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Research Article

Linear positive operators constructed by using Beta-type bases

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Abstract

Starting from a discrete linear approximation process that has the ability to turn polynomials into polynomials of the same degree, we introduce an integral generalization by using Beta-type bases. Some properties of this new sequence of operators are investigated in unweighted and weighted spaces of functions defined on unbounded interval. In our construction particular cases are outlined.

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1. Introduction

An old area of mathematical research called Approximation Theory has a great potential for applications to a wide variety of issues. The study of the linear methods of approximation, which are given by sequences of linear and positive operators, has become a firmly rooted part of the domain mentioned above.

The starting point of this note is a general positive approximation process of discrete type and expressed by series. The functions that are approximated are defined on unbounded intervals. Since a linear substitution maps the interior of such intervals onto $(0,\infty)$ or $\mathbb{R}$, we will choose the benchmark interval $I=[0,\infty)=\mathbb{R}_+$, as it accumulates the problems caused by a finite endpoint and by the unboundedness of the other endpoint. Also, each operator of this class uses an equidistant network with a flexible step of the form $\Delta_n=(k\lambda_n)_{n\geq 1}$, where $(\lambda_n)_{n\geq 1}$ is a strictly decreasing positive sequence of real numbers with the property

$$\lim_{n\to\infty}\lambda_n=0. \tag{1.1}$$

The operators we are referring to are designed as follows

$$(L_nf)(x)=\sum_{k=0}^{\infty}a_k(\lambda_n;x)f(k\lambda_n),\ n\in\mathbb{N},\ x\in\mathbb{R}_+, \tag{1.2}$$

where $a_k(\lambda_n;\cdot):\mathbb{R}_+\to\mathbb{R}_+$ is continuous function for each $(n,k)\in\mathbb{N}\times\mathbb{N}_0$.

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Here $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and f belongs to a function space, say $\mathcal{S}(\mathbb{R}_+)$, for which the right-hand side of relation (1.2) is well defined.

Using a Beta functions base, we introduce and explore an integral generalization of the operators defined by (1.2). The construction of the new class is presented in the next section. Their approximation properties such as convergence towards identity operator and the rate of convergence are investigated in different function spaces.

The use of Beta function in construction of approximation processes is not new. For different special classes of operators, the results have published over time. Among the most recent papers appeared in 2018, we mention [14] and [16]. In the first one, Beta type extension of Jakimovski-Leviatan operators is studied. In the second one, the authors defined a new type of Bernstein operator by using Beta type bases. Also, in [1], genuine α-Bernstein - Durrmeyer operators as a composition of α-Bernstein operators and Beta operators are introduced.

2. The operators

Set e_j the monomial of j-th degree, $e_0(x) = 1$ and $e_j(x) = x^j$, $j \geq 1$. Let Π_j^* be the set of all algebraic polynomials of degree j, $j \in \mathbb{N}_0$. Regarding L_n, $n \in \mathbb{N}$, operators we assume $\Pi := \bigcup_{j\geq 0} \Pi_j^* \subset \mathcal{S}(\mathbb{R}_+)$,

$$L_n e_0 = e_0 \text{ and } L_n e_j \in \Pi_j^*,\ j \in \mathbb{N}. \tag{2.1}$$

Due to the linearity of L_n, the above conditions imply that these operators preserve the constants and map polynomials into polynomials of the same degree. Throughout the paper, based on relations (2.1), we consider the following representations of the polynomials $L_n e_j$

$$(L_n e_j)(x) = \sum_{i=0}^{j} \alpha_{i,j}(n) x^i,\ \alpha_{j,j}(n) \neq 0,\ x \in \mathbb{R}_+ \text{ and } j \in \mathbb{N}. \tag{2.2}$$

Since we are working on the assumption that the sequence $(L_n)_{n\geq 1}$ is an approximation process, this will involve at least the pointwise convergence of $(L_n e_j)_{n\geq 1}$ to e_j for each $j \in \mathbb{N}$, in other words,

$$\lim_{n\to\infty} \alpha_{i,j}(n) = \delta_{i,j},\ 0 \leq i \leq j \text{ and } j \in \mathbb{N}, \tag{2.3}$$

where $\delta_{i,j}$ is Kronecker symbol.

Our requirements in (2.1) are not very restrictive. Many classical operators enjoy these properties. For instance:

i) *Szász-Mirakjan operators* satisfy (2.1), according to Micchelli's result [13, p. 71].

ii) *Baskakov operators* fulfill (2.1), see Sikkema [15, p. 335].

iii) *Mastroianni operators* [12] representing a generalization of Baskakov approximation process. For comparison, [2, pages 344, 350] may be consulted. Relations (2.1) can easily be deduced from the cited works.

In all the above examples $\lambda_n = 1/n$, $n \geq 1$, takes place.

iv) *Bernstein-Chlodovsky operators* introduced by Chlodovsky [3] can be found in [11, p. 36] or in [2, pages 347, 351]. In this case $\lambda_n = b_n/n$, where $(b_n)_{n\geq 1}$ is a sequence of strictly positive real numbers such that $\lim_{n\to\infty} b_n = \infty$ and (1.1) holds. Also, in this case, the requirements (2.1) are obviously satisfied. We propose to study the following integral variant of L_n, $n \geq 1$, operators, while meeting the assumption (2.1).

$$(D_n f)(x) = a_0(\lambda_n; x) f(0) + \sum_{k=1}^{\infty} \frac{a_k(\lambda_n; x)}{B(k, n+1)} \int_0^\infty f(n\lambda_n t) \frac{t^{k-1}}{(1+t)^{n+k+1}} dt, \tag{2.4}$$

$x \in \mathbb{R}_+$ and B stands for Beta function,

$$B(p,q) = \int_0^\infty t^{p-1}(1+t)^{-(p+q)}dt,\ (p,q) \in \mathbb{R}_+^* \times \mathbb{R}_+^*.$$

In the above $f \in \mathcal{F}(\mathbb{R}_+)$, the space of all real valued functions defined on $\mathbb{R}_+$ with the property that the series from the right hand side of relation (2.4) is convergent. We infer that f must be a Lebesgue measurable function on $\mathbb{R}_+$.

Clearly, D_n, $n \in \mathbb{N}$, are linear and positive operators. Moreover,

$$(D_n e_0)(x) = (L_n e_0)(x) = 1,\ x \in \mathbb{R}_+,\ n \geq 1, \tag{2.5}$$

see (2.1).

Using relation (1.2), by a straightforward calculation, we obtain the following identities

$$(D_n e_1)(x) = (L_n e_1)(x),\ x \in \mathbb{R}_+,\ n \geq 1, \tag{2.6}$$

$$(D_n e_2)(x) = \frac{n}{n-1}(L_n e_2)(x) + \frac{n\lambda_n}{n-1}(L_n e_1)(x),\ x \in \mathbb{R}_+,\ n \geq 2. \tag{2.7}$$

Currently, in approximation by linear positive operators, particular attention is paid to those which reproduce affine functions, property implied by the relations $L_n e_j = e_j$, $j \in \{0,1\}$. From (2.5) and (2.6) we notice that if L_n operators have this property, also D_n operators enjoy it.

We specify that particular cases of operators described by formula (2.4) have been investigated by various authors. To illustrate this, we present two examples.

Starting from Jain operators [8] defined as follows

$$(P_n^{[\beta]} f)(x) = e^{-nx} + nx\sum_{k=1}^\infty \frac{(nx+k\beta)^{k-1}}{k!}e^{-(nx+k\beta)} f\left(\frac{k}{n}\right),\ x \geq 0, \tag{2.8}$$

where the parameter $\beta \in [0,1)$, in [16] an integral version with Beta bases functions has been investigated. If in (2.8) we choose $\beta = 0$, $P_n^{[0]}$, $n \in \mathbb{N}$, turn into well known Szász-Mirakjan operators. An integral extension of $P_n^{[0]}$ operators following the model shown in (2.5), was achieved in [7].

The second example is taken from the recent paper [9]. Acar and Kumar introduced and studied the following generalization of Baskakov-Durrmeyer operators in the sense of D.D. Stancu

$$(L_{n,a}^{\alpha,\beta} f)(x) = \sum_{k=1}^\infty \frac{W_{n,k}^a(x)}{B(k,n)}\int_0^\infty \frac{t^{k-1}}{(1+t)^{n+k}} f\left(\frac{nt+\alpha}{n+\beta}\right)dt + W_{n,0}^a(x) f\left(\frac{\alpha}{n+\beta}\right),$$

where

$$W_{n,k}^a(x) = e^{-\frac{ax}{1+x}}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}},\ P_k(n,a) = \sum_{i=0}^k \binom{k}{i}(n)_i a^{k-i},$$

$0 \leq \alpha \leq \beta$, $a > 0$ and $(n)_i$ is the Pochammer symbol given by

$$(n)_0 = 1,\ (n)_i = n(n+1)\ldots(n+i-1) \text{ for } i \in \mathbb{N}.$$

At this point we introduce the j-th central moment of D_n operators, $n \geq j \in \mathbb{N}$, i.e.,

$$\mathcal{M}_j(D_n; x) = (D_n \varphi_x^j)(x),\ \text{where } \varphi_x(t) = t - x,\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Examining identities (2.5)-(2.7) and taking into account (2.2), we get the first two central moments

$$\mathcal{M}_j(D_n; x) = \sum_{l=0}^j \beta_{l,j}(n)x^j,\ x \geq 0,\ n \geq 2,\ j \in \{1,2\},$$

where

$$\beta_{0,1}(n) = \alpha_{0,1}(n),\ \beta_{1,1}(n) = \alpha_{1,1}(n) - 1, \tag{2.9}$$

$$\begin{cases} \beta_{0,2}(n) = \dfrac{n}{n-1}(\alpha_{0,2}(n) + \lambda_n \alpha_{0,1}(n)), \\ \beta_{1,2}(n) = \dfrac{n}{n-1}(\alpha_{1,2}(n) + \lambda_n \alpha_{1,1}(n)) - 2\alpha_{0,1}(n), \\ \beta_{2,2}(n) = \dfrac{n}{n-1}\alpha_{2,2}(n) - 2\alpha_{1,1}(n) + 1. \end{cases} \tag{2.10}$$

Considering the notation $c(n) = \max\limits_{0\le l\le 2} |\beta_{l,2}(n)|$, $n \ge 2$, we can write

$$\mathcal{M}_2(D_n; x) \le c(n)(1 + x + x^2), \ x \ge 0. \tag{2.11}$$

Taking in view (1.1) and (2.3), $\lim\limits_{n\to\infty} c(n) = 0$ takes place.

3. Approximation properties

Our goal is to show how the approximation properties are transferred from L_n to D_n, $n \in \mathbb{N}$, operators. As usual, $B(\mathbb{R}_+)$ and $C(\mathbb{R}_+)$ stand for bounded, respectively continuous real valued functions defined on $\mathbb{R}_+$. Set $C_B(\mathbb{R}_+) = B(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ which is endowed with the norm of the uniform convergence (sup-norm) denoted by $\|\cdot\|_\infty$.

Remark 3.1. Clearly, $C_B(\mathbb{R}_+) \subset \mathcal{F}(\mathbb{R}_+)$. Since (2.5) takes place, for any function $f \in C_B(\mathbb{R}_+)$ we deduce $\|D_n f\|_\infty \le \|f\|_\infty$, in other words, the operators D_n, $n \in \mathbb{N}$, are non-expansive.

We investigate the summation integral operators in functions spaces characterized by the polynomial weight $\rho(x) = 1 + x^2$, $x \ge 0$. We indicate the following notations used throughout the paper.

$$B_\rho(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{R} \mid |f(x)| \le M_f \rho(x), \ x \ge 0\},$$

where M_f is a constant depending only on f. $B_\rho(\mathbb{R}_+)$ is a normed space with respect to the norm $\|\cdot\|_\rho$,

$$\|f\|_\rho = \sup_{x\ge 0} \frac{|f(x)|}{\rho(x)}.$$

Further, we denote by $C_\rho(\mathbb{R}_+)$, $C^*_\rho(\mathbb{R}_+)$ the following subspaces of $B_\rho(\mathbb{R}_+)$

$$C_\rho(\mathbb{R}_+) = \{f \in B_\rho(\mathbb{R}_+) : \ f \text{ is continuous on } \mathbb{R}_+\},$$

$$C^*_\rho(\mathbb{R}_+) = \{f \in C_\rho(\mathbb{R}_+) : \ \lim_{x\to\infty} \frac{f(x)}{\rho(x)} \text{ exists and is finite}\},$$

respectively.

Theorem 3.2. *Let L_n, $n \in \mathbb{N}$, be defined by (1.2) such that conditions (2.1)-(2.3) are fulfilled. For any compact $K \subset \mathbb{R}_+$, the operators D_n defined by (2.4) possess the property*

$$\lim_{n\to\infty} \|D_n f - f\|_K = 0, \ f \in C^*_\rho(\mathbb{R}_+), \tag{3.1}$$

where, for a function h, $\|h\|_K = \sup\limits_{t\in K} |h(t)|$.

Proof. We consider the lattice homomorphism $T_K : C(\mathbb{R}) \to C(K)$ defined by $T_K(f) = f|_K$. Relations (2.5), (2.6), (2.7) and the requirements formulated in the hypothesis of the theorem ensure the identities

$$\lim_{n\to\infty} T_K(D_n e_j) = T_K(e_j), \ j \in \{0, 1, 2\},$$

uniformly on K. Since $C^*_\rho(\mathbb{R}_+)$ is isomorphic to $C(K)$ and $\{e_0, e_1, e_2\}$ is a Korovkin set in $C^*_\rho(\mathbb{R}_+)$, the universal Korovkin-type criterion, see [2, Theorem 4.1.4(vi)] implies

$$(D_n f)(x) \to f(x) \text{ uniformly on } K \text{ as } n \to \infty.$$

Consequently, the proof of (3.1) is completed. □

One of the most recently studied subject in Approximation Theory is the approximation of functions using statistical convergence or matrix summability method. The advantage of replacing the uniform convergence by statistical convergence consists in the fact that the second convergence is efficient in summing divergent sequences which may have unbounded subsequences. The first research which deals with the statistical convergence for positive linear operators was attempted in 2002 by Gadjiev and Orhan. Based on their result, [6, Theorem 1], substituting (2.3) with the relations

$$st - \lim_{n\to\infty} \alpha_{i,j}(n) = \delta_{i,j},\ 0 \le i \le j \text{ and } j \in \mathbb{N},$$

we get

$$st - \lim_{n\to\infty} \|D_n f - f\|_K = 0,\ f \in C^*_\rho(\mathbb{R}_+),$$

for any compact $K \subset \mathbb{R}_+$.

In the above, a real sequence $(x_n)_{n\ge1}$ satisfies $st - \lim_n x_n = L$ if, for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} :\ |x_n - L| \ge \varepsilon\}) = 0$$

holds. In general, for any set $S \subseteq \mathbb{N}$, the density of S is denoted by $\delta(S)$ and is defined by

$$\delta(S) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_S(k),$$

provided the limit exists, where χ_S is the characteristic function of S.

We recall the notion of modulus of smoothness associated to a function $f \in C(\mathbb{R}_+)$ on the compact interval $[0,a]$, $a > 0$ fixed. Denoting by $\omega(f;\cdot)_{[0,a]}$, it is defined as follows

$$\omega(f;\delta)_{[0,a]} = \sup\{|f(t) - f(x)| :\ |t-x| \le \delta,\ (t,x) \in [0,a]\times[0,a]\},$$

where $\delta \ge 0$. As we can see, ω gives the maximum oscillation of f in any interval of length not exceeding δ.

Among the properties of $\omega(f;\cdot)_{[0,a]}$ we recall the following

$$\omega(f;|t-x|)_{[0,a]} \le \left(1 + \frac{(t-x)^2}{\delta^2}\right)\omega(f;\delta)_{[0,a]},\ \delta > 0, \tag{3.2}$$

and $(t,x) \in [0,a]\times[0,a]$, see, e.g., [10].

Using this modulus we establish the rate of convergence of the sequence $(D_n f)_{n\ge1}$ in a specific case, as shown in the following

Theorem 3.3. *Let L_n, $n \in \mathbb{N}$, be defined by (1.2) such that conditions (2.1)-(2.3) are fulfilled. For any compact $[0,b]$, $b > 0$ fixed, the operators D_n, $n \ge 2$, defined by (2.4) verify the following inequality*

$$|(D_n f)(x) - f(x)| \le M_{f,b}\delta_n^2(x) + 2\omega(f;\delta_n(x))_{[0,b+1]},\ x \in [0,b], \tag{3.3}$$

where $f \in C_\rho(\mathbb{R}_+)$, $M_{f,b}$ is a constant depending on f and b, and

$$\delta_n(x) = \sqrt{c(n)(x^2+x+1)}. \tag{3.4}$$

Proof. Let $x \in [0,b]$ arbitrarily fixed. Let t belong to $\mathbb{R}_+$.
If $t \le b+1$, then $|t-x| \le b+1$. With the help of (3.2) we can write

$$\begin{aligned} |f(t) - f(x)| &\le \sup_{|u-v|\le|t-x|} |f(u) - f(v)| = \omega(f;|t-x|)_{[0,b+1]} \\ &\le \left(1 + \frac{(t-x)^2}{\delta^2}\right)\omega(f;\delta)_{[0,b+1]},\ \delta > 0. \end{aligned} \tag{3.5}$$

If $t > b+1$, then $t - x > 1$. Since $f \in C_\rho(\mathbb{R}_+)$, we get

$$\begin{aligned}|f(t) - f(x)| &\le M_f(2 + x^2 + t^2)\\ &= M_f((2+2x^2) + 2x(t-x) + (t-x)^2)\\ &\le M_f((2+2x^2)(t-x)^2 + 2x(t-x)^2 + (t-x)^2)\\ &\le M_f\Big(\sup_{x\in[0,b]}(3+2x^2+2x)\Big)(t-x)^2\\ &= M_{f,b}(t-x)^2, \end{aligned}\tag{3.6}$$

where $M_{f,b}$ is a constant depending only on f and b.
Combining (3.5) and (3.6), we obtain

$$|f(t) - f(x)| \le M_{f,b}(t-x)^2 + \left(1 + \frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta)_{[0,b+1]},\ \delta > 0. \tag{3.7}$$

Since D_n is linear positive operator, it is monotone. Identity $D_n e_0 = e_0$ and inequalities (3.7), (2.11) allow us to write successively

$$\begin{aligned}|(D_nf)(x) - f(x)| &= |D_n(f - f(x); x)| \le D_n(|f - f(x)|; x)\\ &\le D_n\left(M_{f,b}\varphi_x^2 + \left(e_0 + \frac{\varphi_x^2}{\delta^2}\right)\omega(f;\delta)_{[0,b+1]}; x\right)\\ &\le M_{f,b}c(n)(x^2+x+1)\left(1 + \frac{c(n)(x^2+x+1)}{\delta^2}\right)\omega(f;\delta)_{[0,b+1]}.\end{aligned}$$

Choosing $\delta = \delta_n(x)$, see (3.4), we arrive at (3.3) and the proof is ended. □

In the following we will denote by C a positive constant which occurs different values in different relations, its main characteristic being that it does not depend on n.

Due to (2.3), all sequences from relation (2.10), $(b_{i,j}(n))_{n\ge1}$, $0 \le i \le j \le 2$, are bounded with respect to n. Consequently, $0 \le L_n\rho \le C\rho$. This relation and the definition of D_n operators imply $0 \le D_n\rho \le C\rho$. Taking into account the result established in [4, Eq. (4)], the previous inequality represents the necessary and sufficient condition which allows us to draw the following

Remark 3.4. Each operator D_n maps $C_\rho(\mathbb{R}_+)$ into $B_\rho(\mathbb{R}_+)$ and

$$\|D_n\|_{C_\rho(\mathbb{R}_+)\to B_\rho(\mathbb{R}_+)} = \|D_n\rho\|_\rho.$$

Theorem 3.5. *Let L_n, $n \in \mathbb{N}$, be defined by (1.2) such that conditions (2.1)-(2.3) are fulfilled. The operators D_n, $n \ge 2$, defined by (2.4) satisfy the following property*

$$\lim_{n\to\infty} \|D_nf - f\|_\rho = 0 \tag{3.8}$$

for every function $f \in C_\rho^*(\mathbb{R}_+)$.

Proof. Considering the statement drawn in Remark 3.4, we will apply a result set by Gadjiev [4, Theorem 2]. According to this result, in order to obtain (3.8), it is sufficient to prove the following relations

$$\lim_{n\to\infty} \|D_ne_1^k - e_1^k\|_\rho = 0,\ k = 0, 1, 2. \tag{3.9}$$

We were based on the fact that in our case $\rho = e_0 + e_1^2$.
Due to (2.5), for $k = 0$ identity (3.9) is evident.
Using (2.6) and (2.9), we can write

$$\begin{aligned}\|D_ne_1 - e_1\|_\rho &= \sup_{x\ge0}\frac{|(L_ne_1)(x) - x|}{1+x^2} = \sup_{x\ge0}\frac{|\alpha_{0,1}(n) + (\alpha_{1,1}(n)-1)x|}{1+x^2}\\ &\le |\alpha_{0,1}(n)| + |\alpha_{1,1}(n) - 1|.\end{aligned}$$

In view of (2.3), for $k = 1$ relation (3.9) holds.
Then, by calling (2.7) and (2.10), we have

$$\|D_n e_2 - e_2\|_\rho$$

$$= \sup_{x\ge 0} \frac{\left|\frac{n}{n-1}(\alpha_{0,2}(n) + \alpha_{1,2}(n)x + \alpha_{2,2}(n)x^2) + \frac{n\lambda_n}{n-1}(\alpha_{0,1}(n) + \alpha_{1,1}(n)x) - x^2\right|}{1+x^2}$$

$$\le \left|\frac{n}{n-1}\alpha_{2,2}(n) - 1\right| + \frac{n}{n-1}|\alpha_{1,2}(n) + \lambda_n\alpha_{1,1}(n)| + \frac{n}{n-1}|\alpha_{0,2}(n) + \lambda_n\alpha_{0,1}(n)|.$$

Relations (1.1) and (2.3) guarantee the veracity of the identity (3.9) in the case $k = 2$. Thus, the limit from (3.8) is completely proved. $\square$

Since $(1+x^2)^{1+\gamma} \ge \rho(x)$, $(x,\gamma) \in \mathbb{R}_+ \times \mathbb{R}_+^*$, from Theorem 3.5 we enunciate

Remark 3.6. For any $f \in C_\rho^*(\mathbb{R}_+)$ and $\gamma > 0$,

$$\lim_{n\to\infty} \sup_{x\ge 0} \frac{|(D_n f)(x) - f(x)|}{(1+x^2)^{1+\gamma}} = 0$$

takes place. In the particular case of Baskakov-Durrmeyer-Stancu type operators, the above relation was included in [9, Theorem 3.4].

Finally we present an estimate of error of approximation by using a more general weight ρ and a suitable modulus. Keeping the same notation, from this moment, we consider the weight ρ given by

$$\rho(x) = 1 + ax + bx^2,\ x \ge 0, \tag{3.10}$$

where $a \ge 1$ and $b > 0$. Since $\rho(0) = 1$ and $\inf_{x\ge 0} \rho'(x) \ge 1$, in accordance with [5], we can use the weighted modulus $\Omega_\rho(f;\cdot)$ defined by

$$\Omega_\rho(f;\delta) = \sup_{\substack{x,t\in\mathbb{R}_+ \\ |\rho(t)-\rho(x)|\le\delta}} \frac{|f(t) - f(x)|}{(|\rho(t) - \rho(x)| + 1)\rho(x)},\ \delta > 0, \tag{3.11}$$

for each $f \in C_\rho(\mathbb{R}_+)$.
Among the properties of this modulus proved by Gadjiev and Aral [5, Lemmas 4,5] we recall

$$\lim_{\delta\to 0} \Omega_\rho(f;\delta) = 0, \tag{3.12}$$

$$|f(t) - f(x)| \le 2\rho(x)(1+\delta)^2 \left(1 + \frac{(\rho(t) - \rho(x))^2}{\delta^2}\right) \Omega_\rho(f;\delta), \tag{3.13}$$

for each $f \in C_\rho^*(\mathbb{R}_+)$ and $\delta > 0$.
Regarding the weight ρ, from (3.10) we deduce

$$\frac{e_k}{\rho} \le 1,\ k \in \{1,2\},\ \text{and}\ \frac{e_k}{\rho^2} \le 1,\ k \in \{1,2,3,4\}. \tag{3.14}$$

$D_n e_j$, $j \le 4$, are polynomials with coefficients expressed in terms of $(\alpha_{i,j})_{0\le i\le j}$, $j \le 4$, subject to conditions (2.3). We assume that $\rho^2 \in \mathcal{F}(\mathbb{R}_+)$. These facts combined with (3.14) allow us to establish the relations

$$\|D_n\rho - \rho\|_\rho \le \mu_1(n),\ \|D_n\rho^2 - \rho^2\|_{\rho^2} \le \mu_2(n), \tag{3.15}$$

for $n \ge 4$, where

$$\lim_{n\to\infty} \mu_k(n) = 0,\ k \in \{1,2\}. \tag{3.16}$$

Theorem 3.7. *Let L_n, $n \in \mathbb{N}$, be defined by (1.2) such that conditions (2.1)-(2.3) are fulfilled. The operators D_n, $n \geq 4$, defined by (2.4) satisfy the following property*

$$\|D_n f - f\|_{\rho^3} \leq 4(1+\delta_n^*)^2 \Omega_\rho(f;\delta_n^*), \;\; f \in C_\rho^*(\mathbb{R}_+),$$

where ρ is given at (3.10), Ω_ρ is defined at (3.11) and

$$\delta_n^* = \sqrt{\mu_2(n) + 2\mu_1(n)}, \tag{3.17}$$

in accordance with (3.15).

Proof. Like how we reasoned Remark 3.4, we deduce that D_n is a mapping from $C_\rho(\mathbb{R}_+)$ into $B_\rho(\mathbb{R}_+)$ and also a mapping from $C_{\rho^2}(\mathbb{R}_+)$ into $B_{\rho^2}(\mathbb{R}_+)$. By using relations (3.13) and (2.5) we obtain the following estimate

$$\frac{|(D_n f)(x) - f(x)|}{\rho(x)} \leq 4(1+\delta)^2 \delta^{-2} D_n((\rho - \rho(x))^2; x)\Omega_\rho(f;\delta), \tag{3.18}$$

where $f \in C_\rho^*(\mathbb{R}_+)$. Further, we can write

$$D_n((\rho-\rho(x))^2;x) \leq |(D_n\rho^2)(x) - \rho^2(x)| + 2\rho(x)|(D_n\rho)(x) - \rho(x)|.$$

Using the inequalities from (3.15), for $n \geq 4$ and $x \geq 0$, we have

$$D_n((\rho-\rho(x))^2;x) \leq (\mu_2(n) + 2\mu_1(n))\rho^2(x).$$

Returning at (3.18) and choosing $\delta := \delta_n^*$, see (3.17), the conclusion of the theorem is fully motivated. $\square$

Remark 3.8. In view of (3.16) and (3.12), the above theorem implies

$$\lim_{n\to\infty} \|D_n f - f\|_{\rho^3} = 0, \;\; f \in C_\rho^*(\mathbb{R}_+).$$

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References

[1] T. Acar, A.M. Acu and N. Manav, *Approximation of functions by genuine Bernstein-Durrmeyer type operators*, J. Math. Ineq. **12** (4), 975–987, 2018.

[2] F. Altomare and M. Campiti, *Korovkin-type Approximation Theory and its Applications*, de Gruyter Series Studies in Mathematics, Vol. 17, Walter de Gruyter & Co., Berlin, New York, 1994.

[3] I. Chlodovsky, *Sur le développement des fonctions définies dans un interval infini en séries de polynômes de N.S. Bernstein*, Compositio Math. **4**, 380–393, 1937.

[4] A.D. Gadjiev, *Theorems of Korovkin type*, Mat. Zametki, **20** (5), 781–786 (in Russian), 1976; Mathematical Notes, **20** (5), 995-998 (English translation), 1976.

[5] A.D. Gadjiev and A. Aral, *The estimates of approximation by using a new type of weighted modulus of continuity*, Comput. Math. Appl. **54**, 127–135, 2007.

[6] A.D. Gadjiev and C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32**, 129–138, 2002.

[7] V. Gupta and M.A. Noor, *Convergence of derivatives for certain mixed Szász-Beta operators*, J. Math. Anal. Appl. **321**, 1–9, 2006.

[8] G.C. Jain, *Approximation of functions by a new class of linear operators*, J. Aust. Math. Soc. **13** (3), 271–276, 1972.

[9] A.S. Kumar and T. Acar, *Approximation by generalized Baskakov-Durrmeyer-Stancu type operators*, Rend. Circ. Mat. Palermo, Series 2, **65** (3), 411–424, 2016.

[10] G.G. Lorentz, *Approximation of functions*, Holt, Rinehart and Winston, Inc., New York, 1966.

[11] G.G. Lorentz, *Bernstein Polynomials*, 2nd Ed., Chelsea Publ. Comp., New York, NY, 1986.
[12] G. Mastroianni, *Su un operatore lineare e positivo*, Rend. Acc. Sc. Fis. Mat. Napoli, **46**, 161,-176, 1979.
[13] C.A. Micchelli, *Saturation classes and iterates of operators*, Dissertation, Stanford, 1969.
[14] M. Mursaleen and M. Nasiruzzaman, *Approximation of modified Jakimovski-Leviatan-Beta type operators*, Constr. Math. Anal. **1** (2), 88–98, 2018.
[15] P.C. Sikkema, *On some linear positive operators*, Indag. Math. **32**, 327–337, 1970.
[16] S. Tarabie, *On Jain-Beta linear operators*, Appl. Math. Inform. Sci. **6** (2), 213–216, 2012.