On total mean curvatures of foliated half-lightlike submanifolds in semi-Riemannian manifolds

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Abstract

We derive total mean curvature integration formulas of a three co-dimensional foliation $\mathcal{F}^n$ on a screen integrable half-lightlike submanifold, $M^{n+1}$ in a semi-Riemannian manifold $M^{n+3}$. We give generalized differential equations relating to mean curvatures of a totally umbilical half-lightlike submanifold admitting a totally umbilical screen distribution, and show that they are generalizations of those given by [K. L. Duggal and B. Sahin, Differential geometry of lightlike submanifolds, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2010].

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1. Introduction

The rapidly growing importance of lightlike submanifolds in semi-Riemannian geometry, particularly Lorentzian geometry, and their applications to mathematical physics—like in general relativity and electromagnetism motivated the study of lightlike geometry in semi-Riemannian manifolds. More precisely, lightlike submanifolds have been shown to represent different black hole horizons (see [4] and [5] for details). Among other motivations for investing in lightlike geometry by many physicists is the idea that the universe we are living in can be viewed as a 4-dimensional hypersurface embedded in $(4 + m)$-dimensional spacetime manifold, where $m$ is any arbitrary integer. There are significant differences between lightlike geometry and Riemannian geometry as shown in [4] and [5], and many more references therein. Some of the pioneering work on this topic is due to Duggal-Bejancu [4], Duggal-Sahin [5] and Kupeli [7]. It is upon those books that many other researchers, including but not limited to [3], [6], [8], [9], [10], [11], have extended their theories.

Lightlike geometry rests on a number of operators, like shape and algebraic invariants derived from them, such as trace, determinants, and in general the $r$-th mean curvature $S_r$. There is a great deal of work so far on the case $r = 1$ (see some in [4], [5] and many

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more) and as far as we know, very little has been done for the case $r > 1$. This is partly due to the non-linearity of $S_r$ for $r > 1$, and hence very complicated to study. A great deal of research on higher order mean curvatures $S_r$ in Riemannian geometry has been done with numerous applications, for instance see [2] and [1]. This gap has motivated our introduction of lightlike geometry of $S_r$ for $r > 1$. In this paper we have considered a half-lightlike submanifold admitting an integrable screen distribution, of a semi-Riemannian manifold. On it we have focused on a codimension 3 foliation of its screen distribution and thus derived integral formulas of its total mean curvatures (see Theorems 4.9 and 4.10). Furthermore, we have considered totally umbilical half-lightlike submanifolds, with a totally umbilical screen distribution and generalized Theorem 4.3.7 of [5] (see Theorem 5.2 and its Corollaries). The paper is organized as follows; In Section 2 we summarize the basic information on Newton transformations of a foliation $\mathcal{F}$ of the screen distribution. Section 4 focuses on integration formulae of $\mathcal{F}$ and their consequences. In Section 5 we discuss screen umbilical half-lightlike submanifolds and generalizations of some well-known results of [5].

2. Preliminaries

Let $(M^{n+1}, g)$ be a two-co-dimensional submanifold of a semi-Riemannian manifold $(\bar{M}^{n+3}, \bar{g})$, where $g = \bar{g}|_{TM}$. The submanifold $(M^{n+1}, g)$ is called a half-lightlike if the radical distribution $\text{Rad} \, TM = TM \cap TM^\perp$ is a vector subbundle of the tangent bundle $TM$ and the normal bundle $TM^\perp$ of $M$, with rank one. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\text{Rad} \, TM$ in $TM$, and also choose a screen transversal bundle $T(TM^\perp)$, which is semi-Riemannian and complementary to $\text{Rad} \, TM$ in $TM^\perp$. Then,

$$TM = \text{Rad} \, TM \perp S(TM), \quad TM^\perp = \text{Rad} \, TM \perp S(TM^\perp). \quad (2.1)$$

We will denote by $\Gamma(\Xi)$ the set of smooth sections of the vector bundle $\Xi$. It is well-known from [4] and [5] that for any null section $E$ of $\text{Rad} \, TM$, there exists a unique null section $N$ of the orthogonal complement of $S(TM^\perp)$ in $S(TM)$ such that $g(E, N) = 1$, it follows that there exists a lightlike transversal vector bundle $\text{ltr}(TM)$ locally spanned by $N$. Let $W \in \Gamma(S(TM^\perp))$ be a unit vector field, then $\bar{g}(N,N) = \bar{g}(N,Z) = \bar{g}(N,W) = 0$, for any $Z \in \Gamma(S(TM))$.

Let $\text{ltr}(TM)$ be complementary (but not orthogonal) vector bundle to $TM$ in $\bar{TM}$. Then we have the following decompositions of $\text{ltr}(TM)$ and $\bar{TM}$

$$\text{ltr}(TM) = \mathcal{T}(TM) \perp S(TM^\perp), \quad (2.2)$$
$$\bar{TM} = S(TM) \perp S(TM^\perp) \perp \{\text{Rad} \, TM \oplus \text{ltr}(TM)\}. \quad (2.3)$$

It is important to note that the distribution $S(TM)$ is not unique, and is canonically isomorphic to the factor vector bundle $TM/\text{Rad} \, TM$ [4]. Let $P$ be the projection of $TM$ on to $S(TM)$. Then the local Gauss-Weingarten equations of $M$ are the following;

$$\nabla_XY = \nabla_XY + B(X,Y)N + D(X,Y)W, \quad (2.4)$$
$$\nabla_XN = -A_NX + \tau(X)N + \rho(X)W, \quad (2.5)$$
$$\nabla_XW = -A_WX + \phi(X)N, \quad (2.6)$$
$$\nabla_XPY = \nabla_XPY + C(X,PY)E, \quad (2.7)$$
$$\nabla_XE = -A^*_XE - \tau(X)E, \quad (2.8)$$

for all $E \in \Gamma(\text{Rad} \, TM)$, $N \in \Gamma(\text{ltr}(TM))$ and $W \in \Gamma(S(TM^\perp))$, where $\nabla$ and $\nabla^*$ are induced linear connections on $TM$ and $S(TM)$, respectively, $B$ and $D$ are called the local second fundamental forms of $M$, $C$ is the local second fundamental form on $S(TM)$. 
Furthermore, \{A_N, A_W\} and \(A^*_E\) are the shape operators on \(TM\) and \(S(TM)\) respectively, and \(\tau, \rho, \phi\) and \(\delta\) are differential 1-forms on \(TM\). Notice that \(\nabla^*\) is a metric connection on \(S(TM)\) while \(\nabla\) is generally not a metric connection. In fact, \(\nabla\) satisfies the following relation

\[
(\nabla_X g)(Y, Z) = B(X, Y)\lambda(Z) + B(X, Z)\lambda(Y),
\]

for all \(X, Y, Z \in \Gamma(TM)\), where \(\lambda\) is a 1-form on \(TM\) given \(\lambda(\cdot) = \bar{g}(\cdot, N)\). It is well-known from [4] and [5] that \(B\) and \(D\) are independent of the choice of \(S(TM)\) and they satisfy

\[
B(X, E) = 0, \quad D(X, E) = -\phi(X), \quad \forall X \in \Gamma(TM).
\]

The local second fundamental forms \(B, D\) and \(C\) are related to their shape operators by the following equations

\[
g(A^*_E X, Y) = B(X, Y), \quad \bar{g}(A^*_E X, N) = 0, \quad (2.11)
g(A_W X, Y) = \varepsilon(D(X, Y) + \phi(X)\lambda(Y), \quad (2.12)
g(A_N X, PY) = C(X, PY), \quad \bar{g}(A_N X, N) = 0, \quad (2.13)
\]

\[
\bar{g}(A^*_E X, N) = \varepsilon\rho(X), \quad \text{where} \quad \varepsilon = \bar{g}(W, W), \quad (2.14)
\]

for all \(X, Y \in \Gamma(TM)\). From equations (2.11) we deduce that \(A^*_E\) is \(S(TM)\)-valued, self-adjoint and satisfies \(A^*_E E = 0\). Let \(\bar{R}\) denote the curvature tensor of \(\bar{M}\), then

\[
\bar{g}(\bar{R}(X, Y)PZ, N) = g((\nabla_X A_N)Y, PZ) - g((\nabla_Y A_N)X, PZ)
+ \tau(Y)C(X, PZ) - \varepsilon\tau(X)C(Y, PZ)\{\rho(Y)D(X, PZ) - \rho(X)D(Y, PZ)\}, \quad \forall X, Y, Z \in \Gamma(TM).
\]

A half-lightlike submanifold \((M, g)\) of a semi-Riemannian manifold \(\bar{M}\) is said to be totally umbilical [5] if on each coordinate neighborhood \(U\) there exist smooth functions \(\mathcal{H}_1\) and \(\mathcal{H}_2\) on \(\text{tr}(TM)\) and \(S(TM^+)\) respect such that

\[
B(X, Y) = \mathcal{H}_1 g(X, Y), \quad D(X, Y) = \mathcal{H}_2 g(X, Y), \quad \forall X, Y \in \Gamma(TM).
\]

Furthermore, when \(M\) is totally umbilical then the following relations follows by straightforward calculations

\[
A^*_E X = \mathcal{H}_1 PX, \quad P(A_W X) = \varepsilon\mathcal{H}_2 PX, \quad D(X, E) = 0, \quad \rho(E) = 0, \quad (2.17)
\]

for all \(X, Y \in \Gamma(TM)\).

Next, we suppose that \(M\) is a half-lightlike submanifold of \(\bar{M}\), with an integrable screen distribution \(S(TM)\). Let \(M'\) be a leaf of \(S(TM)\). Notice that for any screen integrable half-lightlike \(M\), the leaf \(M'\) of \(S(TM)\) is a co-dimension 3 submanifold of \(\bar{M}\) whose normal bundle is \(\{\text{Rad} TM \oplus \text{tr}(TM)\} \perp S(TM^+)\). Now, using (2.4) and (2.7) we have

\[
\nabla_X Y = \nabla^*_X Y + C(X, PY)E + B(X, Y)N + D(X, Y)W, \quad (2.18)
\]

for all \(X, Y \in \Gamma(TM')\). Since \(S(TM)\) is integrable, then its leave is semi-Riemannian and hence we have

\[
\nabla_X Y = \nabla^*_X Y + h'(X, Y), \quad \forall X, Y \in \Gamma(TM'), \quad (2.19)
\]

where \(h'\) and \(\nabla^*\) are second fundamental form and the Levi-Civita connection of \(M'\) in \(\bar{M}\). From (2.18) and (2.19) we can see that

\[
h'(X, Y) = C(X, PY)E + B(X, Y)N + D(X, Y)W, \quad (2.20)
\]

for all \(X, Y \in \Gamma(TM')\). Since \(S(TM)\) is integrable, then it is well-known from [5] that \(C\) is symmetric on \(S(TM)\) and also \(A_N\) is self-adjoint on \(S(TM)\) (see Theorem 4.1.2 for details). Thus, \(h'\) given by (2.20) is symmetric on \(TM'\).
Let $L \in \Gamma([\text{Rad} TM \oplus \text{ltr}(TM)] \perp S(TM^\perp))$, then we can decompose $L$ as

$$L = aE + bN + cW,$$  \hspace{1cm} (2.21)

for non-vanishing smooth functions on $\overline{M}$ given by $a = \overline{g}(L, N)$, $b = \overline{g}(L, E)$ and $c = \varepsilon \overline{g}(L, W)$. Suppose that $\overline{g}(L, L) > 0$, then using (2.21) we obtain a unit normal vector $\tilde{W}$ to $M'$ given by

$$\tilde{W} = \frac{1}{\overline{g}(L, L)}(aE + bN + cW) = \frac{1}{\overline{g}(L, L)} L.$$  \hspace{1cm} (2.22)

Next we define a $(1,1)$ tensor $A_{\tilde{W}}$ in terms of the operators $A_E^*, A_N$ and $A_W$ by

$$A_{\tilde{W}} X = \frac{1}{\overline{g}(L, L)}(aA_E^* X + bA_N X + cA_W X),$$  \hspace{1cm} (2.23)

for all $X \in \Gamma(TM)$. Notice that $A_{\tilde{W}}$ is self-adjoint on $S(TM)$. Applying $\nabla_X$ to $\tilde{W}$ and using equations (2.23) (2.4) and (2.11)-(2.13), we have

$$\overline{g}(A_{\tilde{W}} X, PY) = -\overline{g}(\nabla_X \tilde{W}, PY), \text{ \ for all } X, Y \in \Gamma(TM).$$  \hspace{1cm} (2.24)

Let $\nabla^* \tilde{W}$ be the connection on the normal bundle $\{\text{Rad} TM \oplus \text{ltr}(TM)\} \perp S(TM^\perp)$. Then from (2.24) we have

$$\nabla_X \tilde{W} = -A_{\tilde{W}} X + \nabla^*_X \tilde{W}, \text{ \ for all } X \in \Gamma(TM),$$  \hspace{1cm} (2.25)

where

$$\nabla^*_X \tilde{W} = -\frac{1}{\overline{g}(L, L)} X(\overline{g}(L, L))\tilde{W} + \frac{1}{\overline{g}(L, L)} \{[X(a) - a\tau(X)]E + \{X(b) + b\tau(X) + c\phi(X)\}N + \{X(c) + aD(X, E) + b\rho(X)\}W\}.$$

**Example 2.1.** Let $\overline{M} = (\mathbb{R}_1^5, \overline{g})$ be a semi-Riemannian manifold, where $\overline{g}$ is of signature $(-, +, +, +, +)$ with respect to canonical basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5)$, where $(x_1, \cdots, x_5)$ are the usual coordinates on $\overline{M}$. Let $M$ be a submanifold of $\overline{M}$ and given parametrically by the following equations

$$x_1 = \phi_1, \quad x_2 = \sin \phi_2 \sin \phi_3, \quad x_3 = \phi_1, \quad x_4 = \cos \phi_2 \sin \phi_3, \quad x_5 = \cos \phi_3, \text{ \ where } \phi_2 \in [0, 2\pi] \text{ \ and } \phi_3 \in (0, \pi/2).$$

Then we have

$$TM = \text{span}\{E, Z_1, Z_2\} \text{ \ and \ } \text{ltr}(TM) = \text{span}\{N\},$$

where

$$E = \partial x_1 + \partial x_3, \quad Z_1 = \cos \phi_3 \partial x_2 - \sin \phi_2 \sin \phi_3 \partial x_5, \quad Z_2 = \cos \phi_3 \partial x_4 - \cos \phi_2 \sin \phi_3 \partial x_5 \text{ \ and } N = \frac{1}{2}(-\partial x_1 + \partial x_3).$$

Also, by straightforward calculations, we have

$$W = \sin \phi_2 \sin \phi_3 \partial x_2 + \cos \phi_2 \sin \phi_3 \partial x_4 + \cos \phi_3 \partial x_5.$$  

Thus, $S(TM^\perp) = \text{span}\{W\}$ and hence $M$ is a half-lightlike submanifold of $\overline{M}$. Furthermore we have $[Z_1, Z_2] = \cos \phi_2 \sin \phi_3 \partial x_2 - \sin \phi_2 \sin \phi_3 \partial x_4$, which leads to $[Z_1, Z_2] = \cos \phi_2 \tan \phi_3 Z_1 - \sin \phi_2 \tan \phi_3 Z_2 \in \Gamma(S(TM))$. Thus, $M$ is a screen integrable half-lightlike submanifold of $\overline{M}$. Finally, it is easy to see that $A_N$ is self-adjoint operator on $S(TM)$.

In the next sections we shall consider screen integrable half-lightlike submanifolds of semi-Riemannian manifold $\overline{M}$ and derive special integral formulas for a foliation of $S(TM)$, whose normal vector is $\tilde{W}$ and with shape operator $A_{\tilde{W}}$. 
3. Newton transformations of $\mathcal{A}_{\hat{W}}$

Let $(\tilde{M}^{m+3}, \tilde{g})$ be a semi-Riemannian manifold and let $(M^{n+1}, g)$ be a screen integrable half-lightlike submanifold of $\tilde{M}$. Then $S(TM)$ admits a foliation and let $\mathcal{F}$ be a such foliation. Then, the leaves of $\mathcal{F}$ are co-dimension three submanifolds of $\tilde{M}$, whose normal bundle is $S(TM)^{\perp}$. Let $\hat{W}$ be unit normal vector to $\mathcal{F}$ such that the orientation of $\tilde{M}$ coincides with that given by $\mathcal{F}$ and $\hat{W}$. The Levi-Civita connection $\nabla$ on the tangent bundle of $\tilde{M}$ induces a metric connection $\nabla'$ on $\mathcal{F}$. Furthermore, $h'$ and $\mathcal{A}_{\hat{W}}'$ are the second fundamental form and shape operator of $\mathcal{F}$. Notice that $\mathcal{A}_{\hat{W}}$ is self-adjoint on $T\mathcal{F}$ and at each point $p \in \mathcal{F}$ has $n$ real eigenvalues (or principal curvatures) $\kappa_1(p), \ldots, \kappa_n(p)$.

Attached to the shape operator $\mathcal{A}_{\hat{W}}$ are $n$ algebraic invariants

$$S_r = \sigma_r(\kappa_1 , \ldots , \kappa_n), \quad 1 \leq r \leq n,$$

where $\sigma_r : M^n \to \mathbb{R}$ are symmetric functions given by

$$\sigma_r(\kappa_1 , \ldots , \kappa_n) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \kappa_{i_1} \cdots \kappa_{i_r}.$$ (3.1)

Then, the characteristic polynomial of $\mathcal{A}_{\hat{W}}$ is given by

$$\det(\mathcal{A}_{\hat{W}}^2 - tI) = \sum_{\alpha=0}^{n} (-1)^{\alpha} S_{r-\alpha} t^{n-\alpha},$$

where $I$ is the identity in $\Gamma(T\mathcal{F})$. The normalized $r$-th mean curvature $H_r$ of $M'$ is defined by

$$H_r = \left( \frac{n}{r} \right) S_r \quad \text{and} \quad H_0 = 1. \quad (a \ \text{constant function} \ 1).$$

In particular, when $r = 1$ then $H_1 = \frac{1}{n} \text{tr}(\mathcal{A}_{\hat{W}}^2)$ which is the mean curvature of a $\mathcal{F}$. On the other hand, $H_2$ relates directly with the (intrinsic) scalar curvature of $\mathcal{F}$. Moreover, the functions $S_r$ ($H_r$ respectively) are smooth on the whole $M$ and, for any point $p \in \mathcal{F}$, $S_r$ coincides with the $r$-th mean curvature at $p$. In this paper, we shall use $S_r$ instead of $H_r$.

Next, we introduce the Newton transformations with respect to the operator $\mathcal{A}_{\hat{W}}$. The Newton transformations $T_r : \Gamma(T\mathcal{F}) \to \Gamma(T\mathcal{F})$ of a foliation $\mathcal{F}$ of a screen integrable half-lightlike submanifold $M$ of an $(n+3)$-dimensional semi-Riemannian manifold $\tilde{M}$ with respect to $\mathcal{A}_{\hat{W}}$ are given by by the inductive formula

$$T_0 = I, \quad T_r = (-1)^r S_r I + \mathcal{A}_{\hat{W}} \circ T_{r-1}, \quad 1 \leq r \leq n. \quad (3.2)$$

By Cayley-Hamilton theorem, we have $T_n = 0$. Moreover, $T_r$ are also self-adjoint and commutes with $\mathcal{A}_{\hat{W}}$. Furthermore, the following algebraic properties of $T_r$ are well-known (see [2], [1] and references therein for details).

$$\text{tr}(T_r) = (-1)^r (n - r) S_r, \quad (3.3)$$

$$\text{tr}(\mathcal{A}_{\hat{W}} \circ T_r) = (-1)^r (r + 1) S_{r+1}, \quad (3.4)$$

$$\text{tr}(\mathcal{A}_{\hat{W}}^2 \circ T_r) = (-1)^{r+1} (-S_1 S_{r+1} + (r + 2) S_{r+2}), \quad (3.5)$$

$$\text{tr}(T_r \circ \nabla^X(\mathcal{A}_{\hat{W}})) = (-1)^r X(S_{r+1}) = (-1)^r g(\nabla X S_{r+1}, X), \quad (3.6)$$

for all $X \in \Gamma(T\mathcal{M})$. We will also need the following divergence formula for the operators $T_r$

$$\text{div}^\nabla (T_r) = \text{tr}(\nabla T_r) = \sum_{\beta=1}^{n} (\nabla^X Z_\beta) T_r) Z_\beta, \quad (3.7)$$

where $\{Z_1, \ldots, Z_n\}$ is a local orthonormal frame field of $T\mathcal{F}$. 

4. Integration formulas for $\mathcal{F}$

This section is devoted to derivation of integral formulas of foliation $\mathcal{F}$ of $S(TM)$ with a unit normal vector $\hat{W}$ given by (2.22). By the fact that $\nabla$ is a metric connection then $\bar{g}(\nabla_{\hat{W}} \hat{W}, W) = 0$. This implies that the vector field $\nabla_{\hat{W}} \hat{W}$ is always tangent to $\mathcal{F}$. Our main goal will be to compute the divergence of the vectors $\nabla_{\hat{W}} \hat{W}$ and $\nabla_{\hat{W}} \hat{W} + (-1)^{r} S_{r+1} \hat{W}$. The following technical lemmas are fundamentally important to this paper. Let $\{E, Z_{i}, N, W\}$, for $i = 1, \cdots, n$ be a quasi-orthonormal field of frame of $TM$, such that $S(TM) = \text{span}\{Z_{i}\}$ and $\epsilon_{i} = \bar{g}(Z_{i}, Z_{i})$.

**Lemma 4.1.** Let $M$ be a screen integrable half-lightlike submanifold of $\mathbb{M}^{n+3}$ and let $M'$ be a foliation of $S(TM)$. Let $A_{\hat{W}}$ be its shape operator, where $\hat{W}$ is a unit normal vector to $\mathcal{F}$. Then

$$\bar{g}((\nabla'_{X} A_{\hat{W}})Y, Z) = \bar{g}(Y, (\nabla'_{X} A_{\hat{W}})Z),$$

$$\bar{g}((\nabla'_{Y} T_{r})Y, Z) = \bar{g}(Y, (\nabla'_{Y} T_{r})Z),$$

for all $X, Y, Z \in \Gamma(TM)$.

**Proof.** By simple calculations we have

$$\bar{g}((\nabla'_{X} A_{\hat{W}})Y, Z) = \bar{g}(Y, (\nabla'_{X} A_{\hat{W}})Y) - \bar{g}(\nabla'_{X} Y, A_{\hat{W}} Z).$$

(4.1)

Using the fact that $\nabla'$ is a metric connection and the symmetry of $A_{\hat{W}}$, (4.1) gives

$$\bar{g}((\nabla'_{X} A_{\hat{W}})Y, Z) = \bar{g}(Y, \nabla'_{X}(A_{\hat{W}} Z)) - \bar{g}(Y, A_{\hat{W}} (\nabla'_{X} Z)).$$

(4.2)

Then, from (4.2) we deduce the first relation of the lemma. A proof of the second relation follows in the same way, which completes the proof. $\square$

**Lemma 4.2.** Let $M$ be a screen integrable half-lightlike submanifold of $\mathbb{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Let $A_{\hat{W}}$ be its shape operator, where $\hat{W}$ is a unit normal vector to $\mathcal{F}$. Denote by $\overline{R}$ the curvature tensor of $\mathcal{M}$. Then

$$\text{div} \nabla' (T_{0}) = 0,$$

$$\text{div} \nabla' (T_{r}) = A_{\hat{W}} \text{div} \nabla' (T_{r-1}) + \sum_{i=1}^{n} \epsilon_{i} (\overline{R}(\hat{W}, T_{r-1} Z_{i}) Z_{i})',$$

where $(\overline{R}(\hat{W}, X) Z)_{r}'$ denotes the tangential component of $\overline{R}(\hat{W}, X) Z$ for $X, Z \in \Gamma(TM)$. Equivalently, for any $Y \in \Gamma(TM)$ then

$$\bar{g}(\text{div} \nabla' (T_{r}), Y) = \sum_{j=1}^{r} \sum_{i=1}^{n} \epsilon_{i} \bar{g}(\overline{R}(T_{r-1} Z_{i}, \hat{W})(-A_{\hat{W}})^{j-1} Y, Z_{i}).$$

(4.3)

**Proof.** The first equation of the lemma is obvious since $T_{0} = I$. We turn to the second relation. By direct calculations using the recurrence relation (3.2) we derive

$$\text{div} \nabla' (T_{r}) = (-1)^{r} \text{div} \nabla' (S_{r} I) + \text{div} \nabla' (A_{\hat{W}} \circ T_{r-1})$$

$$= (-1)^{r} \nabla' S_{r} + A_{\hat{W}} \text{div} \nabla' (T_{r-1}) + \sum_{i=1}^{n} \epsilon_{i} (\nabla'_{Z_{i}} A_{\hat{W}}) T_{r-1} Z_{i}.$$

(4.4)

Using Codazzi equation

$$\bar{g}(\overline{R}(X, Y) Z, \hat{W}) = \bar{g}((\nabla'_{X} A_{\hat{W}}) X, Z) - \bar{g}((\nabla'_{X} A_{\hat{W}}) Y, Z),$$

for any $X, Y, Z \in \Gamma(TM)$ and Lemma 4.1, we have

$$\bar{g}((\nabla'_{Z_{i}} A_{\hat{W}}) Y, T_{r-1} Z_{i}) = \bar{g}((\nabla'_{Y} A_{\hat{W}}) Z_{i}, T_{r-1} Z_{i}) + \bar{g}(\overline{R}(Y, Z_{i}) T_{r-1} Z_{i}, \hat{W})$$

$$= \bar{g}(T_{r-1} (\nabla'_{Y} A_{\hat{W}}) Z_{i}, Z_{i}) + \bar{g}(\overline{R}(\hat{W}, T_{r-1} Z_{i}) Z_{i}, Y),$$

(4.5)
for any $Y \in \Gamma(T\mathcal{F})$. Then applying (4.4) and (4.5) we get
\[
g(\text{div}^\mathcal{V}(T_r), Y) = (-1)^r g(\nabla Y, r) + \text{tr}(T_{r-1}(\nabla Y, A\hat{W}))
+ g(\text{div}^\mathcal{V}(T_{r-1}), Y) + g(Y, \sum_{i=1}^{n} e_i R(\hat{W}, T_{r-1}Z_i)Z_i). \tag{4.6}
\]
Then, applying (4.6) and (3.6) we get the second equation of the lemma. Finally, (4.3) follows immediately by an induction argument.

Notice that when the ambient manifold is a space form of constant sectional curvature, then $(R(\hat{W}, X)Y)' = 0$, for each $X, Y \in \Gamma(T\mathcal{F})$. Hence, from Lemma (4.2) we have $\text{div}^\mathcal{V}(T_r) = 0$.

**Lemma 4.3.** Let $M$ be a screen integrable half-lightlike submanifold of $\mathcal{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Let $A\hat{W}$ be its shape operator, where $\hat{W}$ is a unit normal vector to $\mathcal{F}$. Let $\{Z_i\}$ be a local field such $(\nabla_X Z_i)p = 0$, for $i = 1, \cdots, n$ and any vector field $X \in \Gamma(T\mathcal{M})$. Then at $p \in \mathcal{F}$ we have
\[
g(\nabla Z_i, \nabla\hat{W}, Z_j) = g(A^2_{\hat{W}}Z_i, Z_j) - g(R(Z_i, \hat{W})Z_j, \hat{W})
- g((\nabla_{\hat{W}}A\hat{W})Z_i, Z_j) + g(\nabla_{\hat{W}}\hat{W}, Z_i)g(Z_j, \nabla_{\hat{W}}\hat{W}).
\]

**Proof.** Applying $\nabla Z_i$ to $g(\nabla_{\hat{W}}\hat{W}, Z_j)$ and $g(\hat{W}, \nabla_{\hat{W}}\hat{W}, Z_j)$ in turn and then using the two resulting equations, we have
\[
- g(\nabla_{\hat{W}}\hat{W}, \nabla Z_i, Z_j) = g(\nabla Z_i, \nabla_{\hat{W}}\hat{W}, Z_j) + g(\nabla Z_i, \nabla_{\hat{W}}\hat{W}, Z_j)
+ g(\hat{W}, \nabla_{\hat{W}}\hat{W}, Z_j) Z_j. \tag{4.7}
\]
Furthermore, by direct calculations using $(\nabla_X Z_i)p = 0$ we have
\[
g((\nabla_{\hat{W}}A\hat{W})Z_i, Z_j) = g(\nabla_{\hat{W}}\hat{W}, Z_i)g(Z_j, \nabla_{\hat{W}}\hat{W}),
\]
and hence
\[
g(A^2_{\hat{W}}Z_i, Z_j) - g(R(Z_i, \hat{W})Z_j, \hat{W}) - g((\nabla_{\hat{W}}A\hat{W})Z_i, Z_j)
= g(A^2_{\hat{W}}Z_i, Z_j) - g(R(Z_i, \hat{W})Z_j, \hat{W})
- g(\nabla_{\hat{W}}\hat{W}, Z_i)g(Z_j, \nabla_{\hat{W}}\hat{W})
- g(\nabla Z_i, \nabla_{\hat{W}}\hat{W}, Z_j) + g(\nabla_{Z_i, \nabla_{\hat{W}}\hat{W}} Z_j, \hat{W}). \tag{4.8}
\]
Now, applying (4.7), the condition at $p$ and the following relations
\[
\nabla Z_i, \hat{W} = \sum_{k=1}^{n} e_k g(\nabla Z_i, \hat{W}, Z_k)Z_k, \quad \nabla_{\hat{W}}\hat{W}, Z_j = g(\nabla_{\hat{W}}\hat{W}, Z_j, \hat{W}),
\]
and $g(A^2_{\hat{W}}Z_i, Z_j) = -\sum_{k=1}^{n} e_k g(\nabla Z_i, \hat{W}, Z_k)g(\nabla Z_k, \hat{W})$ to the last line of (4.8) and the fact that $S(TM)$ is integrable we get
\[
g(A^2_{\hat{W}}Z_i, Z_j) - g(R(Z_i, \hat{W})Z_j, \hat{W}) - g((\nabla_{\hat{W}}A\hat{W})Z_i, Z_j)
= g(\nabla Z_i, \nabla_{\hat{W}}\hat{W}, Z_j) - g(\nabla_{\hat{W}}\hat{W}, Z_i)g(Z_j, \nabla_{\hat{W}}\hat{W}),
\]
from which the lemma follows by rearrangement.

Notice that, using parallel transport, we can always construct a frame field from the above lemma.
Proposition 4.4. Let $M$ be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold $\overline{M}$ and let $\mathcal{F}$ be a foliation of $S(TM)$. Then
\[
\text{div}^\gamma(T_r \nabla^\gamma \hat{W}) = g(\text{div}^\gamma(T_r), \nabla^\gamma \hat{W}) + (-1)^{r+1} \hat{W}(S_{r+1})
\]
\[+ (-1)^{r+1} (-S_1 S_{r+1} + (r + 2) S_{r+2}) - \sum_{i=1}^{n} \epsilon_i g(\mathcal{R}(Z_i, \hat{W}) T_r Z_i, \hat{W})
\]
\[+ g(\nabla^\gamma \hat{W}, T_r \nabla^\gamma \hat{W}),
\]
where $\{Z_i\}$ is a field of frame tangent to the leaves of $\mathcal{F}$.

Proof. From (3.7), we deduce that
\[
\text{div}^\gamma(T_r, Z) = g(\text{div}^\gamma(T_r), Z) + \sum_{i=1}^{n} \epsilon_i g(\nabla^\gamma Z_i, T_r Z_i), \tag{4.9}
\]
for all $Z \in \Gamma(T\mathcal{F})$. Then using (4.9), Lemmas 4.2 and 4.3, we obtain the desired result. Hence the proof. \hfill \square

From Proposition 4.4 we have

Theorem 4.5. Let $M$ be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then
\[
\text{div}^\gamma(T_r \nabla^\gamma \hat{W}) = g(\text{div}^\gamma(T_r), \nabla^\gamma \hat{W}) + (-1)^{r+1} \hat{W}(S_{r+1})
\]
\[+ (-1)^{r+1} (-S_1 S_{r+1} + (r + 2) S_{r+2})
\]
\[+ g(\nabla^\gamma \hat{W}, T_r \nabla^\gamma \hat{W}),
\]

Proof. A proof follows easily from Proposition 4.4 by recognizing the fact that
\[
\text{div}^\gamma(T_r \nabla^\gamma \hat{W}) = \text{div}^\gamma(T_r, \nabla^\gamma \hat{W})
\]
\[= g(\nabla^\gamma \hat{W}, T_r \nabla^\gamma \hat{W}),
\]
which completes the proof. \hfill \square

Theorem 4.6. Let $M$ be a screen integrable half-lightlike submanifold of $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then,
\[
\text{div}^\gamma(T_r \nabla^\gamma \hat{W}) + (-1)^{r} S_{r+1} \hat{W} = g(\text{div}^\gamma(T_r), \nabla^\gamma \hat{W})
\]
\[+ (-1)^{r+1} (r + 2) S_{r+2} - \sum_{i=1}^{n} \epsilon_i g(\mathcal{R}(Z_i, \hat{W}) T_r Z_i, \hat{W}).
\]

Proof. By straightforward calculations we have
\[
S_1 = \text{tr}(A_{\hat{W}})
\]
\[= - \sum_{i=1}^{n} \epsilon_i g(\nabla Z_i, \hat{W}, Z_i)
\]
\[= - \sum_{i=1}^{n+1} \epsilon_i g(\nabla Z_i, \hat{W}, Z_i)
\]
\[= -\text{div}^\gamma(\hat{W}),
\]
where $Z_{n+1} = \hat{W}$. From this equation we deduce
\[
\text{div}^\gamma(S_{r+1} \hat{W}) = -S_1 S_{r+1} \hat{W} (S_{r+1}). \tag{4.10}
\]
Then from (4.10) and Theorem 4.5 we get our assertion, hence the proof. \hfill \square
Next, we let $dV$ denote the volume form $\mathcal{M}$. Then from Theorem 4.6 we have the following

**Corollary 4.7.** Let $M$ be a screen integrable half-lightlike submanifold of a compact semi-Riemannian manifold $\mathcal{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then
\[
\int_{\mathcal{M}} \overline{g}(\nabla^\mathcal{F}(T_r), \nabla^\mathcal{F}(\overline{W}))dV = \int_{\mathcal{M}} ((-1)^{r}(r + 2)S_{r+2} + \sum_{i=1}^{n} \epsilon_i \overline{g}(\overline{R}(Z_i, \overline{W})T_rZ_i, \overline{W}))dV.
\]

Setting $r = 0$ in the above corollary we get

**Corollary 4.8.** Let $M$ be a screen integrable half-lightlike submanifold of a compact semi-Riemannian manifold $\mathcal{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$ with mean curvatures $S_r$. Then for $r = 0$ we have
\[
\int_{\mathcal{M}} 2S_2dV = \int_{\mathcal{M}} \overline{Ric}(\overline{W}, \overline{W})dV,
\]
where $\overline{Ric}(\overline{W}, \overline{W}) = \sum_{i=1}^{n} \epsilon_i \overline{g}(\overline{R}(Z_i, \overline{W})\overline{W}, Z_i)$.

Notice that the equation in Corollary 4.8 is the lightlike analogue of (3.5) in [2] for co-dimension one foliations on Riemannian manifolds.

Next, we will discuss some consequences of the integral formulas developed so far.

A semi-Riemannian manifold $\mathcal{M}$ of constant sectional curvature $c$ is called a **semi-Riemannian space form** [4, 5] and is denoted by $\mathcal{M}(c)$. Then, the curvature tensor $\overline{R}$ of $\mathcal{M}(c)$ is given by
\[
\overline{R}(\overline{X}, \overline{Y})\overline{Z} = c\{\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y}\}, \quad \forall \overline{X}, \overline{Y}, \overline{Z} \in \Gamma(\mathcal{M}).
\]

**Theorem 4.9.** Let $M$ be a screen integrable half-lightlike submanifold of a compact semi-Riemannian space form $\mathcal{M}(c)$ of constant sectional curvature $c$. Let $\mathcal{F}$ be a co-dimension three foliation of its screen distribution $S(TM)$. If $V$ is the total volume of $\mathcal{M}$, then
\[
\int_{\mathcal{M}} S_r dV = \begin{cases} 
0, & r = 2k + 1, \\
c^2 \left(\frac{n}{2}\right)^{2r} V, & r = 2k,
\end{cases}
\]
for positive integers $k$.

**Proof.** By setting $\overline{X} = Z_i, \overline{Y} = \overline{W}$ and $Z = T_rZ_i$ in (4.11) we deduce that
\[
\overline{R}(Z_i, \overline{W})T_rZ_i = -c\overline{g}(Z_i, T_rZ_i)\overline{W}.
\]
Then substituting this equation in Corollary 4.7 we obtain
\[
\int_{\mathcal{M}} \overline{g}(\nabla^\mathcal{F}(T_r), \nabla^\mathcal{F}(\overline{W}))dV = \int_{\mathcal{M}} ((-1)^{r}(r + 2)S_{r+2} - c\overline{g}(T_r))dV.
\]
Since $\mathcal{M}$ is of constant sectional curvature $c$, then Lemma 4.2 implies that $T_r = 0$ for any $r$ and hence the above equation simplifies to
\[
(r + 2) \int_{\mathcal{M}} S_{r+2} dV = c(n - r) \int_{\mathcal{M}} S_r dV.
\]
Since $S_1 = -\nabla^\mathcal{F}((\overline{W})$ and that $\mathcal{M}$ is compact, then $\int_{\mathcal{M}} S_1 dV = 0$. Using this fact together with (4.13), mathematical induction gives $\int_{\mathcal{M}} S_r dV = 0$ for all $r = 2k + 1$ (i.e., $r$ odd).
For \( r \) even we will consider \( r = 2m \) and \( n = 2l \) (i.e., both \( M \) and \( \mathcal{M} \) are odd dimensional). With these conditions, (4.13) reduces to
\[
\int_{\mathcal{M}} S_{2m+2} dV = c\frac{l-m}{1+m} \int_{\mathcal{M}} S_{2m} dV.
\]
Now setting \( m = 0, 1, \cdots \) and \( S_0 = 1 \) in (4.14) we obtain
\[
\int_{\mathcal{M}} S_{2} dV = clV, \quad \int_{\mathcal{M}} S_{4} dV = c^2 \frac{(l-1)l}{2} V,
\]
and more generally
\[
\int_{\mathcal{M}} S_{2k} dV = c^k \frac{(l-k+1)(l-k+2)(l-k+3)\cdots l}{k!} V.
\]
Hence, our assertion follows from 4.15, which completes the proof. \( \square \)

Next, when \( \mathcal{M} \) is Einstein i.e., \( \mathcal{Ric} = \mu g \) we have the following.

**Theorem 4.10.** Let \( M \) be a screen integrable half-lightlike submanifold of an Einstein compact semi-Riemannian manifold \( \mathcal{M} \). Let \( \mathcal{F} \) be a co-dimension three foliation of its screen distribution \( S(TM) \) with totally umbilical leaves. If \( V \) is the total volume of \( \mathcal{M} \), then
\[
\int_{\mathcal{M}} S_r dV = \begin{cases} 0, & r = 2k + 1, \\ \left(\frac{\mu}{n}\right)^{\frac{n}{2}} \left(\frac{\mu}{2}\right)^{\frac{n}{2}} V, & r = 2k, \end{cases}
\]
for positive integers \( k \).

**Proof.** Suppose that \( A_N X = \frac{1}{n} S_r \mathbb{I} \). Then by direct calculations using the formula for \( T_r \) we derive \( T_r = (-1)^{r+1} \frac{(n-r)}{n} S_r \mathbb{I} \). Then, under the assumptions of the theorem we obtain \( \mathcal{Ric}(\mathcal{W}, \nabla_{\mathcal{W}} \mathcal{W}) = 0 \) and \( \mathcal{Ric}(\mathcal{W}, \mathcal{W}) = \mu \) and hence, Corollary 4.7 reduces to
\[
n(r + 2) \int_{\mathcal{M}} S_{r+2} dV = \lambda(n-r) \int_{\mathcal{M}} S_r dV.
\]
Notice that (4.17) is similar to (4.13) and hence following similar steps as in the previous theorem we get \( \int_{\mathcal{M}} S_r dV = 0 \) for \( r \) odd and for \( r \) even we get
\[
\int_{\mathcal{M}} S_{2k} dV = \left(\frac{\mu}{n}\right)^{\frac{n}{2}} \frac{(l-k+1)(l-k+2)(l-k+3)\cdots l}{k!} V,
\]
which complete the proof. \( \square \)

**5. Screen umbilical half-lightlike submanifolds**

In this section we consider totally umbilical half-lightlike submanifolds of semi-Riemannian manifold, with a totally umbilical screen distribution and thus, give a generalized version of Theorem 4.3.7 of [5] and its Corollaries, via Newton transformations of the operator \( A_N \).

A screen distribution \( S(TM) \) of a half-lightlike submanifold \( M \) of a semi-Riemannian manifold \( \mathcal{M} \) is said to be totally umbilical [5] if on any coordinate neighborhood \( \mathbb{U} \) there exist a function \( K \) such that
\[
C(X, PY) = Kg(X, PY), \quad \forall X, Y \in \Gamma(TM).
\]
In case \( K = 0 \), we say that \( S(TM) \) is totally geodesic. Furthermore, if \( S(TM) \) is totally umbilical then by straightforward calculations we have
\[
A_N X = PX, \quad C(E, PX) = 0, \quad \forall X \in \Gamma(TM).
\]
Let \{E, Z_i\}, for \(i = 1, \cdots, n\), be a quasi-orthonormal frame field of \(TM\) which diagonalizes \(A_N\). Let \(l_0, l_1, \cdots, l_n\) be the respective eigenvalues (or principal curvatures). Then as before, the \(r\)-th mean curvature \(S_r\) is given by

\[
S_r = \sigma_r(l_0, \cdots, l_n) \quad \text{and} \quad S_0 = 1.
\]

The characteristic polynomial of \(A_N\) is given by

\[
\det(A_N - t\mathbb{I}) = \sum_{\alpha=0}^{n} (-1)^\alpha S_\alpha t^{n-\alpha},
\]

where \(\mathbb{I}\) is the identity in \(\Gamma(TM)\). The normalized \(r\)-th mean curvature \(H_r\) of \(M\) is defined by \(\binom{n}{r} H_r = S_r\) and \(H_0 = 1\). The Newton transformations \(T_r : \Gamma(TM) \to \Gamma(TM)\) of \(A_N\) are given by the inductive formula

\[
T_0 = \mathbb{I}, \quad T_r = (-1)^r S_r \mathbb{I} + A_N \circ T_{r-1}, \quad 1 \leq r \leq n.
\]  

By Cayley-Hamilton theorem, we have \(T_{n+1} = 0\). Also, \(T_r\) satisfies the following properties.

\[
\begin{align*}
\text{tr}(T_r) &= (-1)^r (n + 1 - r)S_r, \quad (5.4) \\
\text{tr}(A_N \circ T_r) &= (-1)^r (r + 1)S_{r+1}, \quad (5.5) \\
\text{tr}(A_N \circ T_r) &= (-1)^{r+1} (-S_1S_{r+1} + (r + 2)S_{r+2}), \quad (5.6) \\
\text{tr}(T_r \circ \nabla_X A_N) &= (-1)^r X(S_{r+1}), \quad (5.7)
\end{align*}
\]

for all \(X \in \Gamma(TM)\).

**Proposition 5.1.** Let \((M, g)\) be a totally umbilical half-lightlike submanifold of a semi-Riemannian manifold \(\mathcal{M}\) of constant sectional curvature \(c\). Then

\[
g(\text{div}^\nabla(T_r, X) = (-1)^{r-1} \lambda(X) E(S_r) - \tau(X) \text{tr}(A_N \circ T_{r-1})
\]

\[
- c\lambda(X) \text{tr}(T_{r-1}) + g(\text{div}^\nabla(T_{r-1}, A_N X) + g((\nabla E A_N)T_{r-1} E, X)
\]

\[
+ \sum_{i=1}^{n} \epsilon_i \{-\lambda(X) B(Z_i, A_N(T_{r-1}Z_i))
\]

\[
+ \varepsilon \tau(Z_i) C(X, T_{r-1}Z_i) \{\rho(X) D(Z_i, T_{r-1}Z_i) - \rho(Z_i) D(X, T_{r-1}Z_i)\}\},
\]

for any \(X \in \Gamma(TM)\).

**Proof.** From the recurrence relation (5.3), we derive

\[
g(\text{div}^\nabla(T_r, X) = (-1)^{r} PX(S_r) + g((\nabla E A_N)T_{r-1} E, X)
\]

\[
+ g(\text{div}^\nabla(T_{r-1}, A_N X) + \sum_{i=1}^{n} \epsilon_i g((\nabla Z_i A_N)T_{r-1}Z_i, X), \quad (5.8)
\]

for any \(X \in \Gamma(TM)\). But

\[
g((\nabla Z_i A_N)T_{r-1}Z_i, X) = g(T_{r-1}Z_i, (\nabla Z_i A_N) X) + g((\nabla Z_i A_N)T_{r-1}Z_i, X)
\]

\[
- g((\nabla Z_i A_N) T_{r-1}Z_i) + g(A_N(\nabla Z_i X), T_{r-1}Z_i)
\]

\[
- g(A_N(\nabla Z_i T_{r-1}Z_i), X), \quad (5.9)
\]

for all \(X \in \Gamma(TM)\). \(\square\)

Then applying (2.9) to (5.9) while considering the fact that \(A_N\) is screen-valued, we get

\[
g((\nabla Z_i A_N)T_{r-1}Z_i, X) = g(T_{r-1}Z_i, (\nabla Z_i A_N) X) - \lambda(X) B(Z_i, A_N(T_{r-1}Z_i)). \quad (5.10)
\]
Furthermore, using (2.15) and (4.11), the first term on the right hand side of (5.10) reduces to
\[ g(T_{r-1}Z_i, (\nabla Z, A_N)X) = -c\lambda(X)g(Z_i, T_{r-1}Z_i) + g(\nabla_X A_N)Z_i, T_{r-1}Z_i) \]
\[ - \tau(X)C(Z_i, T_{r-1}Z_i) + c\tau(Z_i)C(X, T_{r-1}Z_i)(\rho(X)D(Z_i, T_{r-1}Z_i) \]
\[ - \rho(X)D(X, T_{r-1}Z_i), \]
for any \( X \in \Gamma(TM) \). Finally, replacing (5.11) in (5.10) and then put the resulting equation in (5.8) we get the desired result.

Next, from Proposition 5.1 we have the following.

**Theorem 5.2.** Let \((M, g)\) be a half-lightlike submanifold of a semi-Riemannian manifold \(\overline{M}(c)\) of constant curvature \(c\) with a proper totally umbilical screen distribution \(S(TM)\). If \(M\) is also totally umbilical, then the \(r\)-th mean curvature \(S_r\), for \(r = 0, 1, \ldots, n\), with respect to \(A_N\) are solution of the following differential equation
\[ E(S_r+1) - \tau(E)(r + 1)S_{r+1} - c(-1)^r(n + 1 - r)S_r = \mathcal{H}_1(r + 1)S_{r+1}. \]

**Proof.** Replacing \(X\) with \(E\) in the Proposition 5.1 and then using (2.16) and (5.2) we obtain, for all \(r = 0, 1, \ldots, n\),
\[ E(S_r+1) - (-1)^r\tau(E)\text{tr}(A_N \circ T_r) - c(-1)^r\text{tr}(T_r) = (-1)^r\mathcal{H}_1\text{tr}(A_N \circ T_r), \]
from which the result follows by applying (5.4) and (5.5).

**Corollary 5.3.** Under the hypothesis of Theorem 5.2, the induced connection \(\nabla\) on \(M\) is a metric connection, if and only if, the \(r\)-th mean curvature \(S_r\) with respect to \(A_N\) are solution of the following equation
\[ E(S_r+1) - \tau(E)(r + 1)S_{r+1} - c(-1)^r(n + 1 - r)S_r = 0. \]

Also the following holds.

**Corollary 5.4.** Under the hypothesis of Theorem 5.2, \(\overline{M}(c)\) is a semi-Euclidean space, if and only if, the \(r\)-th mean curvature \(S_r\) with respect to \(A_N\) are solution of the following equation
\[ E(S_r+1) - \tau(E)(r + 1)S_{r+1} = \mathcal{H}_1(r + 1)S_{r+1}. \]

Notice that Theorem 5.2 and Corollary 5.3 are generalizations of Theorem 4.3.7 and Corollary 4.3.8, respectively, given in [5].

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**References**


