), \ \ \forall\, X, Y \in \Gamma(TM). \tag{2.24}$$

Let $\nabla^{*\perp}$ be the connection on the normal bundle $\{\mathrm{Rad}\, TM \oplus l\mathrm{tr}(TM)\} \perp S(TM^\perp)$. Then from (2.24) we have

$$\overline{\nabla}_X \widehat{W} = -\mathcal{A}_{\widehat{W}}X + \nabla^{*\perp}_X \widehat{W}, \ \ \forall\, X \in \Gamma(TM), \tag{2.25}$$

where

$$\begin{aligned}\nabla^{*\perp}_X \widehat{W} = & -\frac{1}{\overline{g}(L, L)}X(\overline{g}(L, L))\widehat{W} + \frac{1}{\overline{g}(L, L)}\left[\{X(a) - a\tau(X)\}E\right.\\ & \left. + \{X(b) + b\tau(X) + c\phi(X)\}N + \{X(c) + aD(X, E) + b\rho(X)\}W\right].\end{aligned}$$

Example 2.1. Let $\overline{M} = (\mathbb{R}^5_1, \overline{g})$ be a semi-Riemannian manifold, where $\overline{g}$ is of signature $(-, +, +, +, +)$ with respect to canonical basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \ \partial x_5)$, where $(x_1, \cdots, x_5)$ are the usual coordinates on $\overline{M}$. Let M be a submanifold of $\overline{M}$ and given parametrically by the following equations

$$\begin{aligned} x_1 =& \varphi_1, \ \ x_2 = \sin\varphi_2 \sin\varphi_3, \ \ x_3 = \varphi_1, \ \ x_4 = \cos\varphi_2 \sin\varphi_3,\\ & x_5 = \cos\varphi_3, \ \ \text{where} \ \ \varphi_2 \in [0, 2\pi] \ \ \text{and} \ \ \varphi_3 \in (0, \pi/2).\end{aligned}$$

Then we have $TM = \mathrm{span}\{E, Z_1, Z_2\}$ and $l\mathrm{tr}(TM) = \mathrm{span}\{N\}$, where

$$\begin{aligned} & E = \partial x_1 + \partial x_3, \ \ Z_1 = \cos\varphi_3 \partial x_2 - \sin\varphi_2 \sin\varphi_3 \partial x_5,\\ & Z_2 = \cos\varphi_3 \partial x_4 - \cos\varphi_2 \sin\varphi_3 \partial x_5 \ \ \text{and} \ \ N = \frac{1}{2}(-\partial x_1 + \partial x_3).\end{aligned}$$

Also, by straightforward calculations, we have

$$W = \sin\varphi_2 \sin\varphi_3 \partial x_2 + \cos\varphi_2 \sin\varphi_3 \partial x_4 + \cos\varphi_3 \partial x_5.$$

Thus, $S(TM^\perp) = \mathrm{span}\{W\}$ and hence M is a half-lightlike submanifold of $\overline{M}$. Furthermore we have $[Z_1, Z_2] = \cos\varphi_2 \sin\varphi_3 \partial x_2 - \sin\varphi_2 \sin\varphi_3 \partial x_4$, which leads to $[Z_1, Z_2] = \cos\varphi_2 \tan\varphi_3 Z_1 - \sin\varphi_2 \tan\varphi_3 Z_2 \in \Gamma(S(TM))$. Thus, M is a screen integrable half-lightlike submanifold of $\overline{M}$. Finally, it is easy to see that A_N is self-adjoint operator on $S(TM)$.

In the next sections we shall consider screen integrable half-lightlike submanifolds of semi-Riemannian manifold $\overline{M}$ and derive special integral formulas for a foliation of $S(TM)$, whose normal vector is $\widehat{W}$ and with shape operator $\mathcal{A}_{\widehat{W}}$.

3. Newton transformations of $\mathcal{A}_{\widehat{W}}$

Let $(\overline{M}^{m+3}, \overline{g})$ be a semi-Riemannian manifold and let (M^{n+1}, g) be a screen integrable half-lightlike submanifold of $\overline{M}$. Then $S(TM)$ admits a foliation and let $\mathcal{F}$ be a such foliation. Then, the leaves of $\mathcal{F}$ are co-dimension three submanifolds of $\overline{M}$, whose normal bundle is $S(TM)^{\perp}$. Let $\widehat{W}$ be unit normal vector to $\mathcal{F}$ such that the orientation of $\overline{M}$ coincides with that given by $\mathcal{F}$ and $\widehat{W}$. The Levi-Civita connection $\overline{\nabla}$ on the tangent bundle of $\overline{M}$ induces a metric connection ∇' on $\mathcal{F}$. Furthermore, h' and $\mathcal{A}_{\widehat{W}}$ are the second fundamental form and shape operator of $\mathcal{F}$. Notice that $\mathcal{A}_{\widehat{W}}$ is self-adjoint on $T\mathcal{F}$ and at each point $p \in \mathcal{F}$ has n real eigenvalues (or principal curvatures) $\kappa_1(p), \cdots, \kappa_n(p)$. Attached to the shape operator $\mathcal{A}_{\widehat{W}}$ are n algebraic invariants

$$S_r = \sigma_r(\kappa_1, \cdots, \kappa_n), \quad 1 \leq r \leq n,$$

where $\sigma_r : M'^n \to \mathbb{R}$ are symmetric functions given by

$$\sigma_r(\kappa_1, \cdots, \kappa_n) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \kappa_{i_1} \cdots \kappa_{i_r}. \tag{3.1}$$

Then, the characteristic polynomial of $\mathcal{A}_{\widehat{W}}$ is given by

$$\det(\mathcal{A}_{\widehat{W}} - t\mathbb{I}) = \sum_{\alpha=0}^{n} (-1)^{\alpha} S_r t^{n-\alpha},$$

where $\mathbb{I}$ is the identity in $\Gamma(T\mathcal{F})$. The normalized r-th mean curvature H_r of M' is defined by

$$H_r = \binom{n}{r}^{-1} S_r \quad \text{and} \quad H_0 = 1. \text{ (a constant function 1).}$$

In particular, when $r = 1$ then $H_1 = \frac{1}{n}\mathrm{tr}(\mathcal{A}_{\widehat{W}})$ which is the *mean curvature* of a $\mathcal{F}$. On the other hand, H_2 relates directly with the (intrinsic) scalar curvature of $\mathcal{F}$. Moreover, the functions S_r (H_r respectively) are smooth on the whole M and, for any point $p \in \mathcal{F}$, S_r coincides with the r-th mean curvature at p. In this paper, we shall use S_r instead of H_r.

Next, we introduce the Newton transformations with respect to the operator $\mathcal{A}_{\widehat{W}}$. The Newton transformations $T_r : \Gamma(T\mathcal{F}) \to \Gamma(T\mathcal{F})$ of a foliation $\mathcal{F}$ of a screen integrable half-lightlike submanifold M of an $(n+3)$-dimensional semi-Riemannian manifold $\overline{M}$ with respect to $A_{\widehat{W}}$ are given by by the inductive formula

$$T_0 = \mathbb{I}, \quad T_r = (-1)^r S_r \mathbb{I} + \mathcal{A}_{\widehat{W}} \circ T_{r-1}, \quad 1 \leq r \leq n. \tag{3.2}$$

By Cayley-Hamilton theorem, we have $T_n = 0$. Moreover, T_r are also self-adjoint and commutes with $\mathcal{A}_{\widehat{W}}$. Furthermore, the following algebraic properties of T_r are well-known (see [2], [1] and references therein for details).

$$\mathrm{tr}(T_r) = (-1)^r (n-r) S_r, \tag{3.3}$$

$$\mathrm{tr}(\mathcal{A}_{\widehat{W}} \circ T_r) = (-1)^r (r+1) S_{r+1}, \tag{3.4}$$

$$\mathrm{tr}(\mathcal{A}^2_{\widehat{W}} \circ T_r) = (-1)^{r+1} (-S_1 S_{r+1} + (r+2) S_{r+2}), \tag{3.5}$$

$$\mathrm{tr}(T_r \circ \nabla'_X \mathcal{A}_{\widehat{W}}) = (-1)^r X(S_{r+1}) = (-1)^r \overline{g}(\nabla' S_{r+1}, X), \tag{3.6}$$

for all $X \in \Gamma(T\overline{M})$. We will also need the following divergence formula for the operators T_r

$$\mathrm{div}^{\nabla'}(T_r) = \mathrm{tr}(\nabla' T_r) = \sum_{\beta=1}^{n} (\nabla'_{Z_\beta} T_r) Z_\beta, \tag{3.7}$$

where $\{Z_1, \cdots, Z_n\}$ is a local orthonormal frame field of $T\mathcal{F}$.

4. Integration formulas for $\mathcal{F}$

This section is devoted to derivation of integral formulas of foliation $\mathcal{F}$ of $S(TM)$ with a unit normal vector $\widehat{W}$ given by (2.22). By the fact that $\overline{\nabla}$ is a metric connection then $\overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W},\widehat{W})=0$. This implies that the vector field $\overline{\nabla}_{\widehat{W}}\widehat{W}$ is always tangent to $\mathcal{F}$. Our main goal will be to compute the divergence of the vectors $T_r\overline{\nabla}_{\widehat{W}}\widehat{W}$ and $T_r\overline{\nabla}_{\widehat{W}}\widehat{W}+(-1)^rS_{r+1}\widehat{W}$. The following technical lemmas are fundamentally important to this paper. Let $\{E,Z_i,N,W\}$, for $i=1,\cdots,n$ be a quasi-orthonormal field of frame of $T\overline{M}$, such that $S(TM)=\operatorname{span}\{Z_i\}$ and $\epsilon_i=\overline{g}(Z_i,Z_i)$.

Lemma 4.1. *Let M be a screen integrable half-lightlike submanifold of $\overline{M}^{n+3}$ and let M' be a foliation of $S(TM)$. Let $\mathcal{A}_{\widehat{W}}$ be its shape operator, where $\widehat{W}$ is a unit normal vector to $\mathcal{F}$. Then*

$$\overline{g}((\nabla'_X\mathcal{A}_{\widehat{W}})Y,Z)=\overline{g}(Y,(\nabla'_X\mathcal{A}_{\widehat{W}})Z),$$
$$\overline{g}((\nabla'_XT_r)Y,Z)=\overline{g}(Y,(\nabla'_XT_r)Z),$$

for all $X,Y,Z\in\Gamma(T\mathcal{F})$.

Proof. By simple calculations we have

$$\overline{g}((\nabla'_X\mathcal{A}_{\widehat{W}})Y,Z)=\overline{g}(\nabla'_X(\mathcal{A}_{\widehat{W}}Y),Z)-\overline{g}(\nabla'_XY,\mathcal{A}_{\widehat{W}}Z). \tag{4.1}$$

Using the fact that ∇' is a metric connection and the symmetry of $\mathcal{A}_{\widehat{W}}$, (4.1) gives

$$\overline{g}((\nabla'_X\mathcal{A}_{\widehat{W}})Y,Z)=\overline{g}(Y,\nabla'_X(\mathcal{A}_{\widehat{W}}Z))-\overline{g}(Y,\mathcal{A}_{\widehat{W}}(\nabla'_XZ)). \tag{4.2}$$

Then, from (4.2) we deduce the first relation of the lemma. A proof of the second relation follows in the same way, which completes the proof. □

Lemma 4.2. *Let M be a screen integrable half-lightlike submanifold of $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Let $\mathcal{A}_{\widehat{W}}$ be its shape operator, where $\widehat{W}$ is a unit normal vector to $\mathcal{F}$. Denote by $\overline{R}$ the curvature tensor of $\overline{M}$. Then*

$$\operatorname{div}^{\nabla'}(T_0)=0,$$
$$\operatorname{div}^{\nabla'}(T_r)=\mathcal{A}_{\widehat{W}}\operatorname{div}^{\nabla'}(T_{r-1})+\sum_{i=1}^{n}\epsilon_i(\overline{R}(\widehat{W},T_{r-1}Z_i)Z_i)',$$

where $(\overline{R}(\widehat{W},X)Z)'$ denotes the tangential component of $\overline{R}(\widehat{W},X)Z$ for $X,Z\in\Gamma(T\mathcal{F})$. Equivalently, for any $Y\in\Gamma(T\mathcal{F})$ then

$$\overline{g}(\operatorname{div}^{\nabla'}(T_r),Y)=\sum_{j=1}^{r}\sum_{i=1}^{n}\epsilon_i\overline{g}(\overline{R}(T_{r-1}Z_i,\widehat{W})(-\mathcal{A}_{\widehat{W}})^{j-1}Y,Z_i). \tag{4.3}$$

Proof. The first equation of the lemma is obvious since $T_0=\mathbb{I}$. We turn to the second relation. By direct calculations using the recurrence relation (3.2) we derive

$$\begin{aligned}\operatorname{div}^{\nabla'}(T_r)&=(-1)^r\operatorname{div}^{\nabla'}(S_r\mathbb{I})+\operatorname{div}^{\nabla'}(\mathcal{A}_{\widehat{W}}\circ T_{r-1})\\&=(-1)^r\nabla'S_r+\mathcal{A}_{\widehat{W}}\operatorname{div}^{\nabla'}(T_{r-1})+\sum_{i=1}^{n}\epsilon_i(\nabla'_{Z_i}\mathcal{A}_{\widehat{W}})T_{r-1}Z_i.\end{aligned} \tag{4.4}$$

Using Codazzi equation

$$\overline{g}(\overline{R}(X,Y)Z,\widehat{W})=\overline{g}((\nabla'_Y\mathcal{A}_{\widehat{W}})X,Z)-\overline{g}((\nabla'_X\mathcal{A}_{\widehat{W}})Y,Z),$$

for any $X,Y,Z\in\Gamma(T\mathcal{F})$ and Lemma 4.1, we have

$$\begin{aligned}\overline{g}((\nabla'_{Z_i}\mathcal{A}_{\widehat{W}})Y,T_{r-1}Z_i)&=\overline{g}((\nabla'_Y\mathcal{A}_{\widehat{W}})Z_i,T_{r-1}Z_i)+\overline{g}(\overline{R}(Y,Z_i)T_{r-1}Z_i,\widehat{W})\\&=\overline{g}(T_{r-1}(\nabla'_Y\mathcal{A}_{\widehat{W}})Z_i,Z_i)+\overline{g}(\overline{R}(\widehat{W},T_{r-1}Z_i)Z_i,Y),\end{aligned} \tag{4.5}$$

for any $Y \in \Gamma(T\mathcal{F})$. Then applying (4.4) and (4.5) we get

$$\overline{g}(\operatorname{div}^{\nabla'}(T_r), Y) = (-1)^r \overline{g}(\nabla' S_r, Y) + \operatorname{tr}(T_{r-1}(\nabla'_Y \mathcal{A}_{\widehat{W}})) + \overline{g}(\operatorname{div}^{\nabla'}(T_{r-1}), Y) + \overline{g}(Y, \sum_{i=1}^{n} \epsilon_i \overline{R}(\widehat{W}, T_{r-1} Z_i) Z_i). \tag{4.6}$$

Then, applying (4.6) and (3.6) we get the second equation of the lemma. Finally, (4.3) follows immediately by an induction argument. □

Notice that when the ambient manifold is a space form of constant sectional curvature, then $(\overline{R}(\widehat{W}, X)Y)' = 0$, for each $X, Y \in \Gamma(T\mathcal{F})$. Hence, from Lemma (4.2) we have $\operatorname{div}^{\nabla'}(T_r) = 0$.

Lemma 4.3. *Let M be a screen integrable half-lightlike submanifold of $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Let $\mathcal{A}_{\widehat{W}}$ be its shape operator, where $\widehat{W}$ is a unit normal vector to $\mathcal{F}$. Let $\{Z_i\}$ be a local field such $(\nabla'_X Z_i)p = 0$, for $i = 1, \cdots, n$ and any vector field $X \in \Gamma(T\overline{M})$. Then at $p \in \mathcal{F}$ we have*

$$g(\nabla'_{Z_i} \nabla_{\widehat{W}} \widehat{W}, Z_j) = g(\mathcal{A}^2_{\widehat{W}} Z_i, Z_j) - \overline{g}(\overline{R}(Z_i, \widehat{W}) Z_j, \widehat{W}) - \overline{g}((\nabla'_{\widehat{W}} \mathcal{A}_{\widehat{W}}) Z_i, Z_j) + g(\nabla_{\widehat{W}} \widehat{W}, Z_i) g(Z_j, \nabla_{\widehat{W}} \widehat{W}).$$

Proof. Applying $\overline{\nabla}_{Z_i}$ to $g(\overline{\nabla}_{\widehat{W}} \widehat{W}, Z_j)$ and $\overline{g}(\widehat{W}, \overline{\nabla}_{\widehat{W}} Z_j)$ in turn and then using the two resulting equations, we have

$$-\overline{g}(\overline{\nabla}_{\widehat{W}} \widehat{W}, \overline{\nabla}_{Z_i} Z_j) = g(\overline{\nabla}_{Z_i} \overline{\nabla}_{\widehat{W}} \widehat{W}, Z_j) + \overline{g}(\overline{\nabla}_{Z_i} \widehat{W}, \overline{\nabla}_{\widehat{W}} Z_j) + \overline{g}(\widehat{W}, \overline{\nabla}_{Z_i} \overline{\nabla}_{\widehat{W}} Z_j). \tag{4.7}$$

Furthermore, by direct calculations using $(\nabla'_X Z_i)p = 0$ we have

$$\overline{g}((\nabla'_{\widehat{W}} \mathcal{A}_{\widehat{W}}) Z_i, Z_j) = \overline{g}(\overline{\nabla}_{\widehat{W}} \widehat{W}, \overline{Z_i} Z_j) + \overline{g}(\widehat{W}, \overline{\nabla}_{\widehat{W}} \overline{Z_i} Z_j),$$

and hence

$$\begin{aligned} g(\mathcal{A}^2_{\widehat{W}} Z_i, Z_j) &- \overline{g}(\overline{R}(Z_i, \widehat{W}) Z_j, \widehat{W}) - \overline{g}((\nabla'_{\widehat{W}} \mathcal{A}_{\widehat{W}}) Z_i, Z_j) \\ &= g(\mathcal{A}^2_{\widehat{W}} Z_i, Z_j) - \overline{g}(\overline{R}(Z_i, \widehat{W}) Z_j, \widehat{W}) \\ &\quad - \overline{g}(\overline{\nabla}_{\widehat{W}} \widehat{W}, \overline{Z_i} Z_j) - \overline{g}(\widehat{W}, \overline{\nabla}_{\widehat{W}} \overline{Z_i} Z_j) \\ &= g(\mathcal{A}^2_{\widehat{W}} Z_i, Z_j) - \overline{g}(\overline{\nabla}_{Z_i} Z_j, \overline{\nabla}_{\widehat{W}} \widehat{W}) \\ &\quad - \overline{g}(\overline{\nabla}_{Z_i} \overline{\nabla}_{\widehat{W}} Z_j, \widehat{W}) + \overline{g}(\overline{\nabla}_{[Z_i, \widehat{W}]} Z_j, \widehat{W}). \end{aligned} \tag{4.8}$$

Now, applying (4.7), the condition at p and the following relations

$$\overline{\nabla}_{Z_i} \widehat{W} = \sum_{k=1}^{n} \epsilon_k \overline{g}(\overline{\nabla}_{Z_i} \widehat{W}, Z_k) Z_k, \quad \overline{\nabla}_{\widehat{W}} Z_j = \overline{g}(\overline{\nabla}_{\widehat{W}} Z_j, \widehat{W}) \widehat{W},$$

and $g(\mathcal{A}^2_{\widehat{W}} Z_i, Z_j) = -\sum_{k=1}^{n} \epsilon_k \overline{g}(\overline{\nabla}_{Z_i} \widehat{W}, Z_k) \overline{g}(\overline{\nabla}_{Z_k} Z_j, \widehat{W})$ to the last line of (4.8) and the fact that $S(TM)$ is integrable we get

$$\begin{aligned} g(\mathcal{A}^2_{\widehat{W}} Z_i, Z_j) &- \overline{g}(\overline{R}(Z_i, \widehat{W}) Z_j, \widehat{W}) - \overline{g}((\nabla'_{\widehat{W}} \mathcal{A}_{\widehat{W}}) Z_i, Z_j) \\ &= g(\nabla'_{Z_i} \nabla_{\widehat{W}} \widehat{W}, Z_j) - g(\nabla_{\widehat{W}} \widehat{W}, Z_i) g(Z_j, \nabla_{\widehat{W}} \widehat{W}), \end{aligned}$$

from which the lemma follows by rearrangement. □

Notice that, using parallel transport, we can always construct a frame field from the above lemma.

Proposition 4.4. *Let M be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold $\overline{M}$ and let $\mathcal{F}$ be a foliation of $S(TM)$. Then*

$$\begin{aligned}\operatorname{div}^{\nabla'}(T_r\nabla_{\widehat{W}}\widehat{W}) &= \overline{g}(\operatorname{div}^{\nabla'}(T_r),\nabla_{\widehat{W}}\widehat{W}) + (-1)^{r+1}\widehat{W}(S_{r+1})\\ &\quad + (-1)^{r+1}(-S_1S_{r+1}+(r+2)S_{r+2}) - \sum_{i=1}^{n}\epsilon_i\overline{g}(\overline{R}(Z_i,\widehat{W})T_rZ_i,\widehat{W})\\ &\quad + \overline{g}(\nabla_{\widehat{W}}\widehat{W},T_r\nabla_{\widehat{W}}\widehat{W}),\end{aligned}$$

where $\{Z_i\}$ is a field of frame tangent to the leaves of $\mathcal{F}$.

Proof. From (3.7),we deduce that

$$\operatorname{div}^{\nabla'}(T_rZ) = \overline{g}(\operatorname{div}^{\nabla'}(T_r),Z) + \sum_{i=1}^{n}\epsilon_i\overline{g}(\nabla'_{Z_i}Z,T_rZ_i), \tag{4.9}$$

for all $Z\in\Gamma(T\mathcal{F})$. Then using (4.9), Lemmas 4.2 and 4.3, we obtain the desired result. Hence the proof. □

From Proposition 4.4 we have

Theorem 4.5. *Let M be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then*

$$\begin{aligned}\operatorname{div}^{\overline{\nabla}}(T_r\nabla_{\widehat{W}}\widehat{W}) &= \overline{g}(\operatorname{div}^{\nabla'}(T_r),\nabla_{\widehat{W}}\widehat{W}) + (-1)^{r+1}\widehat{W}(S_{r+1})\\ &\quad + (-1)^{r+1}(-S_1S_{r+1}+(r+2)S_{r+2})\\ &\quad - \sum_{i=1}^{n}\epsilon_i\overline{g}(\overline{R}(Z_i,\widehat{W})T_rZ_i,\widehat{W}).\end{aligned}$$

Proof. A proof follows easily from Proposition 4.4 by recognizing the fact that

$$\begin{aligned}\operatorname{div}^{\overline{\nabla}}(T_r\overline{\nabla}_{\widehat{W}}\widehat{W}) &= \operatorname{div}^{\nabla'}(T_r\nabla_{\widehat{W}}\widehat{W})\\ &\quad - \overline{g}(\nabla_{\widehat{W}}\widehat{W},T_r\nabla_{\widehat{W}}\widehat{W}),\end{aligned}$$

which completes the proof. □

Theorem 4.6. *Let M be a screen integrable half-lightlike submanifold of $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then,*

$$\begin{aligned}\operatorname{div}^{\overline{\nabla}}(T_r\nabla_{\widehat{W}}\widehat{W}+(-1)^rS_{r+1}\widehat{W}) &= \overline{g}(\operatorname{div}^{\nabla'}(T_r),\nabla_{\widehat{W}}\widehat{W})\\ &\quad + (-1)^{r+1}(r+2)S_{r+2} - \sum_{i=1}^{n}\epsilon_i\overline{g}(\overline{R}(Z_i,\widehat{W})T_rZ_i,\widehat{W}).\end{aligned}$$

Proof. By straightforward calculations we have

$$\begin{aligned}S_1 &= \operatorname{tr}(\mathcal{A}_{\widehat{W}})\\ &= -\sum_{i=1}^{n}\epsilon_i\overline{g}(\overline{\nabla}_{Z_i}\widehat{W},Z_i)\\ &= -\sum_{i=1}^{n+1}\epsilon_i\overline{g}(\overline{\nabla}_{Z_i}\widehat{W},Z_i)\\ &= -\operatorname{div}^{\overline{\nabla}}(\widehat{W}),\end{aligned}$$

where $Z_{n+1}=\widehat{W}$. From this equation we deduce

$$\operatorname{div}^{\overline{\nabla}}(S_{r+1}\widehat{W}) = -S_1S_{r+1} + \widehat{W}(S_{r+1}). \tag{4.10}$$

Then from (4.10) and Theorem 4.5 we get our assertion, hence the proof. □

Next, we let dV denote the volume form $\overline{M}$. Then from Theorem 4.6 we have the following

Corollary 4.7. *Let M be a screen integrable half-lightlike submanifold of a compact semi-Riemannian manifold $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then*

$$\int_{\overline{M}} \overline{g}(\mathrm{div}^{\nabla'}(T_r), \overline{\nabla}_{\widehat{W}}\widehat{W})dV = \int_{\overline{M}}((-1)^r(r+2)S_{r+2} + \sum_{i=1}^{n} \epsilon_i \overline{g}(\overline{R}(Z_i, \widehat{W})T_r Z_i, \widehat{W})dV.$$

Setting $r = 0$ in the above corollary we get

Corollary 4.8. *Let M be a screen integrable half-lightlike submanifold of a compact semi-Riemannian manifold $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$ with mean curvatures S_r. Then for $r = 0$ we have*

$$\int_{\overline{M}} 2S_2 dV = \int_{\overline{M}} \overline{Ric}(\widehat{W}, \widehat{W})dV,$$

where $\overline{Ric}(\widehat{W}, \widehat{W}) = \sum_{i=1}^{n} \epsilon_i \overline{g}(\overline{R}(Z_i, \widehat{W})\widehat{W}, Z_i)$.

Notice that the equation in Corollary 4.8 is the lightlike analogue of (3.5) in [2] for co-dimension one foliations on Riemannian manifolds.

Next, we will discuss some consequences of the integral formulas developed so far.

A semi-Riemannian manifold $\overline{M}$ of constant sectional curvature c is called a *semi-Riemannian space form* [4, 6] and is denoted by $\overline{M}(c)$. Then, the curvature tensor $\overline{R}$ of $\overline{M}(c)$ is given by

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} = c\{\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y}\}, \quad \forall \overline{X}, \overline{Y}, \overline{Z} \in \Gamma(T\overline{M}). \tag{4.11}$$

Theorem 4.9. *Let M be a screen integrable half-lightlike submanifold of a compact semi-Riemannian space form $\overline{M}(c)$ of constant sectional curvature c. Let $\mathcal{F}$ be a co-dimension three foliation of its screen distribution $S(TM)$. If V is the total volume of $\overline{M}$, then*

$$\int_{\overline{M}} S_r dV = \begin{cases} 0, & r = 2k+1, \\ c^{\frac{r}{2}} \binom{\frac{n}{2}}{\frac{r}{2}} V, & r = 2k, \end{cases} \tag{4.12}$$

for positive integers k.

Proof. By setting $\overline{X} = Z_i$, $\overline{Y} = \widehat{W}$ and $Z = T_r Z_i$ in (4.11) we deduce that

$$\overline{R}(Z_i, \widehat{W})T_r Z_i = -cg(Z_i, T_r Z_i)\widehat{W}.$$

Then substituting this equation in Corollary 4.7 we obtain

$$\int_{\overline{M}} \overline{g}(\mathrm{div}^{\nabla'}(T_r), \overline{\nabla}_{\widehat{W}}\widehat{W})dV = \int_{\overline{M}}((-1)^r(r+2)S_{r+2} - c\mathrm{tr}(T_r))dV.$$

Since $\overline{M}$ is of constant sectional curvature c, then Lemma 4.2 implies that $T_r = 0$ for any r and hence the above equation simplifies to

$$(r+2)\int_{\overline{M}} S_{r+2}dV = c(n-r)\int_{\overline{M}} S_r dV. \tag{4.13}$$

Since $S_1 = -\mathrm{div}^{\overline{\nabla}}(\widehat{W})$ and that $\overline{M}$ is compact, then $\int_{\overline{M}} S_1 dV = 0$. Using this fact together with (4.13), mathematical induction gives $\int_{\overline{M}} S_r dV = 0$ for all $r = 2k+1$ (i.e., r odd).

For r even we will consider $r = 2m$ and $n = 2l$ (i.e., both M and $\overline{M}$ are odd dimensional). With these conditions, (4.13) reduces to

$$\int_{M} S_{2m+2} dV = c\frac{l-m}{1+m}\int_{M} S_{2m} dV. \tag{4.14}$$

Now setting $m = 0, 1, \cdots$ and $S_0 = 1$ in (4.14) we obtain

$$\int_{M} S_2 dV = clV, \quad \int_{M} S_4 dV = c^2\frac{(l-1)l}{2}V,$$

and more generally

$$\int_{M} S_{2k} dV = c^k\frac{(l-k+1)(l-k+2)(l-k+3)\cdots l}{k!}V. \tag{4.15}$$

Hence, our assertion follows from 4.15, which completes the proof. □

Next, when $\overline{M}$ is Einstein i.e., $\overline{Ric} = \mu\overline{g}$ we have the following.

Theorem 4.10. *Let M be a screen integrable half-lightlike submanifold of an Einstein compact semi-Riemannian manifold $\overline{M}$. Let $\mathcal{F}$ be a co-dimension three foliation of its screen distribution $S(TM)$ with totally umbilical leaves. If V is the total volume of $\overline{M}$, then*

$$\int_{\overline{M}} S_r dV = \begin{cases} 0, & r = 2k+1, \\ \left(\frac{\mu}{n}\right)^{\frac{n}{2}}\binom{\frac{n}{2}}{\frac{r}{2}}V, & r = 2k, \end{cases} \tag{4.16}$$

for positive integers k.

Proof. Suppose that $\mathcal{A}_{\widehat{W}} = \frac{1}{n}S_r\mathbb{I}$. Then by direct calculations using the formula for T_r we derive $T_r = (-1)^{r+1}\frac{(n-r)}{n}S_r\mathbb{I}$. Then, under the assumptions of the theorem we obtain $\overline{Ric}(\widehat{W}, \overline{\nabla}_{\widehat{W}}\widehat{W}) = 0$ and $\overline{Ric}(\widehat{W}, \widehat{W}) = \mu$ and hence, Corollary 4.7 reduces to

$$n(r+2)\int_{\overline{M}} S_{r+2} dV = \lambda(n-r)\int_{\overline{M}} S_r dV. \tag{4.17}$$

Notice that (4.17) is similar to (4.13) and hence following similar steps as in the previous theorem we get $\int_{\overline{M}} S_r dV = 0$ for r odd and for r even we get

$$\int_{\overline{M}} S_{2k} dV = \left(\frac{\mu}{n}\right)^k\frac{(l-k+1)(l-k+2)(l-k+3)\cdots l}{k!}V,$$

which complete the proof. □

5. Screen umbilical half-lightlike submanifolds

In this section we consider totally umbilical half-lightlike submanifolds of semi-Riemannian manifold, with a totally umbilical screen distribution and thus, give a generalized version of Theorem 4.3.7 of [6] and its Corollaries, via Newton transformations of the operator A_N.

A screen distribution $S(TM)$ of a half-lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be totally umbilical [6] if on any coordinate neighborhood $\mathcal{U}$ there exist a function K such that

$$C(X, PY) = Kg(X, PY), \quad \forall X, Y \in \Gamma(TM). \tag{5.1}$$

In case $K = 0$, we say that $S(TM)$ is totally geodesic. Furthermore, if $S(TM)$ is totally umbilical then by straightforward calculations we have

$$A_N X = PX, \quad C(E, PX) = 0, \quad \forall X \in \Gamma(TM). \tag{5.2}$$

Let $\{E, Z_i\}$, for $i = 1, \cdots, n$, be a quasi-orthonormal frame field of TM which diagonalizes A_N. Let $l_0, l_1, \cdots, l_n$ be the respective eigenvalues (or principal curvatures). Then as before, the r-th mean curvature S_r is given by

$$S_r = \sigma_r(l_0, \cdots, l_n) \text{ and } S_0 = 1.$$

The characteristic polynomial of A_N is given by

$$\det(A_N - t\mathbb{I}) = \sum_{\alpha=0}^{n} (-1)^{\alpha} S_r t^{n-\alpha},$$

where $\mathbb{I}$ is the identity in $\Gamma(TM)$. The normalized r-th mean curvature H_r of M is defined by $\binom{n}{r} H_r = S_r$ and $H_0 = 1$. The Newton transformations $T_r : \Gamma(TM) \to \Gamma(TM)$ of A_N are given by the inductive formula

$$T_0 = \mathbb{I}, \quad T_r = (-1)^r S_r \mathbb{I} + A_N \circ T_{r-1}, \quad 1 \leq r \leq n. \tag{5.3}$$

By Cayley-Hamiliton theorem, we have $T_{n+1} = 0$. Also, T_r satisfies the following properties.

$$\operatorname{tr}(T_r) = (-1)^r (n + 1 - r) S_r, \tag{5.4}$$

$$\operatorname{tr}(A_N \circ T_r) = (-1)^r (r + 1) S_{r+1}, \tag{5.5}$$

$$\operatorname{tr}(A_N^2 \circ T_r) = (-1)^{r+1} (-S_1 S_{r+1} + (r + 2) S_{r+2}), \tag{5.6}$$

$$\operatorname{tr}(T_r \circ \nabla_X A_N) = (-1)^r X(S_{r+1}), \tag{5.7}$$

for all $X \in \Gamma(TM)$.

Proposition 5.1. *Let (M, g) be a totally umbilical half-lightlike submanifold of a semi-Riemannian manifold $\overline{M}$ of constant sectional curvature c. Then*

$$\begin{aligned} g(\operatorname{div}^{\nabla}(T_r), X) = {} & (-1)^{r-1} \lambda(X) E(S_r) - \tau(X) \operatorname{tr}(A_N \circ T_{r-1}) \\ & - c\lambda(X) \operatorname{tr}(T_{r-1}) + g(\operatorname{div}^{\nabla}(T_{r-1}), A_N X) + g((\nabla_E A_N) T_{r-1} E, X) \\ & + \sum_{i=1}^{n} \epsilon_i \{ -\lambda(X) B(Z_i, A_N(T_{r-1} Z_i)) \\ & + \varepsilon \tau(Z_i) C(X, T_{r-1} Z_i) \{ \rho(X) D(Z_i, T_{r-1} Z_i) - \rho(Z_i) D(X, T_{r-1} Z_i) \} \}, \end{aligned}$$

for any $X \in \Gamma(TM)$.

Proof. From the recurrence relation (5.3), we derive

$$\begin{aligned} g(\operatorname{div}^{\nabla}(T_r), X) = {} & (-1)^r P X(S_r) + g((\nabla_E A_N) T_{r-1} E, X) \\ & + g(\operatorname{div}^{\nabla}(T_{r-1}), A_N X) + \sum_{i=1}^{n} \epsilon_i g((\nabla_{Z_i} A_N) T_{r-1} Z_i, X), \end{aligned} \tag{5.8}$$

for any $X \in \Gamma(TM)$. But

$$\begin{aligned} g((\nabla_{Z_i} A_N) T_{r-1} Z_i, X) = {} & g(T_{r-1} Z_i, (\nabla_{Z_i} A_N) X) + g(\nabla_{Z_i} A_N (T_{r-1} Z_i), X) \\ & - g(\nabla_{Z_i} (A_N X), T_{r-1} Z_i) + g(A_N (\nabla_{Z_i} X), T_{r-1} Z_i) \\ & - g(A_N (\nabla_{Z_i} T_{r-1} Z_i), X), \end{aligned} \tag{5.9}$$

for all $X \in \Gamma(TM)$. □

Then applying (2.9) to (5.9) while considering the fact that A_N is screen-valued, we get

$$g((\nabla_{Z_i} A_N) T_{r-1} Z_i, X) = g(T_{r-1} Z_i, (\nabla_{Z_i} A_N) X) - \lambda(X) B(Z_i, A_N(T_{r-1} Z_i)). \tag{5.10}$$

Furthermore, using (2.15) and (4.11), the first term on the right hand side of (5.10) reduces to

$$\begin{aligned} g(T_{r-1}Z_i,(\nabla_{Z_i}A_N)X) &= -c\lambda(X)g(Z_i,T_{r-1}Z_i) + g((\nabla_X A_N)Z_i,T_{r-1}Z_i) \\ &\quad - \tau(X)C(Z_i,T_{r-1}Z_i) + \varepsilon\tau(Z_i)C(X,T_{r-1}Z_i)\{\rho(X)D(Z_i,T_{r-1}Z_i) \\ &\quad - \rho(X)D(X,T_{r-1}Z_i)\}, \end{aligned} \tag{5.11}$$

for any $X \in \Gamma(TM)$. Finally, replacing (5.11) in (5.10) and then put the resulting equation in (5.8) we get the desired result.

Next, from Proposition 5.1 we have the following.

Theorem 5.2. *Let (M, g) be a half-lightlike submanifold of a semi-Riemannian manifold $\overline{M}(c)$ of constant curvature c, with a proper totally umbilical screen distribution $S(TM)$. If M is also totally umbilical, then the r-th mean curvature S_r, for $r = 0, 1, \cdots, n$, with respect to A_N are solution of the following differential equation*

$$E(S_{r+1}) - \tau(E)(r+1)S_{r+1} - c(-1)^r(n+1-r)S_r = \mathcal{H}_1(r+1)S_{r+1}.$$

Proof. Replacing X with E in the Proposition 5.1 and then using (2.16) and (5.2) we obtain, for all $r = 0, 1, \cdots, n$,

$$E(S_{r+1}) - (-1)^r\tau(E)\mathrm{tr}(A_N \circ T_r) - c(-1)^r\mathrm{tr}(T_r) = (-1)^r\mathcal{H}_1\mathrm{tr}(A_N \circ T_r),$$

from which the result follows by applying (5.4) and (5.5). □

Corollary 5.3. *Under the hypothesis of Theorem 5.2, the induced connection ∇ on M is a metric connection, if and only if, the r-th mean curvature S_r with respect to A_N are solution of the following equation*

$$E(S_{r+1}) - \tau(E)(r+1)S_{r+1} - c(-1)^r(n+1-r)S_r = 0.$$

Also the following holds.

Corollary 5.4. *Under the hypothesis of Theorem 5.2, $\overline{M}(c)$ is a semi-Euclidean space, if and only if, the r-th mean curvature S_r with respect to A_N are solution of the following equation*

$$E(S_{r+1}) - \tau(E)(r+1)S_{r+1} = \mathcal{H}_1(r+1)S_{r+1}.$$

Notice that Theorem 5.2 and Corollary 5.3 are generalizations of Theorem 4.3.7 and Corollary 4.3.8, respectively, given in [6].

Acknowledgment. This work is based on the research supported wholly by the National Research Foundation of South Africa (Grant Numbers: 95931 and 106072).

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