Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (3) (2020), 974 – 983
DOI : 10.15672/hujms.474938

Research Article

A numerical algorithm for solving the Cauchy singular integral equation based on Hermite polynomials

B. Parsa Moghaddam*[1], J.A. Tenreiro Machado[2], Parisa Sattari Shajari[1], Zeynab Salamat Mostaghim[3]

[1] *Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran*
[2] *Department of Electrical Engineering, Institute of Engineering, Rua Dr. António Bernardino de Almeida, 431, Porto 4249-015, Portugal*
[3] *Young Researchers and Elite Club, Lahijan Branch, Islamic Azad University, Lahijan, Iran*

Abstract

A numerical algorithm based on Hermite polynomials for solving the Cauchy singular integral equation in the general form is presented. The Hermite polynomial interpolation of unknown functions is first introduced. The proposed technique is then used for approximating the solution of the Cauchy singular integral equation. This approach requires the solution of a system of linear algebraic equations. Two examples demonstrate the effectiveness of the proposed method.

Mathematics Subject Classification (2010). 30E20, 97N50, 65M70

Keywords. Cauchy singular integral equation, Hermite polynomials, numerical schemes, interpolation theory

1. Introduction

Singular integrals of various types arise when simulating the physical behavior of complex engineering systems. That is also the case of fractional calculus that relies on singular integrals and became an essential topic in the study of phenomena in various disciplines [12,13,19–21,27]. Moreover, many initial and boundary value problems can be casted into solving singular integrals. For example, the problem of surface water wave scattering by a thin vertical barrier, that occurs in the linearised theory of water waves, can be reduced to a homogeneous singular integral equation with Cauchy kernel [6].
In this paper, we consider the Cauchy singular integral equation (CSIE) as follows

$$a(x)w(x)\varphi(x) + b(x)\int_{\alpha}^{\beta} \frac{w(t)\varphi(t)}{t-x}dt - \int_{\alpha}^{\beta} k(x,t)w(t)\varphi(t)dt = f(x), \tag{1.1}$$

where $\alpha < x < \beta$ and $a(x)$, $b(x)$ and $f(x)$ are known real functions. The function $k(x,t)$ is the kernel of the integral equation, $\varphi(t)$ denotes an unknown function and $w(t)$ represents the known weight function. The kernel function is assumed to be continuous and square

*Corresponding Author.
Email addresses: parsa@liau.ac.ir (B.P. Moghaddam), jtm@isep.ipp.pt (J.A.T. Machado), pariamath306@yahoo.com (P.S. Shajari), salamatmostaghim93@liau.ac.ir (Z.S. Mostaghim)
Received: 26.10.2018; Accepted: 11.06.2019

integrable. For $a(x) = 0$ we have a first kind integral equation of (1.1). Otherwise, it is an integral equation of the second kind. This type of integral equation was discussed in [22, 25]. The CSIE (1.1) has several applications, such as the mixed boundary value problem, the elasticity for cracked media, or the solution of contact problem in solid mechanics [6].

Analytical schemes for obtaining the solutions of these problems were proposed for special cases [7, 9, 18]. However, often we need a general numerical method to solve the CSIE (1.1). The single Cauchy kernel problem can be transformed into the Fredholm type integral equation with singular kernel, and may be solved using conventional schemes [1, 3–5, 8, 10, 11, 16, 17, 23, 26, 28].

This paper applies the Hermite interpolation procedure to solve the CSIE (1.1) and is organized as follows. Section 2 introduces the properties of Hermite polynomials. Section 3 develops the numerical technique for solving the CSIE (1.1) in the general form. Section 4 presents several test problems, comparing the numerical and exact solutions, to assess the accuracy and applicability of the proposed technique. Finally, Section 5 highlights the main conclusions.

2. Hermite polynomials interpolation

Let $x_0, x_1, \ldots, x_n$ be real node points. The interpolation conditions at each node x_i, $i = 0, 1, \ldots, n$, for Hermite interpolation are as defined

$$P^{(j)}(x_i) = c_{ij}, \tag{2.1}$$

for $j = 0, 1, \cdots, k_i - 1$ and $i = 0, 1, \cdots, n$. Hence, the total number of conditions for this interpolation procedure is $m + 1 = k_0 + k_1 + \ldots + k_n$. Assume that Π_m is the space of all polynomials of degree at most m. Then, the following theorem guarantees the existence and uniqueness of such interpolation polynomial and its proof is given in [14].

Theorem 2.1. There exists a unique polynomial P in Π_m fulfilling the interpolation conditions in equation (2.1). We can write the Lagrange form of the Hermite interpolation polynomial. Let $x_0, x_1, ..., x_n$ be distinct nodes in $[a, b]$. The Hermite polynomial of degree $2n + 1$ such that

$$H_{2n+1}(x_i) = f(x_i),\ H'_{2n+1}(x_i) = f'(x_i), i = 0, 1, ..., n, \tag{2.2}$$

is given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)h_j(x) + \sum_{j=0}^{n} f'(x_j)g_j(x), \tag{2.3}$$

where $h_j(x) = L_j^2(x)(1 - 2(x - x_j)L'_j(x))$ and $g_j(x) = (x - x_j)L_j^2(x)$ with the convention that $L_j(x)$ represents Lagrange polynomial.

The convergence and norm estimates of the Hermite interpolation at the zeros of the Chebyshev polynomials are investigated by Al-Khaled and Alquran [2].

3. The proposed numerical scheme

In this section we use the properties of the Hermite polynomial interpolation to solve the CSIE (1.1). Let us consider the unknown function $\varphi(t) \in C^1([\alpha, \beta])$, where α and β are real numbers, and that $C^1([\alpha, \beta])$ denotes the set of all continuously differentiable functions on the interval $[\alpha, \beta]$. For approximating $\varphi(t)$, we divide the interval $[\alpha, \beta]$, into n partitions and we assume that $t_0, t_1, ..., t_n$ are the distinct interpolation points where $\alpha = t_0 < t_1 < ... < t_n = \beta$. Now let x be an arbitrary point in $[\alpha, \beta]$ that differs from

$t_0, t_1, ..., t_n$. Assume that Hermite interpolation formula of $\varphi(t)$, at $t_1, t_2, \cdots, t_n$ and x, is given by

$$\begin{aligned} H_{2n+1}(t) &= \sum_{j=1}^{N} \varphi(t_j) L_j^2(t)(1 - 2(t - t_j)L_j'(t_j)) + \sum_{j=1}^{N} \varphi'(t_j)(t - t_j)L_j^2(t) \\ &\quad + \varphi(x)L_{N+1}^2(t)(1 - 2(t - x)L_{N+1}'(x)) + \varphi'(x)(t - x)L_{N+1}^2(t). \end{aligned} \tag{3.1}$$

Let $\sigma_N(t) = \prod_{i=1}^{N}(t - t_j)$. Consequently, we have $\sigma_N'(t_j) = \prod_{i=1, i\neq j}^{N}(t_j - t_i)$. In terms of $\sigma_N(t)$, the Lagrange polynomial will be given as

$$L_j(t) = \frac{\sigma_N(t)(t - x)}{\sigma_N'(t_j)(t - t_j)(t_j - x)},\ j = 1, ..., N. \tag{3.2}$$

We now approximate solution of CSIE (1.1) in two steps. First let us define

$$(S\varphi)(x) = \int_{\alpha}^{\beta} \frac{w(t)\varphi(t)}{t - x} dt,\ \alpha < x < \beta. \tag{3.3}$$

Replacing the unknown function φ by its Hermite interpolation H_{2N+1} in the above relation, one obtains

$$\begin{aligned} (S_N\varphi)(x) &= (S\varphi_N)(x) = (SH_{2N+1})(x) = \int_{\alpha}^{\beta} \frac{w(t)H_{2N+1}(t)}{t - x} dt \\ &= \sum_{j=0}^{N} \varphi(t_j) \int_{\alpha}^{\beta} \frac{w(t)}{t - x} L_j^2(t)(1 - 2(t - t_j)L_j'(t_j))dt \\ &\quad + \sum_{j=0}^{N} \varphi'(t_j) \int_{\alpha}^{\beta} \frac{w(t)}{t - x}(t - t_j)L_j^2(t)dt \\ &\quad + \varphi(x) \int_{\alpha}^{\beta} \frac{w(t)}{t - x} L_{N+1}^2(t)(1 - 2(t - x)L_{N+1}'(x))dt \\ &\quad + \varphi'(x) \int_{\alpha}^{\beta} \frac{w(t)}{t - x}(t - x)L_{N+1}^2(t). \end{aligned} \tag{3.4}$$

For the first-term, we have

$$\begin{aligned} &\sum_{j=0}^{N} \varphi(t_j) \int_{\alpha}^{\beta} \frac{w(t)}{t - x} L_j^2(t)(1 - 2(t - t_j)L_j'(t_j))dt \\ &= \sum_{j=0}^{N} \varphi(t_j) \int_{\alpha}^{\beta} \frac{w(t)L_j^2(t)}{t - x} dt - 2\sum_{j=0}^{N} \varphi(t_j) \int_{\alpha}^{\beta} \frac{w(t)}{t - x} L_j^2(t)(t - t_j)L_j'(t_j)dt. \end{aligned} \tag{3.5}$$

Then

$$\begin{aligned} \sum_{j=0}^{N} \varphi(t_j) \int_{\alpha}^{\beta} \frac{w(t)L_j^2(t)}{t - x} dt &= \sum_{j=1}^{N} \varphi(t_j) \int_{\alpha}^{\beta} \frac{w(t)\sigma_N^2(t)(t - x)}{\sigma_N'^2(t_j)(t - t_j)^2(t_j - x)^2} dt \\ &= \sum_{j=1}^{N} \varphi(t_j)w_j(x),\ x \neq t_j, \end{aligned} \tag{3.6}$$

where

$$w_j(x) = \int_{\alpha}^{\beta} \frac{w(t)\sigma_N^2(t)(t - x)}{\sigma_N'^2(t_j)(t - t_j)^2(t_j - x)^2} dt,\ x \neq t_j. \tag{3.7}$$

Also, we have

$$2\sum_{j=0}^{N}\varphi(t_j)\int_{\alpha}^{\beta}\frac{w(t)}{t-x}L_j^2(t)(t-t_j)L_j'(t_j)dt = \sum_{j=1}^{N}2\varphi(t_j)L_j'(t_j)\int_{\alpha}^{\beta}\frac{w(t)}{t-x}L_j^2(t)(t-t_j)dt$$
$$= \sum_{j=1}^{N}2\varphi(t_j)L_j'(t_j)(S(L_j^2(t)(t-t_j)))(x), \quad (3.8)$$

where

$$(S(L_j^2(t)(t-t_j)))(x) = \int_{\alpha}^{\beta}\frac{w(t)}{t-x}L_j^2(t)(t-t_j)dt, \; x \neq t_j. \quad (3.9)$$

Using (3.6) and (3.8), formula (3.5) can finally be expressed as

$$\sum_{j=0}^{N}\varphi(t_j)\int_{\alpha}^{\beta}\frac{w(t)}{t-x}L_j^2(t)(1-2(t-t_j)L_j'(t_j))dt$$
$$= \sum_{j=1}^{N}\varphi(t_j)(w_j(x)-2L_j'(t_j)(S(L_j^2(t)(t-t_j)))(x)$$
$$= \sum_{j=1}^{N}\varphi(t_j)K_j(x), \; x\neq t_j, \; j=1,2,\ldots,N, \quad (3.10)$$

where

$$K_j(x) = w_j(x) - 2L_j(t_j)(S(L_j^2(t)(t-t_j)))(x), j=1,2,\ldots,N. \quad (3.11)$$

Similarly, for the second-term of (3.4), one can get

$$\sum_{j=0}^{N}\varphi'(t_j)\int_{\alpha}^{\beta}\frac{w(t)}{t-x}(t-t_j)L_j^2(t)dt = \sum_{j=1}^{N}\varphi'(t_j)Z_j(x), \; x\neq t_j, \; j=1,2,...,N, \quad (3.12)$$

where

$$Z_j(x) = \int_{\alpha}^{\beta}\frac{w(t)\sigma_N^2(t)(t-x)}{\sigma_N'(t_j)(t-t_j)(t_j-x)^2}dt, \; x\neq t_j. \quad (3.13)$$

The third-term of (3.4) can be written as

$$\varphi(x)\int_{\alpha}^{\beta}\frac{w(t)}{t-x}L_{N+1}^2(t)(1-2(t-x)L_{N+1}'(x))dt$$
$$= \varphi(x)\int_{\alpha}^{\beta}\frac{w(t)}{t-x}L_{N+1}^2(t)dt - 2\varphi(x)\int_{\alpha}^{\beta}w(t)L_{N+1}'(x)dt. \quad (3.14)$$

Thus

$$\varphi(x)\int_{\alpha}^{\beta}\frac{w(t)}{t-x}L_{N+1}^2(t)dt = \frac{\varphi(x)}{\sigma_N^2(x)}(S\sigma_N^2)(x), \quad (3.15)$$

where $(S\sigma_N^2(x))(x) = \int_{\alpha}^{\beta}\frac{w(t)\sigma_N^2(t)}{t-x}dt$. In addition,

$$2\varphi(x)\int_{\alpha}^{\beta}w(t)L_{N+1}'(x)dt = 2C\varphi(x)L_{N+1}'(x), \quad (3.16)$$

where $C = \int_{\alpha}^{\beta}w(t)dt$.

Then, using (3.15) and (3.16), formula (3.14) can be rewritten as follows

$$\varphi(x)\int_{\alpha}^{\beta}\frac{w(t)}{t-x}L_{N+1}^2(t)(1-2(t-x)L_{N+1}'(x))dt$$
$$= \frac{\varphi(x)}{\sigma_N^2(x)}(S\sigma_N^2)(x) - 2C\varphi(x)L_{N+1}'(x). \quad (3.17)$$

Now we want to compute the fourth-term of (3.4). In this case a direct calculation yields

$$\varphi'(x)\int_\alpha^\beta \frac{w(t)}{t-x}(t-x)L_{N+1}^2(t) = D\frac{\varphi'(x)}{\sigma_N^2}, \tag{3.18}$$

so that $D = \int_\alpha^\beta w(t)\sigma_N^2(t)dt$. From the above calculations, one can get the following formula for approximating the integral

$$\begin{aligned}(S_N\varphi) = \sum_{j=1}^{N}\varphi(t_j)K_j(x) \quad &+ \quad \sum_{j=1}^{N}\varphi'(t_j)Z_j(x)\\ &+ \quad \varphi(x)(\frac{(S\sigma_N^2)(x)}{\sigma_N^2(x)} - 2CL'_{N+1}(x)) + D\frac{\varphi'(x)}{\sigma_N^2(x)}.\end{aligned} \tag{3.19}$$

In the next step, we approximate the term $\int_\alpha^\beta k_0(x,t)w(t)\varphi(t)dt$ of the CSIE (1.1). Let $Q_{2N+1}(t)$ be a Hermite polynomials interpolation of $k_0(x,t)\varphi(t)$ at the nodes $t_1, t_2, ..., t_N$. Then, we have

$$\begin{aligned}Q_{2N+1}(t) \quad = \quad &\sum_{j=1}^{N} k_0(x,t_j)\varphi(t_j)L_j^2(t)(1-2(t-t_j)L'_j(t_j))\\ &+\sum_{j=1}^{N}(\frac{\partial k_0}{\partial t}\varphi(t_j) + k_0(x,t_j)\varphi'(t_j))(t-t_j)L_j^2(t).\end{aligned} \tag{3.20}$$

Substituting relation (3.20) into $\int_\alpha^\beta k_0(x,t)w(t)\varphi(t)dt$, it results

$$\begin{aligned}&\int_\alpha^\beta k_0(x,t)w(t)\varphi(t)dt\\ &= \quad \int_\alpha^\beta w(t)\sum_{j=1}^{N} k_0(x,t_j)\varphi(t_j)L_j^2(t)(1-2(t-t_j)L'_j(t_j))\\ &\quad + \int_\alpha^\beta w(t)\sum_{j=1}^{N}\left(\frac{\partial k_0}{\partial t}(x,t_j)\varphi(t_j) + k_0(x,t_j)\varphi'(t_j)\right)(t-t_j)L_j^2(t).\end{aligned} \tag{3.21}$$

We have

$$\begin{aligned}&\int_\alpha^\beta w(t)\sum_{j=1}^{N} k_0(x,t_j)\varphi(t_j)L_j^2(t)(1-2(t-t_j)L'_j(t_j))\\ &= \quad \sum_{j=1}^{N}\varphi(t_j)\int_\alpha^\beta k_0(x,t_j)w(t)L_j^2(t)dt - \sum_{j=1}^{N}\varphi(t_j)\int_\alpha^\beta 2k_0(x,t_j)w(t)L_j^2(t-t_j)L'_j(t_j)dt\\ &= \quad \sum_{j=1}^{N}(w_j^{(0)}(x) - w_j^{(1)}(x))\varphi(t_j)\end{aligned} \tag{3.22}$$

with

$$\begin{aligned}w_j^{(0)}(x) \quad &= \quad \int_\alpha^\beta k_0(x,t_j)w(t)L_j^2(t)dt,\\ w_j^{(1)}(x) \quad &= \quad 2\int_\alpha^\beta k_0(x,t_j)w(t)L_j^2(t)(t-t_j)L'_j(t_j)dt,\end{aligned} \tag{3.23}$$

and

$$\int_\alpha^\beta w(t)\sum_{j=1}^N\left(\frac{\partial k_0}{\partial t}(x,t_j)\varphi(t_j)+k_0(x,t_j)\varphi'(t_j)\right)(t-t_j)L_j^2(t)$$

$$\begin{aligned}&= \sum_{j=1}^N\varphi(t_j)\int_\alpha^\beta\frac{\partial k_0}{\partial t}(x,t_j)w(t)(t-t_j)L_j^2(t)dt+\sum_{j=1}^N\varphi'(t_j)\int_\alpha^\beta k_0(x,t_j)w(t)(t-t_j)L_j^2(t)dt\\ &= \sum_{j=1}^N\varphi(t_j)w_0^{(2)}(x)+\sum_{j=1}^N\varphi'(t_j)w_j^{(3)}(x),\end{aligned}\tag{3.24}$$

where

$$\begin{aligned}w_j^{(2)}(x) &= \int_\alpha^\beta\frac{\partial k_0}{\partial t}(x,t_j)w(t)(t-t_j)L_j^2(t)dt,\ x\neq t_j,\\ w_j^{(3)}(x) &= \int_\alpha^\beta k_0(x,t_j)w(t)(t-t_j)L_j^2(t)dt,\ x\neq t_j.\end{aligned}\tag{3.25}$$

Thus, we have

$$\int_\alpha^\beta k_0(x,t)w(t)\varphi(t)dt=\sum_{j=1}^N w_j^{(4)}(x)\varphi(t_j)+\sum_{j=1}^N w_j^{(3)}(x)\varphi'(t_j),\tag{3.26}$$

where

$$w_j^{(4)}(x)=w_j^{(0)}(x)-w_j^{(1)}(x)+w_j^{(2)}(x).\tag{3.27}$$

Substituting (3.15) and (3.22) into CSIE (1.1) it yields

$$\begin{aligned}b(x)\sum_{j=1}^N K_j(x)\varphi(t_j) &+ b(x)\sum_{j=1}^N Z_j(x)\varphi'(t_j)-\sum_{j=1}^N w_j^{(4)}(x)\varphi(t_j)-\sum_{j=1}^N w_j^{(3)}(x)\varphi'(t_j)\\ &+ \left\{a(x)w(x)+b(x)\left(\frac{(S\sigma_N^2)(x)}{\sigma_N^2(x)}-2C\right)\right\}\varphi(x)\\ &+ D\frac{b(x)\varphi'(x)}{\sigma_N^2(x)}=f(x),\ x\neq t_j,\ j=1,2,\ldots,N.\end{aligned}\tag{3.28}$$

Choosing $x_i,\ i=1,2,\ldots,2N$, such that $x_i\neq t_j$, for $j=1,2,\cdots,N$, we obtain the following system of linear equations

$$\begin{aligned}b(x_i)\sum_{j=1}^N K_j(x_i)\varphi(t_j) &+ b(x_i)\sum_{j=1}^N Z_j(x_i)\varphi'(t_j)-\sum_{j=1}^N w_j^{(4)}(x_i)\varphi(t_j)-\sum_{j=1}^N w_j^{(3)}(x_i)\varphi'(t_j)\\ &+ \left\{a(x_i)w(x_i)+b(x_i)\left(\frac{(S\sigma_N^2)(x_i)}{\sigma_N^2(x_i)}-2C\right)\right\}\varphi(x_i)\\ &+ \frac{Db(x_i)\varphi'(x_i)}{\sigma_N^2(x_i)}=f(x_i),\qquad i=1,\ldots,2N.\end{aligned}\tag{3.29}$$

Solving (3.29) we can obtain the unknowns $\varphi(t_j)$ and $\varphi'(t_j)$, $j=1,2,\ldots,N$.

4. Illustrative numerical examples

In this section we illustrate of the proposed scheme and assess its feasibility. We consider two examples for which the analytical solution is known. The performance of the suggested scheme is analyzed in the perspective of the percentage absolute error (PAE) defined as:

$$\text{PAE}=\frac{AE}{|\text{Exact solution}|}\times 100\%,\tag{4.1}$$

where AE is the absolute error.

Example 4.1. In this example we choose the kernel function $k(x,t) = -\frac{1}{t+x+6}$, the weight function $w(t) = \frac{1}{1-t}$, and the values $\alpha = -1$ and $\beta = 1$. Considering $a(x) = 1$ and $b(x) = -\frac{1}{\pi}$ we have the following singular integral equation:

$$\frac{\varphi(x)}{1-x} - \frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{(t-x)(1-t)}dt + \int_{-1}^{1}\frac{\varphi(t)}{(t+x+6)(1-t)}dt = f(x), \tag{4.2}$$

where

$$f(x) = \left(\frac{\sqrt{2}}{5}\right)\left(\frac{3}{5}\right)^{\frac{-1}{4}}\left(\frac{1}{x+4} + \frac{\pi}{x+2}\right) - \frac{\pi\sqrt{2}}{x+2}\left(\frac{x+7}{x+5}\right)^{\frac{1}{4}}.$$

It is known that the exact solution of (4.12) is $\varphi(t) = \frac{1}{t+4}$ [15]. We choose the node points $t_i = \cos\left(\frac{\pi i}{M+1}\right)$, $i = 1, 2, \cdots, M$, and the points x_i as the zeros of the Chebyshev and Legendre polynomials, respectively. After obtaining the values of $\varphi(t_j)$ and $\varphi'(t_j)$, $j = 1, 2, \ldots, N$, by means of the Hermite interpolation formula, we can approximate solution of (4.2). Figure 1 shows that approximating solutions of (4.2) with magnitude of the AE, in the interval $[-1, 1]$ for $M = N = 15$. The PAE of (4.2) for $M = N = 2$ and $M = N = 4$ are reported in Table 1.

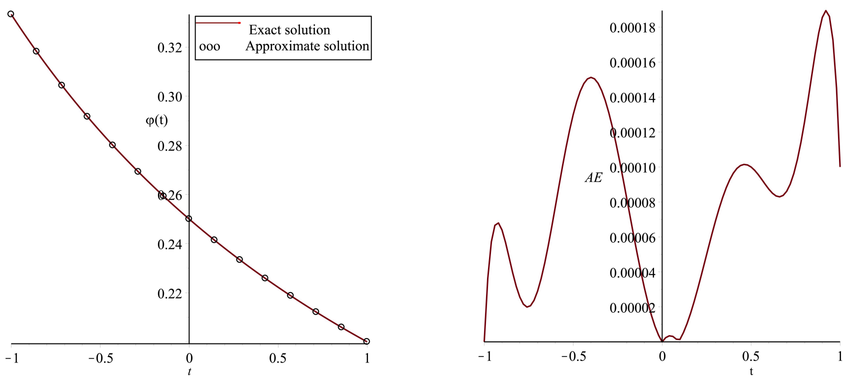

Figure 1. Example 4.1: (left) comparison of the exact and the numerical approximations, (right) magnitude of the AE, with proposed scheme for $M = N = 15$.

Table 1. Example 4.1: Comparison of the PAE, for different numbers of points in the interval $[-1, 1]$ and two identical values of M and N.

	PAE	
t_k	$M = N = 2$	$M = N = 4$
-1.0	2.400×10^{-3}	1.035×10^{-4}
-0.5	–	2.618×10^{-4}
0.0	1.251×10^{-3}	2.011×10^{-4}
0.5	–	1.125×10^{-4}
1.0	2.500×10^{-3}	2.469×10^{-4}

Example 4.2. In this example we consider $a(x) = 0$, $b(x) = 1$, $k(x,t) = 0$, $f(x) = x^4 + 5x^3 + 2x^2 + x - \frac{11}{8}$ and the weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$. Thus, the singular integral equation is given by

$$\int_{-1}^{1}\frac{\varphi(t)}{(t-x)\sqrt{1-t^2}}dt = x^4 + 5x^3 + 2x^2 + x - \frac{11}{8}, \quad -1 < x < 1. \tag{4.3}$$

The exact solution of (4.3) is $\varphi(t) = \dfrac{1}{\pi}\left(t^5 + 5t^4 + \dfrac{3}{2}t^3 - \dfrac{3}{2}t^2 - \dfrac{5}{2}t - \dfrac{9}{8}\right)$ [24].

Likewise to the previous example, Figure 2 presents the exact solution of (4.3), its numerical approximation by means of the proposed algorithm and the magnitude of the AE in the interval $[-1, 1]$ with $M = N = 15$. Moreover, The numerical results of (4.3) are listed in Table 2.

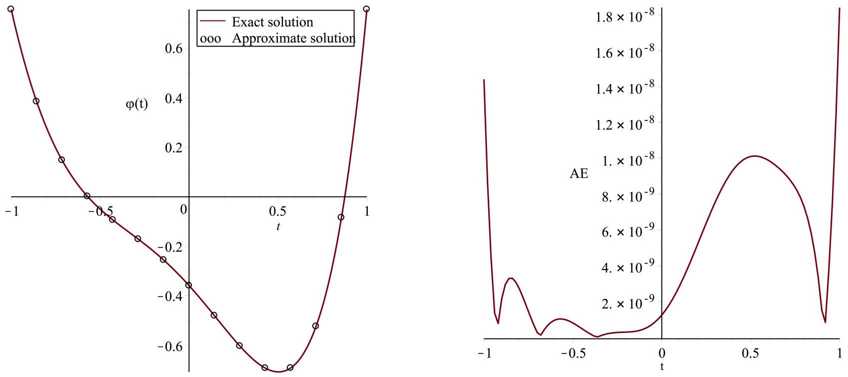

Figure 2. Example 4.2: (left) comparison of the exact and the numerical approximations, (right) magnitude of the AE, with proposed scheme for $M = N = 15$.

Table 2. Example 4.2: Comparison of the PAE, for different numbers of points in the interval $[-1, 1]$ and two identical values of M and N.

	PAE	
t_k	$M = N = 2$	$M = N = 4$
-1.0	3.994×10^{-3}	1.587×10^{-4}
-0.5	–	1.022×10^{-4}
0.0	2.091×10^{-3}	1.040×10^{-4}
0.5	–	1.174×10^{-4}
1.0	1.587×10^{-3}	2.709×10^{-4}

5. Conclusion

In this paper an important class of one-dimensional singular integral equations was considered. A numerical scheme based on the Hermite polynomial interpolation was proposed to solve the general form of the Cauchy singular integral equation. The numerical results for two examples show that proposed numerical algorithm is an accurate and reliable technique.

References

[1] M.A. Abdou and A.A. Nasr, *On the numerical treatment of the singular integral equation of the second kind*, Appl. Math. Comput. **143** (2-3), 373–380, 2003.

[2] K. Al-Khaled and M. Alquran, *Convergence and norm estimates of Hermite interpolation at zeros of Chevyshev polynomials*, SpringerPlus, **5** (1), 2016.

[3] A.A. Badr, *Integro-differential equation with Cauchy kernel*, J. Comput. Appl. Math. **134** (1-2), 191–199, 2001.

[4] P. Baratella and A.P. Orsi, *A new approach to the numerical solution of weakly singular Volterra integral equations*, J. Comput. Appl. Math. **163** (2), 401–418, 2004.

[5] S. Bhattacharya and B.N. Mandal, *Numerical solution of a singular integro-differential equation*, Appl. Math. Comput. **195** (1), 346–350, 2008.
[6] A. Chakrabarti, *Applied Singular Integral Equations*, Science Publishers, 2011.
[7] Z. Chen and Y. Lin, *The exact solution of a linear integral equation with weakly singular kernel*, J. Math. Anal. Appl. **344** (2), 726–734, 2008.
[8] S.M.A. Darwish, *Weakly singular functional-integral equation in infinite dimensional Banach spaces*, Appl. Math. Comput. **136** (1), 123–129, 2003.
[9] R. Estrada and R.P. Kanwal, *Singular integral equations*, Springer Science & Business Media, 2012.
[10] M.A. Golberg, *Numerical Solution of Integral Equations*, Springer US, 1990.
[11] P. Karczmarek, D. Pylak, and M.A. Sheshko, *Application of Jacobi polynomials to approximate solution of a singular integral equation with Cauchy kernel*, Appl. Math. Comput. **181** (1), 694–707, 2006.
[12] F.K. Keshi, B.P. Moghaddam, and A. Aghili, *A numerical approach for solving a class of variable-order fractional functional integral equations*, Comput. Appl. Math. **37** (4), 4821–4834, 2018.
[13] N. Khorrami, A.S. Shamloo, and B.P. Moghaddam, *Nystrom method for solution of fredholm integral equations of the second kind under interval data*, J. Intell. Fuzzy Syst. **36** (3), 2807–2816, 2019.
[14] D. Kincaid and E.W. Cheney, *Numerical Analysis: Mathematics of Scientific Computing*, Am. Math. Soc. 2009.
[15] S. Kumar and A.L. Sangal, *Numerical solution of singular integral equations using cubic spline interpolations*, Indian J. Pure Appl. Math. **35** (3), 415–421, 2004.
[16] P.K. Kythe and P. Puri, *Computational Methods for Linear Integral Equations*, Birkhäuser Boston, 2002.
[17] I. Lifanov, L. Poltavskii, and G. Vainikko, *Hypersingular Integral Equations and Their Applications*, CRC Press, 2003.
[18] B.N. Mandal and G.H. Bera, *Approximate solution of a class of singular integral equations of second kind*, J. Comput. Appl. Math. **206** (1), 189–195, 2007.
[19] B.P. Moghaddam and J.A. Tenreiro Machado, *Extended algorithms for approximating variable order fractional derivatives with applications*, J. Sci. Comput. **71** (3), 1351–1374, 2016.
[20] B.P. Moghaddam and J.A. Tenreiro Machado, *A computational approach for the solution of a class of variable-order fractional integro-differential equations with weakly singular kernels*, Fract. Calc. Appl. Anal. **20** (4), 1023–1042, 2017.
[21] P. Mokhtary, B.P. Moghaddam, A.M. Lopes, and J.A. Tenreiro Machado, *A computational approach for the non-smooth solution of non-linear weakly singular Volterra integral equation with proportional delay*, Numer. Algorithms 1–20, 2019. doi:10.1007/s11075-019-00712-y.
[22] N.I. Muskhelishvili, *Singular integral equations: Boundary problems of functions theory and their application to mathematical physics*, P. Noordhoff, 1953.
[23] A. Polyanin, *Handbook of Integral Equations*, CRC Press, 1998.
[24] A. Setia, *Numerical solution of various cases of Cauchy type singular integral equation*, Appl. Math. Comput. **230**, 200–207, 2014.
[25] M. Sheshko, *Singular integral equations with Cauchy and Hilbert kernels and theirs approximated solutions*, The Learned Society of the Catholic University of Lublin, Lublin, 2003.
[26] B.Q. Tang and X.F. Li, *Solution of a class of Volterra integral equations with singular and weakly singular kernels*, Appl. Math. Comput. **199** (2), 406–413, 2008.
[27] J.A. Tenreiro Machado, F. Mainardi, V. Kiryakova, and T. Atanacković, *Fractional calculus: D'où venons-nous? Que sommes-nous? Où allons-nous?*, Fract. Calc. Appl. Anal. **19** (5), 1074–1104, 2016.

[28] X. Jin, L.M. Keer, and Q. Wang, *A practical method for singular integral equations of the second kind*, Eng. Fract. Mech. **75** (5), 1005–1014, 2008.