Multiplicative order convergence in \(f\)-algebras

Abdullah Aydın

Department of Mathematics, Muş Alparslan University, Muş, Turkey

Abstract

A net \( (x_{\alpha}) \) in an \( f\)-algebra \( E \) is said to be multiplicative order convergent to \( x \in E \) if \( |x_{\alpha} - x| \xrightarrow{u \to} 0 \) for all \( u \in E_+ \). In this paper, we introduce the notions \( mo\)-convergence, \( mo\)-Cauchy, \( mo\)-complete, \( mo\)-continuous, and \( mo\)-KB-space. Moreover, we study the basic properties of these notions.

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1. Introductory facts

In spite of the nature of the classical theory of Riesz algebra and \( f\)-algebra, as far as we know, the concept of convergence in \( f\)-algebras related to multiplication has not been done before. However, there are some close studies under the name unbounded convergence in some kinds of vector lattices; see for example [2–6]. In the light of this information, we define a new concept of the convergence, which is called the \( mo\)-convergence, on \( f\)-algebras. Our aim is to introduce the concept of the \( mo\)-convergence by using the multiplication in \( f\)-algebras and examine the relationship between other types of convergence.

First of all, let us remember some notations and terminologies used in this paper. Let \( E \) be a real vector space. Then \( E \) is called ordered vector space if it has an order relation \( \leq \) (i.e, \( \leq \) is reflexive, antisymmetric, and transitive) that is compatible with the algebraic structure of \( E \) that means \( y \leq x \) implies \( y + z \leq x + z \) for all \( z \in E \) and \( \lambda y \leq \lambda x \) for each \( \lambda \geq 0 \). An ordered vector \( E \) is said to be vector lattice (or, Riesz space) if, for each pair of vectors \( x, y \in E \), the supremum \( x \lor y = \sup\{x, y\} \) and the infimum \( x \land y = \inf\{x, y\} \) both exist in \( E \). Moreover, \( x^+ := x \lor 0 \), \( x^- := (-x) \lor 0 \), and \( |x| := x \lor (-x) \) are called the positive part, the negative part, and the absolute value of \( x \in E \), respectively. Also, two vectors \( x, y \) in a vector lattice are said to be disjoint whenever \( |x| \land |y| = 0 \). A vector lattice \( E \) is called order complete if \( 0 \leq x_{\alpha} \uparrow \leq x \) implies the existence of \( \sup x_{\alpha} \) in \( E \). A subset \( A \) of a vector lattice is called solid whenever \( |x| \leq |y| \) and \( y \in A \) imply \( x \in A \). A solid vector subspace is referred to as an order ideal. An order closed ideal is referred to as a band. A sublattice \( Y \) of a vector lattice is majorizing \( E \) if, for every \( x \in E \), there exists \( y \in Y \) with \( x \leq y \). A partially ordered set \( I \) is called directed if, for each \( a_1, a_2 \in I \), there is another \( a \in I \) such that \( a \geq a_1 \) and \( a \geq a_2 \) (or, \( a \leq a_1 \) and \( a \leq a_2 \)). A function from a directed set \( I \) into a set \( E \) is called a net in \( E \). A net \( (x_{\alpha})_{\alpha \in A} \) in a vector lattice \( X \) is called order convergent (or shortly, o-convergent) to \( x \in X \), if there exists another net

Email addresses: aaydin.aabdullah@gmail.com

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(y_\beta)_{\beta \in B} satisfying y_\beta \downarrow 0, and for any \beta \in B there exists \alpha_\beta \in A such that |x_\alpha - x| \leq y_\beta for all \alpha \geq \alpha_\beta. In this case, we write x_\alpha \xrightarrow{\alpha} x; for more details see for example [1,7,8].

A vector lattice \( E \) under an associative multiplication is said to be a Riesz algebra whenever the multiplication makes \( E \) an algebra (with the usual properties), and in addition, it satisfies the following property: \( x, y \in E_+ \) implies \( xy \in E_+ \). A Riesz algebra \( E \) is called commutative if \( xy = yx \) for all \( x, y \in E \). A Riesz algebra \( E \) is called \( f \)-algebra if \( E \) has additionally property that \( x \wedge y = 0 \) implies \( (xz) \wedge y = (zx) \wedge y = 0 \) for all \( z \in E_+ \); see for example [1]. A vector lattice \( E \) is called Archimedean whenever \( \frac{1}{n} x \downarrow 0 \) holds in \( E \) for each \( x \in E_+ \). Every Archimedean \( f \)-algebra is commutative; see Theorem 140.10 [8]. Assume \( E \) is an Archimedean \( f \)-algebra with a multiplicative unit vector \( e \). Then, by applying Theorem 142.1(v) [8], in view of \( e = ee = e^2 \geq 0 \), it can be seen that \( e \) is a positive vector. In this article, unless otherwise stated, all vector lattices are assumed to be real and Archimedean, and so \( f \)-algebras are commutative.

Recall that a net \( (x_\alpha) \) in a vector lattice \( E \) is unbounded order convergent (or shortly, \( uo \)-convergent) to \( x \in E \) if \( |x_\alpha - x| \wedge u \xrightarrow{o} 0 \) for every \( u \in E_+ \). In this case, we write \( x_\alpha \xrightarrow{\wedge \omega} x \); see for example [6] and [2,3,5]. Motivated from this definition, we give the following notion.

**Definition 1.1.** Let \( E \) be an \( f \)-algebra. A net \( (x_\alpha) \) in \( E \) is said to be multiplicative order convergent to \( x \in E \) (shortly, \( (x_\alpha) \) \( mo \)-converges to \( x \)) if \( |x_\alpha - x| \wedge u \xrightarrow{\omega} 0 \) for all \( u \in E_+ \). Abbreviated as \( x_\alpha \xrightarrow{\mo} x \).

It is clear that \( x_\alpha \xrightarrow{\mo} x \) in an \( f \)-algebra \( E \) implies \( x_\alpha \wedge y \xrightarrow{\mo} xy \) for all \( y \in E \) because of \( |xy| = |x||y| \) for all \( x, y \in E \). Also, in general, the \( mo \)-convergence and \( uo \)-convergence are not the same. To see that we consider the following example.

**Example 1.2.** Let \( E \) be a vector lattice and consider \( \text{Orth}(E) := \{ \mathcal{T} \in L_0(E) : x \perp y \text{ implies } \mathcal{T}x \perp y \} \) the set of orthomorphisms on \( E \). The space \( \text{Orth}(E) \) is not only vector lattice but also an \( f \)-algebra. The \( mo \)-convergence and the \( uo \)-convergence are different in \( \text{Orth}(E) \).

We shall keep in mind the following useful lemma, obtained from the property of \( xy \in E_+ \) for every \( x, y \in E_+ \).

**Lemma 1.3.** If \( y \leq x \) is provided in an \( f \)-algebra \( E \) then \( uy \leq ux \) for all \( u \in E_+ \).

Recall that multiplication by a positive element in \( f \)-algebras is a vector lattice homomorphism, i.e., \( u(x \wedge y) = (ux) \wedge (uy) \) and \( u(x \vee y) = (ux) \vee (uy) \) for every positive element \( u \); see for example Theorem 142.1(i) [8]. We will denote an \( f \)-algebra \( E \) as infinite distributive \( f \)-algebra whenever the following condition holds: if \( \inf(A) \) exists for any subset \( A \) of \( E_+ \) then the infimum of the subset \( uA \) exists and \( \inf(uA) = u \inf(A) \) for each positive vector \( u \in E_+ \). For a net \( (x_\alpha) \downarrow 0 \) in a infinite distributive \( f \)-algebra, the net \( (ux_\alpha) \) is also decreasing to zero for all positive vector \( u \).

**Remark 1.4.** The order convergence implies the \( mo \)-convergence in infinite distributive \( f \)-algebras. The converse holds true in \( f \)-algebras with multiplication unit. Indeed, assume a net \( (x_\alpha)_{\alpha \in A} \) order converges to \( x \) in an infinite distributive \( f \)-algebra \( E \). Then there exists another net \( (y_\beta)_{\beta \in B} \) satisfying \( y_\beta \downarrow 0 \), and, for any \( \beta \in B \), there exists \( \alpha_\beta \in A \) such that \( |x_\alpha - x| \leq y_\beta \). Hence, we have \( |x_\alpha - x|u \leq y_\beta u \) for all \( \alpha \geq \alpha_\beta \) and for each \( u \in E_+ \). Since \( y_\beta \downarrow 0 \), we have \( uy_\beta \downarrow 0 \) for each \( u \in E_+ \) by Lemma 1.3, and \( \inf(uy_\beta) = u \inf(y_\beta) = 0 \) because of \( \inf(y_\beta) = 0 \). Therefore, \( |x_\alpha - x|e \xrightarrow{o} 0 \) for each \( u \in E_+ \). That means \( x_\alpha \xrightarrow{\mo} x \).

For the converse, assume \( E \) is an \( f \)-algebra with multiplication unit \( e \) and \( x_\alpha \xrightarrow{\mo} x \) in \( E \). That is, \( |x_\alpha - x|u \xrightarrow{o} 0 \) for all \( u \in E_+ \). Since \( e \in E_+ \), in particular, choose \( u = e \), and so we have \( |x_\alpha - x| = |x_\alpha - x|e \xrightarrow{o} 0 \), or \( x_\alpha \xrightarrow{o} x \) in \( E \).
By considering Example 141.5 [8], we give the following example.

**Example 1.5.** Let \([a, b]\) be a closed interval in \(\mathbb{R}\) and let \(E\) be a vector lattice of all real continuous functions on \([a, b]\) such that the graph of functions consists of a finite number of line segments. In view of Theorem 141.1 [8], every positive orthomorphism \(\pi\) in \(E\) is trivial orthomorphism, i.e., there is a real number \(\lambda\) such that \(\pi(f) = \lambda f\) for all \(f \in E\). Therefore, a net of positive orthomorphisms \((\pi_\alpha)\) is order convergent to \(\pi\) if and only if it is \(mo\)-convergent to \(\pi\) whenever the multiplication is the natural multiplicative, i.e., \(\pi_1 \pi_2(f) = \pi_1(\pi_2(f))\) for all \(\pi_1, \pi_2 \in \text{Orth}(E)\) and \(f \in E\). Indeed, \(\text{Orth}(E)\) is Archimedean \(f\)-algebra with the identity operator as a unit element; see Theorem 140.4 [8]. So, by applying Remark 1.4, the \(mo\)-convergence implies the order convergence of the net \((\pi_\alpha)\).

Conversely, assume the net of positive orthomorphisms \(\pi_\alpha \xrightarrow{mo} \pi\) in \(\text{Orth}(E)\). Then we have \(\pi_\alpha(f) \xrightarrow{o} \pi(f)\) for all \(f \in E\); see Theorem VIII.2.3 [7]. For fixed \(0 \leq \mu \in \text{Orth}(E)\), there is a real number \(\lambda\) such that \(\mu(f) = \lambda f\) for all \(f \in E\). Since \(|\pi_\alpha(f) - \pi(f)| = |\lambda f - \lambda f| \xrightarrow{o} 0\), we have

\[
|(\pi_\alpha f - \pi f)| \xrightarrow{o} \mu = |\mu \lambda f - \lambda \mu f| = |\lambda f - \lambda f| \xrightarrow{o} 0
\]

for all \(f \in E\). Since \(\mu\) is arbitrary, we get \(\pi_\alpha \xrightarrow{mo} \pi\).

2. Main results

We begin the section with the next list of properties of the \(mo\)-convergence which follows directly from Lemma 1.3, and the inequalities \(|x - y| \leq |x - x_\alpha| + |x_\alpha - y|\) and \(||x|| = ||x_\alpha|| - |x|\) if and only if \(x, x_\alpha \in E\).

**Lemma 2.1.** Let \(x_\alpha \xrightarrow{mo} x\) and \(y_\alpha \xrightarrow{mo} y\) in an \(f\)-algebra \(E\). Then the following holds:

(i) \(x_\alpha \xrightarrow{mo} x\) if and only if \((x_\alpha - x) \xrightarrow{mo} 0\);
(ii) if \(x_\alpha \xrightarrow{mo} x\) then \(y_\beta \xrightarrow{mo} x\) for each subnet \((y_\beta)\) of \((x_\alpha)\);
(iii) suppose \(x_\alpha \xrightarrow{mo} x\) and \(y_\beta \xrightarrow{mo} y\) then \(ax_\alpha + by_\beta \xrightarrow{mo} ax + by\) for any \(a, b \in \mathbb{R}\);
(iv) if \(x_\alpha \xrightarrow{mo} y\) and \(x_\alpha \xrightarrow{mo} y\) then \(x = y\);
(v) if \(x_\alpha \xrightarrow{mo} x\) then \(\alpha \xrightarrow{mo} |x|\).

Recall that an order complete vector lattice \(E^5\) is said to be an order completion of the vector lattice \(E\) whenever \(E\) is Riesz isomorphic to a majorizing order dense vector lattice subspace of \(E^5\). Every Archimedean Riesz space has a unique order completion; see Theorem 2.24 [1].

**Proposition 2.2.** Let \((x_\alpha)\) be a net in an \(f\)-algebra \(E\). Then \(x_\alpha \xrightarrow{mo} 0\) in \(E\) if and only if \(x_\alpha \xrightarrow{mo} 0\) in the order completion \(E^5\) of \(E\).

**Proof.** Assume \(x_\alpha \xrightarrow{mo} 0\) in \(E\). Then \(x_\alpha \xrightarrow{mo} u_\alpha = 0\) in \(E^5\) for all \(u \in E_+, \) and so \(x_\alpha \xrightarrow{mo} 0\) in \(E^5\) for all \(u \in E_+;\) see Corollary 2.9 [6]. Now, let us fix \(v \in E^5_+\). Then there exists \(x_v \in E_+\) such that \(v \leq x_v \) because \(E\) majorizes \(E^5\). Then we have \(x_\alpha \xrightarrow{mo} v\) \(x_\alpha \xrightarrow{mo} x_v \) and \(v \xrightarrow{mo} x_v\). From \(x_\alpha \xrightarrow{mo} 0\) in \(E^5\) it follows that \(x_\alpha \xrightarrow{mo} 0\) in \(E^5\), that is, \(x_\alpha \xrightarrow{mo} 0\) in the order completion \(E^5\) because \(E^5\) is arbitrary.

Conversely, assume \(x_\alpha \xrightarrow{mo} 0\) in \(E^5\). Then, for all \(u \in E^5_+, \) we have \(x_\alpha \xrightarrow{mo} u\) in \(E^5\). In particular, for all \(x \in E^5_+, \) \(x \xrightarrow{mo} 0\) in \(E^5\). By Corollary 2.9 [6], we get \(x_\alpha \xrightarrow{mo} 0\) in \(E\) for all \(x \in E_+.\) Hence \(x_\alpha \xrightarrow{mo} 0\) in \(E.\)

The multiplication in \(f\)-algebra is \(mo\)-continuous in the following sense.

**Theorem 2.3.** Let \(E\) be an infinite distributive \(f\)-algebra, and \((x_\alpha)_{\alpha \in A}\) and \((y_\beta)_{\beta \in B}\) be two nets in \(E\). If \(x_\alpha \xrightarrow{mo} x\) and \(y_\beta \xrightarrow{mo} y\) for some \(x, y \in E\) and each positive element of \(E\) can be written as a multiplication of two positive elements then \(x_\alpha y_\beta \xrightarrow{mo} xy.\)
\textbf{Proof.} Assume \(x_\alpha \xrightarrow{m_0} x\) and \(y_\beta \xrightarrow{m_0} y\). Then \(|x_\alpha - x| u \xrightarrow{\alpha} 0\) and \(|y_\beta - y| u \xrightarrow{\beta} 0\) for every \(u \in E_+\). Let us fix \(u \in E_+\). So, there exist another two nets \((z_\gamma)_{\gamma \in \Gamma} \downarrow 0\) and \((z_\xi)_{\xi \in \Xi} \downarrow 0\) in \(E\) such that, for all \((\gamma, \xi) \in \Gamma \times \Xi\) there are \(\alpha_\gamma \in A\) and \(\beta_\xi \in B\) with \(|x_\alpha - x| u \leq z_\gamma\) and \(|y_\beta - y| u \leq z_\xi\) for all \(\alpha \geq \alpha_\gamma\) and \(\beta \geq \beta_\xi\).

Next, we show the \(m_0\)-convergence of \((x_\alpha y_\beta)\) to \(xy\). By considering the equality \(|xy| = |x||y|\) and Lemma 1.3, we have

\[
|x_\alpha y_\beta - xy| u = |x_\alpha y_\beta - x_\alpha y + x_\alpha y - xy| u \\
\leq |x_\alpha| |y_\beta - y| u + |x_\alpha - x| |y| u \\
\leq |x_\alpha - x| |y_\beta - y| u + |x_\alpha - x| |y| u.
\]

The second and the third terms in the last inequality both order converge to zero as \(\beta \rightarrow \infty\) and \(\alpha \rightarrow \infty\) respectively because of \(|x||y| u \in E_+\), \(x_\alpha \xrightarrow{m_0} x\) and \(y_\beta \xrightarrow{m_0} y\).

Now, we show the convergence of the first term of last inequality. There are two positive elements \(u_1, u_2 \in E_+\) such that \(u = u_1 u_2\) because the positive element of \(E\) can be written as a multiplication of two positive elements. So, we get \(|x_\alpha - x| |y_\beta - y| u = (|x_\alpha - x| u_1) (|y_\beta - y| u_2)\). Since \((z_\gamma)_{\gamma \in \Gamma} \downarrow 0\) and \((z_\xi)_{\xi \in \Xi} \downarrow 0\), the multiplication \((z_\gamma z_\xi) \downarrow 0\).

Indeed, we firstly show that the multiplication is decreasing. For indexes \((\gamma_1, \xi_1)(\gamma_2, \xi_2) \in \Gamma \times \Xi\), we have \(z_{\gamma_2} \leq z_{\gamma_1}\) and \(z_{\xi_2} \leq z_{\xi_1}\) because both of them are decreasing. Since the nets are positive, it follows from \(z_{\xi_2} \leq z_{\xi_1}\) that \(z_{\gamma_2} z_{\xi_2} \leq z_{\gamma_2} z_{\xi_1} \leq z_{\gamma_1} z_{\xi_1}\). As a result \((z_\gamma z_\xi)_{(\gamma, \xi) \in \Gamma \times \Xi} \downarrow 0\). Now, we show that the infimum of multiplication is zero. For a fixed index \(\gamma_0\), we have \(z_{\gamma_0} z_\xi \leq z_{\gamma_0} z_{\xi_1}\) for \(\gamma \geq \gamma_0\) because \((z_\gamma)\) is decreasing. Thus, we get \(\inf(z_{\gamma_0} z_\xi) = 0\) because of \(\inf(z_{\gamma_0} z_{\xi_1}) = 0\). Therefore, we see \(|x_\alpha - x| u_1 (|y_\beta - y| u_2) \xrightarrow{\alpha, \beta} 0\). Hence, we get \(x_\alpha y_\beta \xrightarrow{m_0} xy\).

The lattice operations in an \(f\)-algebra are \(m_0\)-continuous in the following sense.

\textbf{Proposition 2.4.} Let \((x_\alpha)_{\alpha \in \mathcal{A}}\) and \((y_\beta)_{\beta \in \mathcal{B}}\) be two nets in an \(f\)-algebra \(E\). If \(x_\alpha \xrightarrow{m_0} x\) and \(y_\beta \xrightarrow{m_0} y\) then \((x_\alpha \vee y_\beta)_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} \xrightarrow{m_0} x \vee y\). In particular, \(x_\alpha \xrightarrow{m_0} x\) implies \(x_\alpha^{+} \xrightarrow{m_0} x^{+}\).

\textbf{Proof.} Assume \(x_\alpha \xrightarrow{m_0} x\) and \(y_\beta \xrightarrow{m_0} y\). Then there exist two nets \((z_\gamma)_{\gamma \in \Gamma}\) and \((w_\lambda)_{\lambda \in \Lambda}\) in \(E\) satisfying \(z_\gamma \downarrow 0\) and \(w_\lambda \downarrow 0\), and for all \((\gamma, \lambda) \in \Gamma \times \Lambda\) there are \(\alpha_\gamma \in A\) and \(\beta_\lambda \in B\) such that \(|x_\alpha - x| u \leq z_\gamma\) and \(|y_\beta - y| u \leq w_\lambda\) for all \(\alpha \geq \alpha_\gamma\) and \(\beta \geq \beta_\lambda\) and for every \(u \in E_+\). It follows from the inequality \(|a \vee b - a \vee c| \leq |b - c|\) in vector lattices that \(|x_\alpha \vee y_\beta - x \vee y| u \leq |x_\alpha \vee y_\beta - x_\alpha \vee y\| u + |x_\alpha \vee y - x \vee y| u\| u\| u\| u\| u\| u\| u\| u\| u\| u\| u\| u\|

for all \(\alpha \geq \alpha_\gamma\) and \(\beta \geq \beta_\lambda\) and for every \(u \in E_+\). Since \((w_\lambda + z_\gamma) \downarrow 0\), \(|x_\alpha \vee y_\beta - x \vee y| u\| u\| u\| u\| u\| u\| u\| u\| u\| u\| u\|

order converges to 0 for all \(u \in E_+\). That is, \((x_\alpha \vee y_\beta)_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} \xrightarrow{m_0} x \vee y\).

\textbf{Lemma 2.5.} Let \((x_\alpha)\) be a net in an \(f\)-algebra \(E\). Then

(i) \(0 \leq x_\alpha \xrightarrow{m_0} x\) implies \(x_\alpha \xrightarrow{\alpha} x\).

(ii) if \((x_\alpha)\) is monotone and \(x_\alpha \xrightarrow{m_0} x\) then implies \(x_\alpha \xrightarrow{\alpha} x\).

\textbf{Proof.} (i) Assume \(0 \leq x_\alpha \xrightarrow{m_0} x\). Then we have \(x_\alpha = x_\alpha^{+} \xrightarrow{m_0} x^{+} = 0\) by Proposition 2.4. Hence, we get \(x_\alpha \in E_+\).

(ii) We show that \(x_\alpha \uparrow\) and \(x_\alpha \xrightarrow{m_0} x\) implies \(x_\alpha \uparrow x\). Fix an index \(\alpha\). Then we have \(x_\beta - x_\alpha \in X_+\) for \(\beta \geq \alpha\). By (i), \(x_\beta - x_\alpha \xrightarrow{m_0} x - x_\alpha \in X_+\). Therefore, \(x \geq x_\alpha\) for any \(\alpha\). Since \(\alpha\) is arbitrary, then \(x\) is an upper bound of \((x_\alpha)\). Assume \(y\) is another upper bound of \((x_\alpha)\), i.e., \(y \geq x_\alpha\) for all \(\alpha\). So, \(y - x_\alpha \xrightarrow{m_0} y - x \in X_+\), or \(y \geq x\), and so \(x_\alpha \uparrow x\).

The following simple observation is useful in its own right.

\textbf{Proposition 2.6.} Decreasing disjoint sequence in an \(f\)-algebra \(m_0\)-converges to zero.
\textbf{Proof.} Suppose \((x_n)\) is a disjoint decreasing sequence in an \(f\)-algebra \(E\). So, \(|x_n| u\) is also a disjoint sequence in \(E\) for all \(u \in E_+\); see Theorem 142.1(iii) \([8]\). Fix \(u \in E_+\), by Corollary 3.6 \([6]\), we have \(|x_n| u \xrightarrow{\text{mo}} 0\) in \(E\). So, \(|x_n| u \wedge w \xrightarrow{\text{mo}} 0\) in \(E\) for all \(w \in E_+\). Thus, in particular, taking \(w\) as \(|x_n| u\), we get
\[|x_n| u = |x_n| u \wedge |x_n| u = |x_n| u \wedge w \xrightarrow{\text{mo}} 0.\]

Therefore, \(x_n \xrightarrow{\text{mo}} 0\) in \(E\).

For the next two facts, observe the following fact. Let \(E\) be a vector lattice, \(I\) be an order ideal of \(E\) and \((x_\alpha)\) be a net in \(I\). If \(x_\alpha \xrightarrow{\omega} x\) in \(I\) then \(x_\alpha \xrightarrow{\sigma} x\) in \(E\). Conversely, if \((x_\alpha)\) is order bounded in \(I\) and \(x_\alpha \xrightarrow{\sigma} x\) in \(E\) then \(x_\alpha \xrightarrow{\omega} x\) in \(I\).

\textbf{Proposition 2.7.} Let \(E\) be an \(f\)-algebra, \(B\) be a projection band of \(E\) and \(P_B\) be the corresponding band projection. If \(x_\alpha \xrightarrow{\text{mo}} x\) in \(E\) then \(P_B(x_\alpha) \xrightarrow{\text{mo}} P_B(x)\) in both \(E\) and \(B\).

\textbf{Proof.} It is known that \(P_B\) is a lattice homomorphism and \(0 \leq P_B \leq I\). It follows from \(|P_B(x_\alpha) - P_B(x)| = |P_B(x_\alpha - x)| \leq |x_\alpha - x|\) that \(|P_B(x_\alpha) - P_B(x)| \leq |x_\alpha - x| u\) for all \(u \in E_+\). Then it follows easily that \(P_B(x_\alpha) \xrightarrow{\text{mo}} P_B(x)\) in both \(X\) and \(B\).

\textbf{Theorem 2.8.} Let \(E\) be an \(f\)-algebra and \(I\) be an order ideal and sub-\(f\)-algebra of \(E\). For an order bounded net \((x_\alpha)\) in \(I\), \(x_\alpha \xrightarrow{\text{mo}} 0\) in \(I\) if and only if \(x_\alpha \xrightarrow{\text{mo}} 0\) in \(E\).

\textbf{Proof.} Suppose \(x_\alpha \xrightarrow{\text{mo}} 0\) in \(E\). Then for any \(u \in I_+\), we have \(|x_\alpha| u \xrightarrow{\omega} 0\) in \(E\). So, the preceding remark implies \(|x_\alpha| u \xrightarrow{\sigma} 0\) in \(I\) because \(|x_\alpha| u\) is order bounded in \(I\). Therefore, we get \(x_\alpha \xrightarrow{\text{mo}} 0\) in \(I\).

Conversely, assume that \((x_\alpha)\) \text{mo}-converges to zero in \(I\). For any \(u \in I_+\), we have \(|x_\alpha| u \xrightarrow{\omega} 0\) in \(I\), and so in \(E\). Then, by applying Theorem 142.1(iv) \([8]\), we have \(x_\alpha w = 0\) for all \(w \in I^d = \{x \in E : x \perp y \text{ for all } y \in I\}\) and for each \(\alpha\) because \((x_\alpha)\) in \(I\). For any \(u \in I_+\) and each \(0 \leq w \in I^d\), it follows that
\[|x_\alpha|(u + w) = |x_\alpha|u + |x_\alpha|w = |x_\alpha| u \xrightarrow{\sigma} 0\]
in \(E\). So that, for each \(z \in (I \oplus I^d)_+\), we get \(|x_\alpha| z \xrightarrow{\sigma} 0\) in \(E\). It is known that \(I \oplus I^d\) is order dense in \(E\); see Theorem 1.36 \([1]\). Fix \(v \in E_+\). Then there exists some \(u \in (I \oplus I^d)\) such that \(v \leq u\). Thus, we have \(|x_\alpha| v \leq |x_\alpha| u \xrightarrow{\sigma} 0\) in \(E\). Therefore, \(|x_\alpha| v \xrightarrow{\sigma} 0\), and so \(x_\alpha \xrightarrow{\text{mo}} 0\) in \(E\).

The following proposition extends Theorem 3.8 \([2]\) to the general setting.

\textbf{Theorem 2.9.} Let \(E\) be an infinite distributive \(f\)-algebra with a unit \(e\) and \((x_n)\) \(\downarrow\) be a sequence in \(E\). Then \(x_n \xrightarrow{\text{mo}} 0\) if and only if \(|x_n|(u \wedge e) \xrightarrow{\omega} 0\) for all \(u \in E_+\).

\textbf{Proof.} For the forward implication, assume \(x_n \xrightarrow{\text{mo}} 0\). Hence, \(|x_n| u \xrightarrow{\omega} 0\) for all \(u \in E_+\), and so \(|x_n|(u \wedge e) \leq |x_n| u \xrightarrow{\sigma} 0\) because of \(e \in E_+\). Therefore, \(|x_n|(u \wedge e) \xrightarrow{\sigma} 0\).

For the reverse implication, fix \(u \in E_+\). By applying Theorem 2.57 \([1]\) and Theorem 142.1(i) \([8]\), note that
\[|x_n| u \leq |x_n| (u - u \wedge ne) + |x_n| (u \wedge ne) \leq \frac{1}{n} u^2 |x_n| + n |x_n| (u \wedge e)\]
Since \((x_n) \downarrow\) and \(E\) is Archimedean, we have \(\frac{1}{n} u^2 |x_n| \downarrow 0\). Furthermore, it follows from \(|x_n|(u \wedge e) \xrightarrow{\sigma} 0\) for each \(u \in E_+\) that there exists another sequence \((y_m)_{m \in B}\) satisfying \(y_m \downarrow 0\), and for any \(m \in B\), there exists \(n_m\) such that \(|x_n|(u \wedge e) \leq \frac{1}{n} y_m\), or \(n |x_n| (u \wedge e) \leq y_m\) for all \(n \geq n_m\). Hence, we get \(n |x_n|(u \wedge e) \xrightarrow{\sigma} 0\). Therefore, we have \(|x_n| u \xrightarrow{\sigma} 0\), and so \(x_n \xrightarrow{\text{mo}} 0\).

The \text{mo-convergence} passes obviously to any sub-\(f\)-algebra \(Y\) of \(E\), i.e., for any net \((y_\alpha)\) in \(Y\), \(y_\alpha \xrightarrow{\text{mo}} 0\) in \(E\) implies \(y_\alpha \xrightarrow{\text{mo}} 0\) in \(Y\). For the converse, we give the following theorem.
Theorem 2.10. Let $Y$ be a sub-f-algebra of an f-algebra $E$ and $(y_α)$ be a net in $Y$. If $y_α \xrightarrow{mo} 0$ in $Y$ then it mo-converges to zero in $E$ for each of the following cases;

(i) $Y$ is majorizing in $E$;
(ii) $Y$ is a projection band in $E$;
(iii) if, for each $u \in E$, there are element $x, y \in Y$ such that $|u - y| \leq |x|$.

Proof. Assume $(y_α)$ is a net in $Y$ and $y_α \xrightarrow{mo} 0$ in $Y$. Let us fix $u \in E_+$.

(i) Since $Y$ is majorizing in $E$, there exists $v \in Y_+$ such that $u \leq v$. It follows from

$$0 \leq |y_α| u \leq |y_α| v \xrightarrow{α} 0,$$

that $|y_α| u \xrightarrow{α} 0$ in $Y$. That is, $y_α \xrightarrow{mo} 0$ in $E$.

(ii) Since $Y$ is a projection band in $E$, we have $Y = Y^\perp$ and $E = Y \oplus Y^\perp$. Hence $u = u_1 + u_2$ with $u_1 \in Y_+$ and $u_2 \in Y^\perp_+$. Thus, we have $y_α \wedge u_2 = 0$ because $(y_α)$ in $Y$ and $u_2 \in Y^\perp$. Hence, by applying Theorem 142.1(iii) [8], we see $y_α u = 0$ for all index $α$.

It follows from

$$|y_α| u = |y_α| (u_1 + u_2) = |y_α| u_1 \xrightarrow{α} 0$$

that $|y_α| u \xrightarrow{α} 0$ in $E$. Therefore, $y_α \xrightarrow{mo} 0$ in $E$.

(iii) For the given $u \in E_+$, there exists elements $x, y \in Y$ with $|u - y| \leq |x|$. Then

$$|y_α| u \leq |y_α| |u - y| + |y_α| |y| \leq |y_α| |x| + |y_α| |y|.$$

By mo-convergence of $(y_α)$ in $Y$, we have $|y_α| |x| \xrightarrow{α} 0$ and $|y_α| |y| \xrightarrow{α} 0$, and so $|y_α| u \xrightarrow{α} 0$. That means $y_α \xrightarrow{mo} 0$ in $E$ because $u$ is arbitrary in $E_+$.

We continue with some basic notions in f-algebra, which are motivated by their analogies from vector lattice theory.

Definition 2.11. Let $(x_α)_{α \in A}$ be a net in f-algebra $E$. Then

(i) $(x_α)$ is said to be mo-Cauchy if the net $(x_α - x_α')_{(α, α') \in A \times A}$ mo-converges to 0,
(ii) $E$ is called mo-complete if every mo-Cauchy net in $E$ is mo-convergent,
(iii) $E$ is called mo-continuous if $x_α \xrightarrow{α} 0$ implies $x_α \xrightarrow{mo} 0$,
(iv) $E$ is called a mo-KB-space if every order bounded increasing net in $E_+$ is mo-convergent.

Remark 2.12. An f-algebra $E$ is mo-continuous if and only if $x_α \downarrow 0$ in $E$ implies $x_α \xrightarrow{mo} 0$. Indeed, the implication is obvious. For the converse, consider a net $x_α \xrightarrow{α} 0$. Then there exists a net $z_β \downarrow 0$ in $X$ such that, for any $β$ there exists $α_β$ so that $|x_α| \leq z_β$ for all $α \geq α_β$. Hence, by mo-continuity of $E$, we have $z_β \xrightarrow{mo} 0$, and so $x_α \xrightarrow{mo} 0$.

Proposition 2.13. Let $(x_α)$ be a net in an f-algebra $E$. If $x_α \xrightarrow{mo} x$ and $(x_α)$ is an o-Cauchy net then $x_α \xrightarrow{o} x$. Moreover, if $x_α \xrightarrow{mo} x$ and $(x_α)$ is wo-Cauchy then $x_α \xrightarrow{wo} x$.

Proof. Assume $x_α \xrightarrow{mo} x$ and $(x_α)$ is an order Cauchy net in $E$. Then $x_α - x_β \xrightarrow{o} 0$ as $α, β \to \infty$. Thus, there exists another net $z_γ \downarrow 0$ in $E$ such that, for every $γ$, there exists $α_γ$ satisfying

$$|x_α - x_β| \leq z_γ$$

for all $α, β \geq α_γ$. By taking f-limit over $β$ the above inequality and applying Proposition 2.4, i.e., $|x_α - x_β| \xrightarrow{mo} |x_α - x|$, we get $|x_α - x| \leq z_γ$ for all $α \geq α_γ$. That means $x_α \xrightarrow{o} x$. The similar argument can be applied for the wo-convergence case, and so the proof is omitted.

In the case of mo-complete in f-algebras, we have conditions for mo-continuity.

Theorem 2.14. For an mo-complete f-algebra $E$, the following statements are equivalent:

(i) $E$ is mo-continuous;
(ii) $E$ is mo-complete;
(iii) $E$ is mo-KB-space;
(iv) $E$ is mo-continuous.

(ii) if $0 \leq x_\alpha \uparrow \leq x$ holds in $E$ then $x_\alpha$ is a mo-Cauchy net;
(iii) $x_\alpha \downarrow 0$ implies $x_\alpha \xrightarrow{mo} 0$ in $E$.

**Proof.** (i) $\Rightarrow$ (ii) Consider the increasing and bounded net $0 \leq x_\alpha \uparrow \leq x$ in $E$. Then there exists a net $(y_\alpha)$ in $E$ such that $(y_\beta - x_\alpha)_{\alpha,\beta} \downarrow 0$; see Lemma 12.8 [1]. Thus, by applying Remark 2.12, we have $(y_\beta - x_\alpha)_{\alpha,\beta} \xrightarrow{mo} 0$, and so the net $(x_\alpha)$ is mo-Cauchy because of $|x_\alpha - x_\alpha'_{\alpha',\alpha} \leq |x_\alpha - y_\beta| + |y_\beta - x_\alpha'|$.

(ii) $\Rightarrow$ (iii) Suppose that $x_\alpha \downarrow 0$ in $E$, and fix arbitrary $\alpha_0$. Then we have $x_\alpha \leq x_{\alpha_0}$ for all $\alpha \geq \alpha_0$. Thus we can get $0 \leq (x_{\alpha_0} - x_\alpha)_{\alpha \geq \alpha_0} \uparrow \leq x_{\alpha_0}$. So, it follows from (ii) that the net $(x_{\alpha_0} - x_\alpha)_{\alpha \geq \alpha_0}$ is mo-Cauchy, i.e., $(x_\alpha - x_\alpha')_{\alpha \rightarrow \alpha'} \xrightarrow{mo} 0$ as $\alpha_0 \leq \alpha, \alpha' \rightarrow \infty$. Then there exists $x \in E$ satisfying $x_\alpha \xrightarrow{mo} x$ as $\alpha_0 \leq \alpha \rightarrow \infty$ because $E$ is mo-complete. Since $x_\alpha \downarrow$ and $x_\alpha \xrightarrow{mo} 0$, it follows from Lemma 2.5 that $x_\alpha \downarrow 0$, and so we have $x = 0$. Therefore, we get $x_\alpha \xrightarrow{mo} 0$.

(iii) $\Rightarrow$ (i) It is just the implication of Remark 2.12. □

**Corollary 2.15.** Let $E$ be an mo-continuous and mo-complete $f$-algebra. Then $E$ is order complete.

**Proof.** Suppose $0 \leq x_\alpha \uparrow \leq u$ in $E$. We show the existence of supremum of $(x_\alpha)$. By considering Theorem 2.14 (ii), we see that $(x_\alpha)$ is an mo-Cauchy net. Hence, there is $x \in E$ such that $x_\alpha \xrightarrow{mo} x$ because $E$ is mo-complete. It follows from Lemma 2.5 that $x_\alpha \uparrow x$ because of $x_\alpha \uparrow$ and $x_\alpha \xrightarrow{mo} x$. Therefore, $E$ is order complete. □

**Proposition 2.16.** Every mo-KB-space is mo-continuous.

**Proof.** Assume $x_\alpha \downarrow 0$ in $E$. From Theorem 2.14, it is enough to show $x_\alpha \xrightarrow{mo} 0$. Let us fix an index $\alpha_0$, and define another net $y_\alpha := x_{\alpha_0} - x_\alpha$ for $\alpha \geq \alpha_0$. Then it is clear that $0 \leq y_\alpha \uparrow \leq x_{\alpha_0}$, i.e., $(y_\alpha)$ is increasing and order bounded net in $E$. Since $E$ is a mo-KB-space, there exists $y \in E$ such that $y_\alpha \xrightarrow{mo} y$. Thus, by Lemma 2.5, we have $y_\alpha \xrightarrow{alpha} y$. Hence, $y = \sup y_\alpha = \sup(x_{\alpha_0} - x_\alpha) = x_{\alpha_0}$ because of $x_\alpha \downarrow 0$. Therefore, we get $y_\alpha = x_{\alpha_0} - x_\alpha \xrightarrow{mo} x_{\alpha_0}$ or $x_\alpha \xrightarrow{mo} 0$ because of $y_\alpha \xrightarrow{mo} y$. □

**Proposition 2.17.** Every mo-KB-space is order complete.

**Proof.** Suppose $0 \leq x_\alpha \uparrow \leq z$ is an order bounded and increasing net in an mo-KB-space $E$ for some $z \in E_+$. Then $x_\alpha \xrightarrow{mo} x$ for some $x \in E$ because $E$ is mo-KB-space. By Lemma 2.5, we have $x_\alpha \uparrow x$ because of $x_\alpha \uparrow$ and $x_\alpha \xrightarrow{mo} x$. So, $E$ is order complete. □

**Proposition 2.18.** Let $Y$ be an sub-$f$-algebra and order closed sublattice of an mo-KB-space $E$. Then $Y$ is also a mo-KB-space.

**Proof.** Let $(y_\alpha)$ be a net in $Y$ such that $0 \leq y_\alpha \uparrow \leq y$ for some $y \in Y_+$. Since $E$ is a mo-KB-space, there exists $x \in E_+$ such that $y_\alpha \xrightarrow{mo} x$. By Lemma 2.5, we have $y_\alpha \uparrow x$, and so $x \in Y$ because $Y$ is order closed. Thus $Y$ is a mo-KB-space. □

**References**

