Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (3) (2020), 1006 – 1019
DOI : 10.15672/hujms.496068

Research Article

On anti-Kähler manifolds with complex semi-symmetric metric F-connection

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Abstract

In this paper, we construct a complex semi-symmetric metric F-connection on an anti-Kähler manifold. First, we present some results concerning the torsion tensor of the complex semi-symmetric metric F-connection. Finally, we find expressions of the curvature tensor, the conharmonic curvature tensor and the Weyl projective curvature tensor of such connection, and study their properties.

Mathematics Subject Classification (2010). 53B20, 53B15, 53C15

Keywords. anti-Kähler manifold, complex semi-symmetric metric F-connection, curvature tensor, pure tensor, Tachibana operator

1. Introduction

A linear connection $\overline{\nabla}$ on an $n-$dimensional differentiable manifold M is said to be a semi-symmetric connection if its torsion is of the form: $S(X,Y) = p(Y)X - p(X)Y$, where p is a $1-$form. The connection $\overline{\nabla}$ is a metric connection if there is a Riemannian metric g on M such that $\overline{\nabla}g = 0$, otherwise it is non-metric. If the connection $\overline{\nabla}$ is both semi-symmetric and metric, then it is called a semi-symmetric metric connection. Hayden [5] defined and studied semi-symmetric metric connections. After that, Yano [10] proved the theorem: A Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes if and only if Riemannian manifold is conformally flat. As a generalization of semi-symmetric metric connections, Yano and Imai [12] defined a semi-symmetric metric F-connection on a Kähler manifold and obtained some results by using the Bochner curvature tensor.

An anti-Kähler or Kähler-Norden manifold means a triplet (M, g, F) which consists of an $n = 2m$ dimensional differentiable manifold M, an almost complex structure F and a pseudo-Riemannian metric g such that $g(FX,Y) = g(X,FY)$ and $\nabla F = 0$ for all vector field X and Y on M, where ∇ is the Levi-Civita connection of g. Such manifolds also refer to as generalized $B-$manifolds [4] or as almost complex manifolds with Norden metric [1] or as almost complex manifolds with $B-$metric [2].

An almost Hermitian manifold (M, g, F) always admits a unique natural connection ∇^C with a torsion T^C such that $\nabla^C F = 0$, $\nabla^C g = 0$ and $T^C(FX,Y) = T^C(X,FY)$ for all vector fields X, Y on M. This connection known as the canonical Hermitian connection or

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Received: 12.12.2018; Accepted: 24.06.2019

the Chern connection. Analogously to the canonical Hermitian connection on an almost Hermitian manifold, Ganchev and Mihova in [3] defined on an almost complex manifold with Norden metric (M, g, F) a natural connection ∇' (i.e., $\nabla' F = 0$, $\nabla' g = 0$) with a torsion T' satisfying $T'(X,Y,Z)+T'(Y,Z,X)-T'(FX,Y,FZ)-T'(Y,FZ,FX) = 0$. This connection is called canonical and it is proved that it is unique on (M, g, F).

In this paper, we define on an anti-Kähler manifold a canonical connection (i.e, a linear connection such that the anti-Hermitian structures (g, F) are parallel with respect to it) with torsion S locally expressed by $S_{ij}^{\ \ k} = p_j \delta_i^k - p_i \delta_j^k - p_t F_j^{\ t} F_i^{\ k} + p_t F_i^{\ t} F_j^{\ k}$ and study its torsion and curvature properties. We are calling the canonical connection as a complex semi-symmetric metric F-connection. Also, note that the torsion tensor S of the complex semi-symmetric metric F-connection satisfies $S(FX, Y) = S(X, FY) = FS(X, Y)$ for all vector fields X, Y on M. Hence we can say that the considered complex semi-symmetric metric F-connection on an anti-Kähler manifold is different from the canonical connections in [2, 3, 7]. This paper is organized as follows. In section 2, we introduce anti-Kähler manifolds and give a brief account of information of pure tensors, holomorphic tensors and Tachibana operator. Also we construct, using the method of Hayden [5], a complex semi-symetric metric F- connection on an anti-Kähler manifold. In the next section, we investigate conditions for the torsion tensor of the complex semi-symmetric metric F-connection to be holomorphic and recurrent. In the last section, we investigate expressions of the curvature tensor, the conharmonic curvature tensor and the Weyl projective curvature tensor of such connection and study their properties. Also, an example is presented.

2. A complex semi-symmetric metric F-connection

An anti-Kähler manifold is an $n = 2m$ dimensional differentiable manifold M_n equipped with a $(1, 1)-$tensor $F = (F_i^{\ j})$ and a pseudo-Riemannian metric tensor $g = (g_{ij})$ which satisfy the following conditions:

$$F_i^{\ k} F_k^{\ j} = -\delta_i^j, \tag{2.1}$$

$$F_i^{\ k} g_{kj} = F_j^{\ k} g_{ki} \tag{2.2}$$

and

$$\nabla_k F_i^{\ j} = 0.$$

Here we use the notation ∇_k to denote the operator of the Riemannian covariant derivation. Throughout this paper, the notation ∇_k will be used for the same purpose. The condition (2.2) is purity condition of the pseudo-Riemannian metric g with respect to the almost complex structure F. We also note that we get, as a consequence of (2.2), $F_{ij} = F_{ji}$. As it is already known, the almost complex structure F satisfies additionally the condition $g_{kl} F_i^{\ k} F_j^{\ l} = -g_{ij}$ and is trace-free, which can be written as $g^{kl} F_{kl} = 0$, where $F_{kl} = g_{jl} F_k^{\ j}$. Such manifolds are an object of interest of geometers and physicists. In [6], it is proved that the condition $\nabla_k F_i^{\ j} = 0$ is equivalent to the holomorphicity (analyticity) of the anti-Hermitian metric g, that is, $(\phi_F g)_{kij} = 0$, where ϕ_F is the Tachibana operator applied to g.

Let (M_n, g, F) be an anti-Kähler manifold. The following conditions hold [6, 8]:

i) The Levi-Civita connection on (M_n, g, F) is pure with respect to F;
ii) The Riemannian curvature tensor R on (M_n, g, F) is pure with respect to F;
iii) The Riemannian curvature tensor R is holomorphic: $(\phi_F R)_{kijl}^{\ \ \ \ \ t} = 0$, where ϕ_F is the Tachibana operator applied to R.

For any $(p, q)-$tensor K, purity and holomorphicity are defined as follows:

Definition 2.1. If A $(p,q)-$tensor $K = (\ K^{j_1 j_2 \ldots j_p}_{i_1 i_2 \ldots i_q})$ satisfies the condition

$$\begin{aligned} K^{j_1\ldots j_p}_{m i_2 \ldots i_q} F_{i_1}^{\ m} &= K^{j_1\ldots j_p}_{i_1 m \ldots i_q} F_{i_2}^{\ m} = \ldots = K^{j_1 \ldots jp}_{i_1 i_2 \ldots m} F_{i_q}^{\ m} = \\ K^{m j_2 \ldots j_p}_{i_1 \ldots i_q} F_m^{\ j_1} &= K^{j_1 m \ldots j_p}_{i_1 \ldots i_q} F_m^{\ j_2} = \ldots = K^{j_1 j_2 \ldots m}_{i_1 \ldots i_q} F_m^{\ j_p}, \end{aligned}$$

then the tensor K is called as a pure tensor with respect to the tensor F, where $F = (F_i^{\ j})$ is a $(1,1)-$tensor. The Tachibana operator ϕ_F applied to the pure $(p,q)-$tensor K is given by [9]

$$\begin{aligned} & (\phi_F K)^{j_1 \ldots j_p}_{k i_1 \ldots i_q} \qquad (2.3) \\ = \ & F_k^{\ m} \partial_m t^{j_1 \ldots j_p}_{i_1 \ldots i_q} - \partial_k \left(K \circ F\right)^{j_1 \ldots j_p}_{i_1 \ldots i_q} \\ & + \sum_{\lambda=1}^{q} \left(\partial_{i_\lambda} F_k^{\ m}\right) \ K^{j_1 \ldots j_p}_{i_1 \ldots m \ldots i_q} + \sum_{\mu=1}^{p} \left(\partial_k F_m^{\ j_\mu} - \partial_m F_k^{\ j_\mu}\right) \ K^{i_1 \ldots m \ldots i_r}_{j_1 \ldots j_s}, \end{aligned}$$

where

$$\begin{aligned} (K \circ F)^{j_1 \ldots j_p}_{i_1 \ldots i_q} &= K^{j_1 \ldots j_p}_{m i_2 \ldots i_q} F_{i_1}^{\ m} = \ldots = K^{j_1 \ldots jp}_{i_1 i_2 \ldots m} F_{i_q}^{\ m} \\ &= K^{m j_2 \ldots j_p}_{i_1 \ldots i_q} F_m^{\ j_1} = \ldots = K^{j_1 j_2 \ldots m}_{i_1 \ldots i_q} F_m^{\ j_p}. \end{aligned}$$

If the pure tensor K satisfies $\phi_F K = 0$, then it is called as a $\phi-$tensor. If the $(1,1)-$tensor F is a complex structure, then a $\phi-$tensor is a holomorphic (analytic) tensor [9] (for Tachibana operator and its applications, see [8] and [11]).

A linear connection $\overline{\nabla}$ on (M_n, g, F) is said to be a metric F-connection if the following conditions are satisfied:

$$\begin{aligned} i) \ \overline{\nabla}_h g_{ij} &= 0, \qquad (2.4) \\ ii) \ \overline{\nabla}_h F_i^{\ j} &= 0, \end{aligned}$$

where $\overline{\nabla}_h$ denotes the operator of covariant derivation with respect to $\overline{\nabla}$. We consider a complex semi-symmetric metric F-connection $\overline{\nabla}$ whose torsion tensor is in the form:

$$S_{ij}^{\ k} = p_j \delta_i^k - p_i \delta_j^k - p_t F_j^{\ t} F_i^{\ k} + p_t F_i^{\ t} F_j^{\ k}, \qquad (2.5)$$

where p_i are local components of any $1-$form p.

Let $\overline{\Gamma}^k_{ij}$ be the components of the complex semi-symmetric metric F-connection $\overline{\nabla}$. If we put

$$\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + T_{ij}^{\ k}, \qquad (2.6)$$

where Γ^k_{ij} are the components of the Levi-Civita connection ∇ of g and T^k_{ij} are the components of a $(1,2)-$tensor field T on M_n, then the torsion tensor S of $\overline{\nabla}$ is given by

$$S_{ij}^{\ k} = \overline{\Gamma}^k_{ij} - \overline{\Gamma}^k_{ji} = T_{ij}^{\ k} - T_{ji}^{\ k}.$$

Because the connection (2.6) must be provided the first formula of (2.4), by employing the method proposed by Hayden in [5], we find

$$T_{ij}^{\ k} = p_j \delta_i^k - p^k g_{ij} - p_t F_j^{\ t} F_i^{\ k} + p_t F^{kt} F_{ij},$$

where $p^k = p_i g^{ik}$, $F^{kt} = F_i^{\ t} g^{ik}$ and $F_{ij} = F_i^{\ k} g_{jk}$. Hence the connection (2.6) becomes

$$\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + p_j \delta_i^k - p^k g_{ij} - p_t F_j^{\ t} F_i^{\ k} + p_t F^{kt} F_{ij}. \qquad (2.7)$$

Also, using (2.7) we can easily verify

$$\overline{\nabla}_k F_i^{\ j} = 0.$$

Consequently, the components $\overline{\Gamma}^k_{ij}$ of the complex semi-symmetric metric F-connection $\overline{\nabla}$ are in the form (2.7).

3. Torsion properties of the complex semi-symmetric metric F-connection

This section is devoted to the properties of the torsion tensor of the complex semi-symmetric metric F-connection $\overline{\nabla}$.

Proposition 3.1. *On an anti-Kähler manifold (M_n, g, F) equipped with the connection (2.7), the torsion tensor S of the connection (2.7) is pure with respect to F.*

Proof. By using (2.1) and (2.5), it follows that $S^k_{mj} F_i{}^m = S^k_{im} F_j{}^m = S^m_{ij} F_m{}^k$, that is, the torsion tensor S is pure. □

An F-connection is pure if and only if its torsion tensor is pure [8]. Thus we can say that the connection (2.7) is pure with respect to F.

Theorem 3.2. *On an anti-Kähler manifold (M_n, g, F) equipped with the connection (2.7), the torsion tensor S of the connection (2.7) is a holomorphic tensor if the $1-$form p is holomorphic.*

Proof. Let (M_n, g, F) be an anti-Kähler manifold and ∇ be its Levi-Civita connection with components Γ^h_{ij}.

If we apply the Tachibana operator ϕ_F to the torsion tensor S of the connection (2.7), we get

$$\begin{aligned}
&(\phi_F S)_{kij}{}^l \qquad (3.1)\\
&= F_k{}^m(\partial_m S^l_{ij}) - \partial_k(S^m_{ij} F_m{}^l)\\
&= F_k{}^m(\nabla_m S^l_{ij} + \Gamma^s_{mi} S^l_{sj} + \Gamma^s_{mj} S^l_{is} - \Gamma^l_{ms} S^s_{ij})\\
&\quad - F_m{}^l(\nabla_k S^m_{ij} + \Gamma^s_{ki} S^m_{sj} + \Gamma^s_{kj} S^m_{is} - \Gamma^m_{ks} S^s_{ij})\\
&= F_k{}^m(\nabla_m S^l_{ij}) - F_m{}^l(\nabla_k S^m_{ij}).
\end{aligned}$$

Substitution (2.5) into (3.1) gives

$$\begin{aligned}
(\phi_F S)_{kij}{}^l &= [F_k{}^m(\nabla_m p_j) - F_j{}^m(\nabla_k p_m)]\delta^l_i - [F_k{}^m(\nabla_m p_i) - F_i{}^m(\nabla_k p_m)]\delta^l_j\\
&\quad + [F_k{}^m F_i{}^s(\nabla_m p_s) + \nabla_k p_i]F_j{}^l - [F_k{}^m F_j{}^s(\nabla_m p_s) + \nabla_k p_j]F_i{}^l.
\end{aligned}$$

On the other hand, for the $1-$form p, we calculate

$$\begin{aligned}
(\phi_F p)_{kj} &= F_k{}^m(\partial_m p_j) - \partial_k(F_j{}^m p_m)\\
&= F_k{}^m(\nabla_m p_j + \Gamma^s_{mj} p_s) - F_j{}^m(\nabla_k p_m + \Gamma^s_{km} p_s)\\
&= F_k{}^m(\nabla_m p_j) - F_j{}^m(\nabla_k p_m).
\end{aligned}$$

From this, we can say that the $1-$form p is holomorphic if and only if

$$F^m_k(\nabla_m p_j) = F^m_j(\nabla_k p_m). \qquad (3.2)$$

Assumming that the $1-$form p is holomorphic, then (3.1) becomes $(\phi_F S)_{kij}{}^l = 0$, that is, the torsion tensor S is a holomorphic tensor which completes the proof. □

From now on, we will take into account such a special case of complex semi-symmetric metric F-connections which its $1-$form p is holomorphic, that is, the following condition always holds:

$$F_k{}^m(\nabla_m p_j) = F_j{}^m(\nabla_k p_m).$$

As a result of (3.1) and Proposition 3.1, we can write

$$F_k{}^m(\nabla_m S_{ij}{}^l) = F_i{}^m(\nabla_k S_{mj}{}^l) = F_j{}^m(\nabla_k S_{im}{}^l). \qquad (3.3)$$

A $(p, q)-$tensor T is called recurrent with respect to a given linear connection if its components satisfy

$$\nabla_h K^{j_1 \dots j_p}_{i_1 i_2 \dots i_q} = \omega_h K^{j_1 \dots j_p}_{i_1 i_2 \dots i_q},$$

where $\omega = (\omega_h)$ is the recurrence $1-$form.

Theorem 3.3. *On an anti-Kähler manifold (M_n, g, F) equipped with the connection (2.7), the torsion tensor S with respect to the connection (2.7) is recurrent, that is, $\overline{\nabla}_k S^l_{ij} = \omega_k S^l_{ij}$ if and only if the 1−form p is recurrent with respect to $\overline{\nabla}$, where ω_k is the recurrence 1−form.*

Proof. First we prove necessity. Assume that the torsion tensor S is recurrent, that is,

$$\overline{\nabla}_k S^l_{ij} = \omega_k S^l_{ij}.$$

Contracting the above equality with respect to i and l, we obtain

$$\overline{\nabla}_k S_{lj}{}^l = \omega_k S_{lj}{}^l. \tag{3.4}$$

On the other hand, from (2.5) we have

$$S_{lj}{}^l = (n-2)p_j. \tag{3.5}$$

Thus, (3.4) and (3.5) give

$$\overline{\nabla}_k p_j = \omega_k p_j.$$

This means that the 1−form p is recurrent with respect to $\overline{\nabla}$.

In contrast, let us assume that the 1−form p is recurrent with respect to $\overline{\nabla}$. Then covariant differentiation of (2.5) with respect to the connection (2.7) directly gives

$$\begin{aligned}\overline{\nabla}_k S_{ij}{}^l &= \left(\overline{\nabla}_k p_j\right)\delta^l_i - \left(\overline{\nabla}_k p_i\right)\delta^l_j - \left(\overline{\nabla}_k p_t\right)F_j{}^t F_i{}^l + \left(\overline{\nabla}_k p_t\right)F_i{}^t F_j{}^l \\ &= \omega_k p_j \delta^l_i - \omega_k p_i \delta^l_j - \omega_k p_t F_j{}^t F_i{}^l + \omega_k p_t F_i{}^t F_j{}^l \\ &= \omega_k S_{ij}{}^l\end{aligned}$$

which completes the proof. □

Proposition 3.4. *Let (M_n, g, F) be an anti-Kähler manifold equipped with the connection (2.7) and the 1−form p be recurrent with respect to the connection (2.7). Then the recurrence 1−form ω and the 1−form p are collinear, that is, $\omega_k = \alpha p_k$, where α is an arbitrary constant, if and only if the 1−form p is closed, that is, $dp = 0$.*

Proof. Covariant differentiation of the 1−form p with respect to the connection (2.7) yields

$$\overline{\nabla}_k p_j = \nabla_k p_j - p_j p_k + p_m p^m g_{jk} + p_m p_t F_k{}^t F_j{}^m - p_m p_t F^{mt} F_{jk}$$

and

$$\overline{\nabla}_j p_k = \nabla_j p_k - p_j p_k + p_m p^m g_{jk} + p_m p_t F_k{}^t F_j{}^m - p_m p_t F^{mt} F_{jk}$$

from which it follows that

$$\overline{\nabla}_k p_j - \overline{\nabla}_j p_k = \nabla_k p_j - \nabla_j p_k.$$

Since the 1−form p is recurrent with respect to $\overline{\nabla}$, from the above equation we can write

$$\omega_k p_j - \omega_j p_k = \nabla_k p_j - \nabla_j p_k.$$

Then the 1−form p is closed if and only if $\omega_k p_j = \omega_j p_k$ which means that the 1−forms ω and p are collinear, $\omega_k = \alpha p_k$, where α is an arbitrary constant. □

4. Curvature properties of the complex semi-symmetric metric F-connection

This section deals with curvature properties of the connection (2.7). It is known that the curvature tensor $\overline{R}$ of the connection (2.7) is characterized by

$$\overline{R}_{ijk}{}^h = \partial_i \overline{\Gamma}^h_{jk} - \partial_j \overline{\Gamma}^h_{ik} + \overline{\Gamma}^h_{im}\overline{\Gamma}^m_{jk} - \overline{\Gamma}^h_{jm}\overline{\Gamma}^m_{ik}.$$

Then, the curvature tensor $\overline{R}$ is as follows:

$$\begin{aligned}\overline{R}_{ijk}{}^h &= R_{ijk}{}^h - \delta^h_i \pi_{jk} + \delta^h_j \pi_{ik} + g_{ik}\pi_j{}^h - g_{jk}\pi_i{}^h \\ &\quad + F_i{}^h F_k{}^t \pi_{jt} - F_j{}^h F_k{}^t \pi_{it} - F_{ik}F^{ht}\pi_{jt} + F_{jk}F^{ht}\pi_{it},\end{aligned} \tag{4.1}$$

where $R_{ijk}^{\ \ \ h}$ are the components of the Riemannian curvature tensor and

$$\pi_{jk} = \nabla_j p_k - p_j p_k + \frac{1}{2} p^m p_m g_{kj} + p_m p_t F_k^{\ t} \ F_j^{\ m} - \frac{1}{2} p^m p_t F_m^{\ t} F_{jk}. \tag{4.2}$$

Contracting (4.1) with respect to h and k, it follows that $\overline{R}_{ijk}^{\ \ \ \ k} = 0$.

Theorem 4.1. *Let (M_n, g, F) be an anti-Kähler manifold equipped with the connection (2.7). The curvature tensor of the connection (2.7) and the Riemannian curvature tensor of the Levi-Civita connection coincide if the $1-$form p satisfies $\nabla_l p^l + \frac{n-4}{2} p_l p^l = 0$, where $p^l = g^{il} p_i$.*

Proof. By assumption $\overline{R}_{ijk}^{\ \ \ l} = R_{ijk}^{\ \ \ l}$, from (4.1), we find

$$\begin{aligned} 0 \ = \ & -\delta_i^l \pi_{jk} + \delta_j^l \pi_{ik} + g_{ik} \pi_j^{\ l} - g_{jk} \pi_i^{\ l} \\ & - F_i^{\ l} F_k^{\ t} \pi_{jt} + F_j^{\ l} F_k^{\ t} \pi_{it} + F_{ik} F^{lt} \pi_{jt} - F_{jk} F^{lt} \pi_{it}. \end{aligned}$$

Contracting the above with respect to i and l, and then multiplying it by g^{jk}, we have

$$trace\pi = \nabla_l p^l + \frac{n-4}{2} p_l p^l = 0.$$

□

Now, we state and prove two lemmas that we shall need.

Lemma 4.2. *On an anti-Kähler manifold (M_n, g, F) equipped with the connection (2.7), the tensor π given by (4.2) is symmetric if and only if the $1-$form p is closed.*

Proof. It follows immediately from (4.2) that $\pi_{jk} - \pi_{kj} = \nabla_j p_k - \nabla_k p_j = (dp)_{jk}$. This means that the tensor π is symmetric if and only if $dp = 0$, that is, the $1-$form p is closed. □

Lemma 4.3. *On an anti-Kähler manifold (M_n, g, F) equipped with the connection (2.7), the tensor π given by (4.2) is a holomorphic tensor and thus the following relation holds:*

$$(\nabla_m \pi_{ij}) F_k^{\ m} = (\nabla_k \pi_{mj}) F_i^{\ m} = (\nabla_k \pi_{im}) F_j^{\ m}.$$

Proof. Let π be the tensor given by (4.2) on the anti-Kähler manifold (M_n, g, F). The tensor π is pure with respect to F. In fact, using (2.1), (2.2) and (4.2) we have

$$F_k^{\ t} \pi_{it} - F_i^{\ t} \pi_{tk} = (\nabla_i p_t) F_k^{\ t} - (\nabla_t p_k) F_i^{\ t} = 0.$$

We calculate

$$\begin{aligned} (\phi_F \pi)_{kij} \ &= \ F_k^{\ m} (\partial_m \pi_{ij}) - \partial_k (\pi_{im} F_j^{\ m}) \\ &= \ (\nabla_m \pi_{ij}) F_k^{\ m} - (\nabla_k \pi_{im}) F_j^{\ m}. \end{aligned} \tag{4.3}$$

Applying (4.2) into (4.3), standard calculations give

$$(\phi_F \pi)_{kij} = (\nabla_m \nabla_i p_j) F_k^{\ m} - (\nabla_k \nabla_m p_j) F_i^{\ m}. \tag{4.4}$$

If we apply the Ricci identity to the $1-$form p, then we have

$$(\nabla_m \nabla_i p_j) F_k^{\ m} = (\nabla_i \nabla_m p_j) F_k^{\ m} - \frac{1}{2} p_s R_{mij}^{\ \ \ \ s} F_k^{\ m}$$

and

$$(\nabla_k \nabla_i p_m) F_j^{\ m} = (\nabla_i \nabla_k p_m) F_j^{\ m} - \frac{1}{2} p_s R_{kim}^{\ \ \ \ s} F_j^{\ m}.$$

(4.4), with the help of the last two equation can be rewritten as follows:

$$(\phi_F \pi)_{kij} = -\frac{1}{2} p_s (R_{mij}^{\ \ \ \ s} F_k^{\ m} - R_{kim}^{\ \ \ \ s} F_j^{\ m}).$$

This immediately gives $(\phi_F \pi)_{kij} = 0$. Hence, in view of (4.3) and the purity of the tensor π we can write

$$(\nabla_m \pi_{ij}) F_k^{\ m} = (\nabla_k \pi_{mj}) F_i^{\ m} = (\nabla_k \pi_{im}) F_j^{\ m}.$$

This completes the proof. □

Theorem 4.4. *On an anti-Kähler manifold (M_n, g, F) equipped with the connection (2.7), the curvature tensor $\overline{R}$ of the connection (2.7) is a holomorphic tensor and thus the following relation holds:*

$$(\nabla_m \overline{R}_{ijl}^{\ \ \ t}) F_k^{\ m} = (\nabla_k \overline{R}_{mjl}^{\ \ \ \ t} F_i^{\ m}).$$

Proof. Using the purity of the tensor π, it is easy to see that

$$\overline{R}_{ijk}^{\ \ \ m} F_m^{\ l} = \overline{R}_{mjk}^{\ \ \ \ l} F_i^{\ m} = \overline{R}_{imk}^{\ \ \ \ l} F_j^{\ m} = \overline{R}_{ijm}^{\ \ \ \ l} F_k^{\ m},$$

that is, the curvature tensor $\overline{R}$ is pure with respect to F.

Applying the Tachibana operator ϕ_F to the curvature tensor $\overline{R}$, we have

$$\begin{aligned} & (\phi_F \overline{R})_{kijl}^{\ \ \ \ \ t} \\ = \ & F_k^{\ m} (\partial_m \overline{R}_{ijl}^{\ \ \ t}) - \partial_k (\overline{R}_{ijl}^{\ \ \ m} F_m^{\ t}) \\ = \ & F_k^{\ m} (\nabla_m \overline{R}_{ijl}^{\ \ \ t} + \Gamma_{mi}^s \overline{R}_{sjl}^{\ \ \ t} + \Gamma_{mj}^s \overline{R}_{isl}^{\ \ \ t} + \Gamma_{ml}^s \overline{R}_{ijs}^{\ \ \ t} - \Gamma_{ms}^t \overline{R}_{ijl}^{\ \ \ m}) \\ & - F_m^{\ t} (\nabla_k \overline{R}_{ijl}^{\ \ \ m} + \Gamma_{ki}^s \overline{R}_{sjl}^{\ \ \ t} + \Gamma_{kj}^s \overline{R}_{isl}^{\ \ \ t} + \Gamma_{kl}^s \overline{R}_{ijs}^{\ \ \ t} - \Gamma_{ks}^t \overline{R}_{ijl}^{\ \ \ s}) \\ = \ & (\nabla_m \overline{R}_{ijl}^{\ \ \ t}) F_k^{\ m} - (\nabla_k \overline{R}_{ijl}^{\ \ \ m}) F_m^{\ t} \end{aligned} \tag{4.5}$$

from which, by (4.1), we find

$$\begin{aligned} (\phi_F \overline{R})_{kijl}^{\ \ \ \ \ t} = \ & (\phi_F R)_{kijl}^{\ \ \ \ \ t} \\ & + [(\nabla_k \pi_{jm}) F_l^{\ m} - (\nabla_m \pi_{jl}) F_k^{\ m}] \delta_i^t + [(\nabla_m \pi_{il}) F_k^{\ m} - (\nabla_k \pi_{im}) F_l^{\ m}] \delta_j^t \\ & + [(\nabla_k \pi_i^m) F_m^{\ t} - \left(\nabla_m \pi_i^t\right) F_k^{\ m}] g_{jl} - [\left(\nabla_k \pi_j^m\right) F_m^{\ t} - \left(\nabla_m \pi_j^t\right) F_k^{\ m}] g_{il} \\ & + [(\nabla_m \pi_{js}) F_k^{\ m} F_l^{\ s} + \nabla_k \pi_{jl}] F_i^t - [(\nabla_m \pi_{is}) F_k^{\ m} F_l^{\ s} + \nabla_k \pi_{il}] F_j^t \\ & + [(\nabla_m \pi_i^s) F_k^{\ m} F_s^{\ t} + \nabla_k \pi_i^t] F_{jl} - [\left(\nabla_m \pi_j^s\right) F_k^{\ m} F_s^{\ t} + \nabla_k \pi_j^t] F_{il}. \end{aligned}$$

When we take into account lemma 4.3, the last relation becomes $(\phi_F \overline{R})_{kijl}^{\ \ \ \ \ t} = 0$, that is, the curvature tensor $\overline{R}$ is a holomorphic tensor. Thus, by (4.5), we can write

$$(\nabla_m \overline{R}_{ijl}^{\ \ \ t}) F_k^{\ m} = (\nabla_k \overline{R}_{mjl}^{\ \ \ \ t} F_i^{\ m}).$$

□

Multiplying (4.1) by g_{hl}, the curvature $(0,4)-$tensor is given in the form:

$$\begin{aligned} \overline{R}_{ijkl} = \ & R_{ijkl} - g_{il} \pi_{jk} + g_{jl} \pi_{ik} + g_{ik} \pi_{jl} - g_{jk} \pi_{il} + F_{il} F_k^{\ t} \pi_{jt} \\ & - F_{jl} F_k^{\ t} \pi_{it} - F_{ik} F_l^{\ t} \pi_{jt} + F_{jk} F_l^{\ t} \pi_{it}, \end{aligned} \tag{4.6}$$

where R_{ijkl} are the curvature $(0,4)-$tensor of the Levi-Civita connection ∇ of g. We can immediately say that the curvature $(0,4)-$tensor $\overline{R}$ satisfies the following properties:

i) $\overline{R}_{ijkl} = -\overline{R}_{jikl}$,
ii) $\overline{R}_{ijkl} = -\overline{R}_{ijlk}$.

Theorem 4.5. *Let (M_n, g, F) be an anti-Kähler manifold equipped with the connection (2.7) and let us assume that $n \geqslant 6$. The curvature $(0,4)-$tensor $\overline{R}$ of the connection (2.7) holds the followings*

$$\begin{aligned} i) \ \overline{R}_{ijkl} - \overline{R}_{klij} &= 0, \\ ii) \ \overline{R}_{ijkl} + \overline{R}_{kijl} + \overline{R}_{jkil} &= 0 \end{aligned}$$

if and only if the $1-$form p is closed.

Proof. *i*) From (4.6), we obtain

$$\begin{aligned} &\overline{R}_{ijkl}-\overline{R}_{klij} \\ =\ &(\pi_{li}-\pi_{il})\,g_{jk}+(\pi_{kj}-\pi_{jk})\,g_{il}+(\pi_{ik}-\pi_{ki})\,g_{jl} \\ &+(\pi_{jl}-\pi_{lj})\,g_{ik}+F_{il}F_{k}^{\ t}(\pi_{jt}-\pi_{tj})+F_{jl}F_{k}^{\ t}(\pi_{ti}-\pi_{it}) \\ &+F_{ik}F_{l}^{\ t}(\pi_{tj}-\pi_{jt})+F_{jk}F_{l}^{\ t}(\pi_{it}-\pi_{ti}). \end{aligned} \tag{4.7}$$

If we assume that $\overline{R}_{ijkl}-\overline{R}_{klij}=0$, then (4.7) becomes

$$\begin{aligned} 0\ =\ &(\pi_{li}-\pi_{il})\,g_{jk}+(\pi_{kj}-\pi_{jk})\,g_{il}+(\pi_{ik}-\pi_{ki})\,g_{jl} \\ &+(\pi_{jl}-\pi_{lj})\,g_{ik}+F_{il}F_{k}^{\ t}(\pi_{jt}-\pi_{tj})+F_{jl}F_{k}^{\ t}(\pi_{ti}-\pi_{it}) \\ &+F_{ik}F_{l}^{\ t}(\pi_{tj}-\pi_{jt})+F_{jk}F_{l}^{\ t}(\pi_{it}-\pi_{ti}). \end{aligned}$$

Transvecting the above with g^{il}, we find

$$(n-4)\,(\pi_{kj}-\pi_{jk})=0.$$

In view of $n\geqslant 6$, the last relation gives

$$\pi_{jk}-\pi_{kj}=0$$

from which the result follows.

Conversely, using the fact that the 1−form p is closed, from (4.7) it is easy to see that $\overline{R}_{ijkl}-\overline{R}_{klij}=0$.

ii) We omit the proof because it can be established by the same way as in the proof of (*i*). □

Theorem 4.6. *On an anti-Kähler manifold (M_n,g,F), the Ricci tensor of the connection (2.7) is characterized by*

$$\overline{R}_{jk}=R_{jk}+(4-n)\,\pi_{jk}-g_{jk}trace\pi+F_{jk}F^{lt}\pi_{lt},$$

where R_{jk} is the components of the Ricci tensor of the Levi-Civita connection of g. Let us assume that $n\geqslant 6$, then the Ricci tensor is symmetric if and only if the 1−form p is closed.

Proof. Contracting (4.1) with respect to i and l, we obtain

$$\overline{R}_{jk}=R_{jk}+(4-n)\,\pi_{jk}-g_{jk}trace\pi+F_{jk}F^{lt}\pi_{lt}. \tag{4.8}$$

Also, we have

$$\overline{R}_{jk}-\overline{R}_{kj}=(4-n)\,(\pi_{jk}-\pi_{kj}).$$

This implies that $\overline{R}_{jk}-\overline{R}_{kj}=0$ if and only if the 1−form p is closed. □

As a result of Theorem 4.5 and 4.6, we can state:

Theorem 4.7. *Let (M_n,g,F) be an anti-Kähler manifold. The Ricci tensor of the connection (2.7) is symmetric if and only if*

$$\overline{R}_{ijkl}-\overline{R}_{klij}=0$$

or

$$\overline{R}_{ijkl}+\overline{R}_{kijl}+\overline{R}_{jkil}=0.$$

Let $\overline{\tau}$ be the scalar curvature of the connection (2.7), where $\overline{\tau}$ is obtained by contracting the Ricci tensor (4.8): $\overline{\tau}=g^{jk}\overline{R}_{jk}$. The scalar curvature $\overline{\tau}$ is given by

$$\begin{aligned} \overline{\tau}\ &=\ \tau+2(2-n)trace\pi \\ &=\ \tau+2(2-n)(\nabla_l p^l+\frac{n-4}{2}p_lp^l), \end{aligned} \tag{4.9}$$

where τ is the scalar curvature of the Riemannian manifold (M_n,g).

Example 4.8. The pseudo-Euclidean space $\mathbb{R}^{2n}$ is given by pseudo-Euclidean metric

$$\begin{aligned}(g_{\alpha\beta}) &= \begin{pmatrix} g_{ij} & g_{\bar{i}j} \\ g_{i\bar{j}} & g_{\bar{i}\bar{j}} \end{pmatrix} \\ &= \begin{pmatrix} \delta_{ij} & 0 \\ 0 & -\delta_{ij} \end{pmatrix}, i,j = 1,...,n, \quad \bar{i},\bar{j} = n+1,...,2n.\end{aligned}$$

Let $\mathbb{C}^n$ be the complex space. The usual identification r of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ is given by

$$r : z = (z^1, z^2, ..., z^n) \in \mathbb{C}^n \to r(z) = Z = (x^1, x^2, ..., x^n; y^1, y^2, ..., y^n) \in \mathbb{R}^{2n}$$

where $z^k = x^k + iy^k$, $k = 1,...,n$. The canonical complex structure F on $\mathbb{R}^{2n}$ is determined by the matrix

$$\begin{aligned}\left(F_{\alpha}^{\ \beta}\right) &= \begin{pmatrix} F_i^j & F_{\bar{i}}^j \\ F_i^{\bar{j}} & F_{\bar{i}}^{\bar{j}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \delta_i^j \\ -\delta_i^j & 0 \end{pmatrix}, \ i,j = 1,...,n, \quad \bar{i},\bar{j} = n+1,...,2n\end{aligned}$$

or

$$(F_{\alpha\beta}) = \begin{pmatrix} F_{ij} & F_{\bar{i}j} \\ F_{i\bar{j}} & F_{\bar{i}\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix}$$

with respect to the natural basis of $\mathbb{R}^{2n}$. In the example, Greek indices take on values 1 to $2n$. For all Z, W on $\mathbb{R}^{2n}$ the metric g and the complex structure F on $\mathbb{R}^{2n}$ are related by the equality $g(FZ, FW) = -g(Z, W)$, that is, g is pure with respect to F. Hence $(\mathbb{R}^{2n}, g, F)$ is an anti-Kähler Euclidean space. Note that the metric g is of signature (n, n).

We suppose that p_α is a gradient, $p_\alpha = (p_i, p_{\bar{i}}) = (\partial_i f, \partial_{\bar{i}} f)$, f being a holomorphic function. The condition for the function f to be locally holomorphic is given by [6]

$$(\phi_F df)_{\sigma\beta} = F_\sigma^\alpha \partial_\alpha \partial_\beta f - \partial_\sigma (F_\beta^\alpha \partial_\alpha f) + (\partial_\beta F_\sigma^\alpha) \partial_\alpha f = 0.$$

Then, the components of the complex semi-symmetric metric F-connection in $(\mathbb{R}^{2n}, g, F)$ are the followings

$$\begin{aligned}\overline{\Gamma}_{ij}^k &= \overline{\Gamma}_{\bar{i}j}^{\bar{k}} = \overline{\Gamma}_{i\bar{j}}^{\bar{k}} = -\overline{\Gamma}_{\bar{i}\bar{j}}^k = (\partial_j f)\, \delta_i^k - (\partial_h f)\, \delta^{hk} \delta_{ij}, \\ \overline{\Gamma}_{\bar{i}j}^k &= \overline{\Gamma}_{i\bar{j}}^k = \overline{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} = -\overline{\Gamma}_{ij}^{\bar{k}} = \left(\partial_{\bar{j}} f\right) \delta_i^k + (\partial_{\bar{h}} f)\, \delta^{\overline{hk}} \delta_{ij}.\end{aligned}$$

The torsion tensor of the complex semi-symmetric metric F-connection has the components

$$\begin{aligned}S_{ij}^{\ \ k} &= S_{\bar{i}j}^{\ \ \bar{k}} = S_{i\bar{j}}^{\ \ \bar{k}} = -S_{\bar{i}\bar{j}}^{\ \ k} = (\partial_j f)\, \delta_i^k - (\partial_i f)\, \delta_j^k, \\ S_{\bar{i}j}^{\ \ k} &= S_{i\bar{j}}^{\ \ k} = S_{\bar{i}\bar{j}}^{\ \ \bar{k}} = -S_{ij}^{\ \ \bar{k}} = \left(\partial_{\bar{j}} f\right) \delta_i^k - (\partial_{\bar{i}} f)\, \delta_j^k.\end{aligned}$$

One verifies that the torsion tensor S is pure with respect to F and furthermore $(\phi_F S)_{\sigma\alpha\beta}^\gamma = 0$, that is, S is holomorphic.

The components of the curvature tensor $\overline{R}$ of the complex semi-symmetric metric F-connection are the followings

$$\begin{aligned}\overline{R}_{\bar{i}jk}^{\ \ \ \ l} &= \overline{R}_{i\bar{j}k}^{\ \ \ \ l} = \overline{R}_{ij\bar{k}}^{\ \ \ \ l} = \overline{R}_{\bar{i}\bar{j}k}^{\ \ \ \ \bar{l}} \\ &= \overline{R}_{\bar{i}j\bar{k}}^{\ \ \ \ \bar{l}} = \overline{R}_{i\bar{j}\bar{k}}^{\ \ \ \ \bar{l}} = -\overline{R}_{\bar{i}\bar{j}\bar{k}}^{\ \ \ \ l} = -\overline{R}_{ijk}^{\ \ \ \ \bar{l}} \\ &= -\delta_i^l \pi_{\bar{k}j} + \delta_j^l \pi_{\bar{k}i} - \delta_{kj} \pi_{\bar{i}}^l + \delta_{ki} \pi_{\bar{j}}^l,\end{aligned}$$

$$\begin{aligned}
\overline{R}_{i\overline{j}\overline{k}}{}^{\overline{l}} &= \overline{R}_{\overline{i}\overline{j}k}{}^{l} = \overline{R}_{\overline{i}\overline{j}\overline{k}}{}^{\overline{l}} = \overline{R}_{\overline{i}j\overline{k}}{}^{l} \\
&= -\overline{R}_{i\overline{j}k}{}^{\overline{l}} = -\overline{R}_{ijk}{}^{l} = -\overline{R}_{ij\overline{k}}{}^{\overline{l}} = -\overline{R}_{\overline{i}jk}{}^{\overline{l}} \\
&= \delta_i^l \pi_{kj} - \delta_j^l \pi_{ki} + \delta_{kj}\pi_{\overline{i}}^{\overline{l}} - \delta_{ki}\pi_{j}^{l},
\end{aligned}$$

where

$$\begin{aligned}
\pi_{kj} &= -\pi_{\overline{k}\overline{j}} = \partial_k\partial_j f + (\partial_{\overline{k}} f)\left(\partial_{\overline{j}} f\right) - (\partial_k f)(\partial_j f) \\
&\quad + \frac{1}{2}\delta^{hm}\delta_{ij}\left[(\partial_{\overline{h}} f)(\partial_{\overline{m}} f) - (\partial_h f)(\partial_m f)\right],
\end{aligned}$$

$$\pi_{\overline{k}j} = \pi_{k\overline{j}} = \partial_{\overline{k}}\partial_j f - (\partial_{\overline{k}} f)(\partial_j f) - (\partial_k f)\left(\partial_{\overline{j}} f\right) + \delta^{hm}\delta_{ij}(\partial_{\overline{m}} f)(\partial_m f)$$

and $\pi_\sigma^\beta = g^{\alpha\beta}\pi_{\sigma\alpha}$. Simple calculations show that $(\phi_F\pi)_{\sigma\alpha\beta} = 0$. Using this, one checks that the curvature tensor $\overline{R}$ is pure with respect to F and furthermore $(\phi_F\overline{R})_{\sigma\alpha\beta\gamma}^{\ \ \ \ \eta} = 0$, that is, $\overline{R}$ is holomorphic.

The components of the curvature $(0,4)$-tensor $\overline{R}$ are the followings

$$\begin{aligned}
\overline{R}_{\overline{i}jkl} &= \overline{R}_{i\overline{j}kl} = \overline{R}_{ij\overline{k}l} = -\overline{R}_{\overline{i}\overline{j}k\overline{l}} \\
&= -\overline{R}_{\overline{i}j\overline{k}\overline{l}} = -\overline{R}_{i\overline{j}\overline{k}\overline{l}} = -\overline{R}_{\overline{i}\overline{j}\overline{k}l} = \overline{R}_{ijk\overline{l}} \\
&= -\delta_{il}\pi_{\overline{k}j} + \delta_{jl}\pi_{\overline{k}i} - \delta_{kj}\pi_{\overline{i}l} + \delta_{ki}\pi_{\overline{j}l},
\end{aligned}$$

$$\begin{aligned}
\overline{R}_{\overline{i}j\overline{k}l} &= -\overline{R}_{ijkl} = \overline{R}_{ij\overline{k}\overline{l}} = \overline{R}_{\overline{i}jk\overline{l}} \\
&= \overline{R}_{\overline{i}\overline{j}kl} = -\overline{R}_{\overline{i}\overline{j}\overline{k}\overline{l}} = -\overline{R}_{i\overline{j}\overline{k}\overline{l}} = \overline{R}_{i\overline{j}k\overline{l}} \\
&= \delta_{il}\pi_{kj} - \delta_{jl}\pi_{ki} + \delta_{kj}\pi_{il} - \delta_{ki}\pi_{jl}.
\end{aligned}$$

It is a straightforward verification that the conditions

$$\begin{aligned}
\overline{R}_{\sigma\alpha\beta\gamma} &= -\overline{R}_{\alpha\sigma\beta\gamma}, \\
\overline{R}_{\sigma\alpha\beta\gamma} &= -\overline{R}_{\sigma\alpha\gamma\beta}, \\
\overline{R}_{\sigma\alpha\beta\gamma} &= \overline{R}_{\beta\gamma\sigma\alpha},
\end{aligned}$$

$$\overline{R}_{\sigma\alpha\beta\gamma} + \overline{R}_{\alpha\beta\sigma\gamma} + \overline{R}_{\beta\sigma\alpha\gamma} = 0$$

are fulfilled.

For the Ricci tensor $\overline{R}_{\beta\gamma}$, we get

$$\begin{aligned}
\overline{R}_{jk} &= 2(1-n)\pi_{kj} - 2\delta_{kj} trace\pi, \\
\overline{R}_{\overline{j}\overline{k}} &= -2n\pi_{\overline{k}\overline{j}} + 2\delta_{kj}\pi_l^l, \\
\overline{R}_{\overline{j}k} &= \overline{R}_{j\overline{k}} = -2n\pi_{\overline{k}j} - 2\delta_{kj} trace\pi
\end{aligned}$$

from which it follows that the condition $\overline{R}_{\beta\gamma} = \overline{R}_{\gamma\beta}$ is verified, which means that the Ricci tensor of our connection in $(\mathbb{R}^{2n}, g, F)$ is symmetric.

A pseudo-Riemannian manifold is called an Einstein space if the equation

$$R_{jk} = \lambda g_{jk}$$

holds with a scalar function λ. The pseudo-Riemannian manifold with any complex semi-symmetric metric F-connection in which the Ricci tensor satisfies the equation

$$\overline{R}_{(jk)} = \gamma g_{jk}$$

may be called an Einstein space, where γ is a scalar function and $R_{(jk)}$ is symmetric part of Ricci tensor of the complex semi-symmetric metric F-connection.

Theorem 4.9. *Let (M_n, g, F) be an anti-Kähler manifold and the pseudo-Riemannian manifold M_n be an Einstein space with respect to the Levi-Civita connection. Then the pseudo-Riemannian manifold M_n equipped with the connection (2.7) will be an Einstein space with respect to the connection (2.7) if the 1−form p satisfies*

$$\gamma - \lambda = \frac{2(2-n)}{n}(\nabla_l p^l + \frac{n-4}{2} p_l p^l),$$

where λ is a scalar function coming from the Einstein property of Riemannian spaces, that is, $R_{jk} = \lambda g_{jk}$.

Proof. Using (4.8), the symmetric part of the Ricci tensor of the connection (2.7) is given by

$$\begin{aligned} \overline{R}_{(jk)} &= \frac{1}{2}(\overline{R}_{jk} + \overline{R}_{jk}) \\ &= \frac{1}{2}\{2R_{jk} + (4-n)(\pi_{jk} + \pi_{kj}) - 2g_{jk} trace\pi + 2F_{jk}F^{lt}\pi_{lt}\} \\ &= R_{jk} + \frac{4-n}{2}(\pi_{jk} + \pi_{kj}) - g_{jk} trace\pi + F_{jk}F^{lt}\pi_{lt}. \end{aligned}$$

If we transvect the last equation with g^{jk}, then we get

$$\begin{aligned} \overline{R}_{(jk)} g^{jk} &= R_{jk} g^{jk} + (4-2n) trace\pi \\ \gamma g_{jk} g^{jk} &= \lambda g_{jk} g^{jk} + 2(2-n) trace\pi \\ \gamma - \lambda &= \frac{2(2-n)}{n} trace\pi, \end{aligned}$$

where $trace\pi = \nabla_l p^l + \frac{n-4}{2} p_l p^l$. Thus the connection (2.7) is Einstein if the equation $\gamma - \lambda = \frac{2(2-n)}{n}(\nabla_l p^l + \frac{n-4}{2} p_l p^l)$ holds. □

The conharmonic curvature tensor with respect to the connection (2.7) is given by

$$\overline{V}_{ijkl} = \overline{R}_{ijkl} - \frac{1}{n-2}\left[\overline{R}_{jk} g_{il} - \overline{R}_{ik} g_{jl} - \overline{R}_{jl} g_{ik} + \overline{R}_{il} g_{jk}\right].$$

Using (4.6) and (4.8) we have

$$\begin{aligned} &\overline{V}_{ijkl} = V_{ijkl} + F_{il}F_k{}^t\pi_{jt} - F_{jl}F_k{}^t\pi_{it} - F_{ik}F_l{}^t\pi_{jt} + F_{jk}F_l{}^t\pi_{it} \\ &-\frac{1}{n-2}[\left(2\pi_{jk} - g_{jk} trace\pi + F_{jk}F^{mt}\pi_{mt}\right) g_{il} \\ &- \left(2\pi_{ik} - g_{ik} trace\pi + F_{ik}F^{mt}\pi_{mt}\right) g_{jl} \\ &- \left(2\pi_{jl} - g_{jl} trace\pi + F_{jl}F^{mt}\pi_{mt}\right) g_{ik} + \left(2\pi_{il} - g_{il} trace\pi + F_{il}F^{mt}\pi_{mt}\right) g_{jk}], \end{aligned} \tag{4.10}$$

where V_{ijkl} is the conharmonic curvature tensor with respect to the Levi-Civita connection.

Theorem 4.10. *Let (M_n, g, F) be an anti-Kähler manifold equipped with the connection (2.7). If the conharmonic curvature tensor with respect to the connection (2.7) vanishes, then the scalar curvature of the connection (2.7) vanishes.*

Proof. If we assume that $\overline{V}_{ijkl} = 0$, from (4.10) we have

$$\begin{aligned} 0 &= V_{ijkl} + F_{il}F_k{}^t\pi_{jt} - F_{jl}F_k{}^t\pi_{it} - F_{ik}F_l{}^t\pi_{jt} + F_{jk}F_l{}^t\pi_{it} \\ &\quad -\frac{1}{n-2}[\left(2\pi_{jk} - g_{jk}\pi_t^t + F_{jk}F^{mt}\pi_{mt}\right) g_{il} - \left(2\pi_{ik} - g_{ik}\pi_t^t + F_{ik}F^{mt}\pi_{mt}\right) g_{jl} \\ &\quad - \left(2\pi_{jl} - g_{jl}\pi_t^t + F_{jl}F^{mt}\pi_{mt}\right) g_{ik} + \left(2\pi_{il} - g_{il}\pi_t^t + F_{il}F^{mt}\pi_{mt}\right) g_{jk}]. \end{aligned}$$

When we multiply the last equation by g^{il}, using the condition $V_{ijkl}g^{il} = V_{ljk}{}^{l} = -\frac{\tau}{n-2}g_{jk}$, we find

$$\nabla_l p^l + \frac{1}{2(2-n)}\tau + \frac{n-4}{2}p_l p^l = \frac{\overline{\tau}}{2(2-n)} = 0,$$

where $\lambda = \frac{1}{2(2-n)}$, $\mu = \frac{n-4}{2}$ and τ and $\overline{\tau}$ respectively are the scalar curvatures of the Levi-Civita connection and the connection (2.7). This completes the proof. □

Theorem 4.11. *Let (M_n, g, F) be an anti-Kähler manifold equipped with the connection (2.7). If the conharmonic curvature tensors with respect to the connection (2.7) and the Levi-Civita connection coincide, then the $1-$form p satisfies $\nabla_l p^l + \frac{n-4}{2}p_l p^l = 0$.*

Proof. Let $\overline{V}_{ijkl} = V_{ijkl}$, from (4.10) we obtain

$$\begin{aligned} 0 \;=\; & F_{il}F_k{}^t\pi_{jt} - F_{jl}F_k{}^t\pi_{it} - F_{ik}F_l{}^t\pi_{jt} + F_{jk}F_l{}^t\pi_{it} \\ & -\frac{1}{n-2}[\left(2\pi_{jk} - g_{jk}trace\pi + F_{jk}F^{mt}\pi_{mt}\right)g_{il} \\ & -\left(2\pi_{ik} - g_{ik}trace\pi + F_{ik}F^{mt}\pi_{mt}\right)g_{jl} \\ & -\left(2\pi_{jl} - g_{jl}trace\pi + F_{jl}F^{mt}\pi_{mt}\right)g_{ik} + \left(2\pi_{il} - g_{il}trace\pi + F_{il}F^{mt}\pi_{mt}\right)g_{jk}]. \end{aligned} \tag{4.11}$$

Transvecting (4.11) by g^{il}, we have

$$(2-n)\pi_{jk} + g_{jk}trace\pi = 0. \tag{4.12}$$

Transvecting (4.12) by g^{jk}, we find

$$2trace\pi = 0$$

which leads to $\nabla_l p^l + \frac{n-4}{2}p_l p^l = 0$. □

Theorem 4.12. *Let (M_n, g, F) be an anti-Kähler manifold equipped with the connection (2.7). Then the conharmonic curvature tensor with respect to the connection (2.7) has the following properties:*

i) $\overline{V}_{ijkl} = -\overline{V}_{jikl}$,

ii) $\overline{V}_{ijkl} = -\overline{V}_{ijlk}$,

iii) Under the condition of $n \geqslant 6$, $\overline{V}_{ijkl} + \overline{V}_{kijl} + \overline{V}_{jkil} = 0$ if and only if the $1-$form p is closed.

Proof. *i)* Interchanging i and j in (4.10), and then adding it to (4.10), we have

$$\overline{V}_{ijkl} + \overline{V}_{jikl} = V_{ijkl} + V_{jikl}.$$

Since in a Riemannian manifold $V_{ijkl} + V_{jikl} = 0$, we find $\overline{V}_{ijkl} + \overline{V}_{jikl} = 0$. Similarly the proof of (ii) can easily be proven.

iii) From (4.10), using $V_{ijkl} + V_{kijl} + V_{jkil} = 0$ we can write

$$\begin{aligned} & \overline{V}_{ijkl} + \overline{V}_{kijl} + \overline{V}_{jkil} \\ = \; & F_{il}F_k{}^t(\pi_{jt} - \pi_{tj}) - F_{jl}F_k{}^t(\pi_{it} - \pi_{ti}) - F_{kl}F_j{}^t(\pi_{it} - \pi_{ti}) \\ & -\frac{2}{n-2}[(\pi_{jk} - \pi_{kj})g_{il} - (\pi_{ik} - \pi_{ki})g_{jl} + (\pi_{ij} - \pi_{ji})g_{kl}]. \end{aligned} \tag{4.13}$$

It is a direct consequence of (4.13) that $dp = 0$ implies $\overline{V}_{ijkl} + \overline{V}_{kijl} + \overline{V}_{jkil} = 0$.

If we assume that $\overline{V}_{ijkl} + \overline{V}_{kijl} + \overline{V}_{jkil} = 0$. Transvecting the last equation with F^{il} and then it with $F_i{}^k$, we obtain $(n-4)(\pi_{ij} - \pi_{ji}) = 0$ which gives the result. □

The Weyl projective curvature tensor with respect to the connection (2.7) is given by

$$\overline{P}_{ijkl}=\overline{R}_{ijkl}-\frac{1}{n-1}\left[\overline{R}_{jk}g_{il}-\overline{R}_{ik}g_{jl}\right]. \tag{4.14}$$

Substituting the values of $\overline{R}_{ijkl}$ and $\overline{R}_{ik}$ from (4.6) and (4.8) respectively into (4.14), we get

$$\begin{aligned}&\overline{P}_{ijkl} \\ =\ & P_{ijkl}+g_{ik}\pi_{jl}-g_{jk}\pi_{il}-F_{il}F_k^{\ t}\pi_{jt}+F_{jl}F_k^{\ t}\pi_{it} \\ &+F_{ik}F_l^{\ t}\pi_{jt}-F_{jk}F_l^{\ t}\pi_{it}-\frac{1}{n-1}[(3\pi_{jk}-g_{jk}trace\pi+F_{jk}F^{mt}\pi_{mt})g_{il} \\ &-(3\pi_{ik}-g_{ik}trace\pi+F_{ik}F^{mt}\pi_{mt})g_{jl}],\end{aligned} \tag{4.15}$$

where P_{ijkl} is the Weyl projective curvature tensor with respect to the Levi-Civita connection.

Theorem 4.13. *Let (M_n,g,F) be an anti-Kähler manifold equipped with the connection (2.7). If the Weyl projective curvature tensor with respect to the connection (2.7) vanishes, then the $1-$form p is closed, under the condition of $n\geqslant 6$.*

Proof. Let $\overline{P}_{ijkl}=0$. Then from (4.15) we get

$$\begin{aligned}0\ =\ & P_{ijkl}+g_{ik}\pi_{jl}-g_{jk}\pi_{il}-F_{il}F_k^{\ t}\pi_{jt}+F_{jl}F_k^{\ t}\pi_{it}+F_{ik}F_l^{\ t}\pi_{jt} \\ &-F_{jk}F_l^{\ t}\pi_{it}-\frac{1}{n-1}[(3\pi_{jk}-g_{jk}\pi_m^m+F_{jk}F^{mt}\pi_{mt})g_{il} \\ &-(3\pi_{ik}-g_{ik}\pi_m^m+F_{ik}F^{mt}\pi_{mt})g_{jl}].\end{aligned}$$

Transvecting the previous equation by g^{kl}, we have

$$P_{ijk}^{\ \ \ \ k}+(n-4)(\pi_{kj}-\pi_{jk})=0.$$

Since in a Riemannian manifold, the following equation holds: $P_{ijk}^{\ \ \ \ k}=0$, the result immediately follows, under the condition of $n\geqslant 6$. □

Theorem 4.14. *Let (M_n,g,F) be an anti-Kähler manifold equipped with the connection (2.7). Then the Weyl projective curvature tensor with respect to the connection (2.7) has the following properties:*

i) $\overline{P}_{ijkl}=-\overline{P}_{jikl}$,

ii) $\overline{P}_{ijkl}+\overline{P}_{kijl}+\overline{P}_{jkil}=0$ *if and only if the $1-$form p is closed.*

Proof. *i)* Interchanging i and j in (4.15), and then adding it to (4.15), we obtain

$$\overline{P}_{ijkl}+\overline{P}_{jikl}=P_{ijkl}+P_{jikl}.$$

Since in a Riemannian manifold $P_{ijkl}+P_{jikl}=0$, we find $\overline{P}_{ijkl}=-\overline{P}_{jikl}$.

ii) From (4.15), using $P_{ijkl}+P_{kijl}+P_{jkil}=0$ we get

$$\begin{aligned}&\overline{P}_{ijkl}+\overline{P}_{kijl}+\overline{P}_{jkil} \\ =\ & F_{jl}F_k^{\ t}(\pi_{it}-\pi_{ti})-F_{il}F_k^{\ t}(\pi_{jt}-\pi_{tj})-F_{kl}F_j^{\ t}(\pi_{it}-\pi_{ti}) \\ &-\frac{3}{n-1}\left[(\pi_{jk}-\pi_{kj})g_{il}-(\pi_{ik}-\pi_{ki})g_{jl}+(\pi_{ij}-\pi_{ji})g_{kl}\right].\end{aligned} \tag{4.16}$$

It follows directly from (4.16) that $dp=0$ implies $\overline{P}_{ijkl}+\overline{P}_{kijl}+\overline{P}_{jkil}=0$.

Conversely, let us assume that $\overline{P}_{ijkl}+\overline{P}_{kijl}+\overline{P}_{jkil}=0$. When we transvect (4.16) with g^{il}, it reduces to

$$\left(\frac{5n-8}{n-1}\right)(\pi_{kj}-\pi_{jk})=0$$

from which the result follows. □

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